MECCANICA RAZIONALE

Modi normali di vibrazione

→ coordinate normali

Quindi: $\mathcal{L} = \frac{1}{2} \dot{\underline{q}} A(\underline{q}) \dot{\underline{q}} - V(\underline{q})$

→ linearizzare $\underline{x} = \underline{q}_E - \underline{q}$

→ $\tilde{\mathcal{L}} = \frac{1}{2} \dot{\underline{x}} A(\underline{q}_E) \dot{\underline{x}} - \frac{1}{2} \underline{x} \, \mathrm{Hess}\, V|_{\underline{q}_E} \underline{x}$

$$A(\underline{q}_E) \ddot{\underline{x}} + C \underline{x} = \underline{0}$$

$\uparrow$ $\mathrm{Hess}\, V|_{q_E}$

l coord. libere → l eq. differenziali 2° ordine accoppiate fra di loro

$\underline{x} \longrightarrow \underline{\xi}$

fisiche → normali

$$A\ddot{\underline{x}} + C\underline{x} = 0 \longrightarrow \ddot{\xi}_i + \gamma_i \xi_i = 0$$

$i = 1, \dots, l$

$\xi \rightarrow$ risolvere

fisiche

exp

sin, cos

$\downarrow$

moti collettivi

$A\ddot{\underline{x}} + C\underline{x} = 0$ si può risolvere

direttamente :

- frequenze $\det(\lambda^2 A + C) = 0$

[nelle coord normali

$$(\lambda^2 + \gamma_1)(\lambda^2 + \gamma_2) \cdots (\lambda^2 + \gamma_l) = 0$$

λ^2 soddisfa

$$\lambda^2 + \gamma_i = 0$$

che viene da $\ddot{\xi}_i + \gamma_i \xi_i = 0$

per $\xi \sim e^{\lambda t}$]

$\cdot \quad (\lambda^2 A + C)\,\underline{u} = \underline{0}$

equazione agli autovettori (relativi)

$$x(t) = \underline{u}_1 \left(c_1 \sin \omega t + c_2 \cos \omega t \right) + \underline{u}_2 \left(c_1 e^{-\lambda t} + c_2 e^{+\lambda t} \right)$$

$+ \ldots$

Generalizzazione :

$\underline{x} \in R^{\ell}$

$$A \ddot{\underline{x}} + C \underline{x} = \underline{F}(t)$$

Termine forzante

$$\to \quad \underbrace{\ddot{\xi}_i + \gamma_i \xi_i}_{\text{calcolate come prima.}} = f_i(t) \qquad i = 1, \ldots, \ell$$

eq. coeff. costanti con un termine forzante.

In particolare se $\underline{F}(t)$ è una funzione armonica del tempo con una frequenza ω_f:

se ω_f è vicina ad una frequenza propria del sistema $\omega_j = \sqrt{\lambda_j}$, il modo normale j-esimo entra in risonanza.

Per passare alle coordinate normali

→ cerchiamo soluzione particolare del sistema non omogeneo

$$\underline{x} = \sum_{i=1}^{\ell} \underline{u}^{(i)} \, g_i(t)$$

sono quelli che abbiamo già calcolato quando $\underline{F} = \underline{0}$

$$A\ddot{\underline{x}} + C\underline{x} = \sum_{i=1}^{\ell} \left(\ddot{g}_i A \underline{u}^{(i)} + \right.$$

$$\left. + g_i \, C \underline{u}^{(i)} \right) =$$

siccome : $\lambda^2 A \underline{u} + C\underline{u} = \underline{0}$

$$\rightarrow \quad C\underline{u}^{(i)} = -\lambda_{(i)}^2 A \underline{u}^{(i)} =$$

$$= \gamma_i A \underline{u}^{(i)}$$

$$= \sum_{i=1}^{\ell} \left(\ddot{g}_i + \gamma_i g_i \right) A \underline{u}^{(i)} = \underline{F}(T)$$

moltiplichiamo scalarmente per $\underline{u}^{(j)}$

$$\underline{u}^{(j)} \cdot \left(\sum_{i=1}^{\ell} (\ddot{g}_i + \gamma_i g_i) A \underline{u}^{(i)} \right) =$$

$$= \sum_{i=1}^{\ell} (\ddot{g}_i + \gamma_i g_i) \underbrace{\underline{u}^{(j)} A \underline{u}^{(i)}}_{=0 \text{ se } j \neq i}$$

$\ddot{g}_j + \gamma_j g_j$

in questa somma rimane solo un termine

$$= \underline{u}^{(j)} A \underline{u}^{(j)} (\ddot{g}_j + \gamma_j g_j)$$

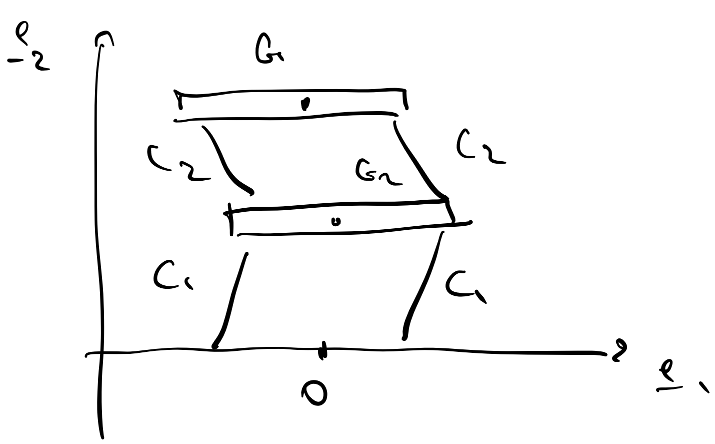
$$= \underline{F} \cdot \underline{u}^{(j)}$$

Troviamo:

$$\boxed{\ddot{g}_j + \gamma_j g_j = \frac{\underline{F} \cdot \underline{u}^{(j)}}{\underline{u}^{(j)} A \underline{u}^{(j)}} = f_j(T)}$$

le eq. sono disaccoppiate.

Esempio



$c_1 = c_2 = c$

$m_1 = 2m$

$m_2 = m$

$x_0(T) = a_f \sin \omega_f T$

$$\underline{A} = \begin{pmatrix} 2m & 0 \\ 0 & m \end{pmatrix} \qquad C = \begin{pmatrix} 4c & -2c \\ -2c & 2c \end{pmatrix}$$

$$\underline{F}(\tau) = \begin{pmatrix} 2m\, a_f\, \omega_f^2 \\ m\, a_f\, \omega_f^2 \end{pmatrix} \sin \omega_f \tau$$

→ Riscalare $\quad t = \sqrt{\dfrac{m}{2c}}\, \tau$

$$A \frac{d^2 \underline{x}}{d\tau^2} + C \underline{x} = \underline{F}$$

dividiamo per $2c$

$$\tilde{A} \frac{d^2}{d\tau^2} \underline{x} + \tilde{C} \underline{x} = \tilde{\underline{F}}$$

dove $\quad \tilde{A} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

$$\tilde{C} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\tilde{\underline{F}}(\tau) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} m\, a_f \frac{\omega_f^2}{2c} \sin \omega_f \sqrt{\frac{m}{2c}}\, \tau$$

$\tilde{\omega}_f^2 \qquad \tilde{\omega}_f$

$$\frac{m}{2c} \frac{d^2}{d\tau^2} = \frac{d^2}{d\left(\sqrt{\frac{2c}{m}}\, \tau\right)^2} = \frac{d^2}{dt^2} \qquad \tau = \sqrt{\frac{2c}{m}}\, t$$

per semplicità : $\tilde{\omega}_f = \omega_f \sqrt{\frac{m}{2c}}$

$$\underline{\tilde{F}}(\tau) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} a_f \tilde{\omega}_f^2 \sin \tilde{\omega}_f \tau$$

$$\tilde{A} \frac{d^2 \underline{x}}{d\tau^2} + \tilde{C} \underline{x} = \underline{\tilde{F}}(\tau)$$

sistema omogeneo :

$$\det \left(\lambda^2 \tilde{A} + \tilde{C} \right) = 0$$

$$\rightarrow \lambda^2 = -1 \pm \frac{1}{\sqrt{2}} < 0$$

$$\left(\lambda^2 \tilde{A} + \tilde{C} \right) \underline{u} = 0$$

$$\rightarrow \underline{u}^{(1)} = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \quad \text{per } \lambda^2_{(1)} = -1 + \frac{1}{\sqrt{2}}$$

$$\rightarrow \underline{u}^{(2)} = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \quad \text{per } \lambda^2_{(2)} = -1 - \frac{1}{\sqrt{2}}$$

$$\underline{x}(\tau) = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \left(k_1 \sin \omega_1 \tau + k_2 \cos \omega_1 \tau \right) + \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \left(k_3 \sin \omega_2 \tau + k_4 \cos \omega_2 \tau \right)$$

Proviamo con una soluzione della forma

sistema omogeneo

$$\underline{x}(\tau) = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} g_1(\tau) + \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} g_2(\tau)$$

funzioni incognite.

$$\tilde{A}\,\frac{d^2\underline{x}}{d\tau^2} + \tilde{C}\,\underline{x} =$$

$$= g_1''(\tau)\,\tilde{A}\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} + g_1(\tau)\,\tilde{C}\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$$

$$+ g_2''(\tau)\,\tilde{A}\begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} + g_2(\tau)\,\tilde{C}\begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$$

$$= g_1''(\tau)\,\tilde{A}\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} + g_1(\tau)\left(-\left(-1+\frac{1}{\sqrt{2}}\right)\tilde{A}\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}\right)$$

$$+ g_2''(\tau)\,\tilde{A}\begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} + g_2(\tau)\left(-\left(-1-\frac{1}{\sqrt{2}}\right)\tilde{A}\begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}\right)$$

$$= \left(g_1'' + \omega_1^2 g_1\right) \tilde{A} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} +$$

$$+ \left(g_2'' + \omega_2^2 g_2\right) \tilde{A} \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$$

Moltiplichiamo scalarmente per

$$\underline{u}^{(1)} = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \quad e \quad \underline{u}^{(2)} = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$$

e usiamo $\begin{cases} \underline{u}^{(1)} \tilde{A} \, \underline{u}^{(2)} = 0 \\ \underline{u}^{(2)} \tilde{A} \, \underline{u}^{(1)} = 0 \end{cases}$

$$\left(g_1'' + \omega_1^2 g_1\right) \begin{pmatrix} 1 & \sqrt{2} \end{pmatrix} \hat{A} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} =$$

$$= \left(g_1'' + \omega_1^2 g_1\right) \begin{pmatrix} 1 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$$

$$= \left(g_1'' + \omega_1^2 g_1\right) 4 \qquad \begin{pmatrix} 1 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 2 \\ \sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} a_f \, \tilde{\omega}_f^2 \sin \tilde{\omega}_f \tau$$

$$\boxed{g_1'' + \omega_1^2 g_1 = \frac{2+\sqrt{2}}{4} a_f \tilde{\omega}_f^2 \sin \tilde{\omega}_f \tau}$$

$$(g_2'' + \omega_2^2 g_2)\,(1 \;\; -\sqrt{2})\,\tilde{A}\begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$$

$$= (g_2'' + \omega_2^2 g_2)\,(1 \;\; -\sqrt{2})\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$$

$$= (g_2'' + \omega_2^2 g_2)\, 4$$

$$= (1 \;\; -\sqrt{2})\begin{pmatrix} 2 \\ 1 \end{pmatrix} a_f \tilde{\omega}_f^2 \sin \tilde{\omega}_f \tau$$

$$\boxed{g_2'' + \omega_2^2 g_2 = \frac{2-\sqrt{2}}{4} a_f \tilde{\omega}_f^2 \sin \tilde{\omega}_f \tau}$$

Notiamo che entrambe queste equazioni hanno la forma:

$$\longrightarrow \quad g_i'' + \omega_i^2 g_i = f_i \sin \tilde{\omega}^2 \tau$$

$$f_1 = \frac{2+\sqrt{2}}{4} a_f \tilde{\omega}_f^2 \,, \quad f_2 = \frac{2-\sqrt{2}}{4} a_f \tilde{\omega}_f^2$$

Ci sono due casi:

- se $\tilde{\omega}_f \neq \omega_i$ $i = 1, 2$

$$g_i(\tau) = A_i \sin \tilde{\omega}_f \tau$$

dove $\underline{\underline{A_i}} = \underbrace{\frac{f_i}{(\omega_i^2 - \tilde{\omega}_f^2)}}$ $i = 1, 2$

- se $\omega_i = \tilde{\omega}_f$ per $i = 1$ oppure 2

$$g_i(\tau) = -\frac{f_i}{2\omega_i} \tau \cos \omega_i \tau$$

$$g_j(\tau) = -\frac{f_i}{\omega_j^2 - \omega_i^2} \sin \omega_j \tau \quad i \neq j$$

Alla fine: la soluzione del sistema è:

$$\underline{x}(\tau) = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \left(\underline{k_1 \sin \omega_1 \tau + k_2 \cos \omega_1 \tau} + \underline{g_1(\tau)} \right)$$

$$+\begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \left(k_3 \sin\omega_2\tau + k_4 \cos\omega_2\tau + g_2(\tau) \right)$$

Idee simili le possiamo usare anche per sistemi meccanici con forze dipendenti dalla velocità

$$A\ddot{\underline{x}} + B\dot{\underline{x}} + C\underline{x} = \underline{0}$$

$\rightarrow$ passare a coordinate normali per $A\ddot{\underline{x}} + C\underline{x} = \underline{0}$

e poi vedere se siamo fortunati e anche B è diagonale

$$\rightarrow \quad \underline{x}(\tau) = \sum u^{(i)} g_i(\tau)$$

e troviamo condizioni per g_i

Altrimenti, prendiamo tutto

$$A\ddot{\underline{x}} + B\dot{\underline{x}} + C\underline{x} = 0$$

e proviamo a sostituire

soluzioni del tipo $\underline{x} = \underline{u}\, e^{\lambda t}$

<u>Quindi</u> : modi normali

$$A\,\ddot{\underline{x}} + C\,\underline{x} = \underline{0}$$

- $\det(\lambda^2 A + C) = 0$
- $(\lambda^2 A + C)\,\underline{u} = 0$

$\Big\}\ x(t) = \ldots$

$\rightarrow\ A\,\ddot{\underline{x}} + C\,\underline{x} = \underline{F}(\tau)$: risonanza

$\rightarrow\ A\,\ddot{\underline{x}} + B\,\dot{\underline{x}} + C\,\underline{x} = \underline{0}$: ?

⋮