

PARENTESI DI POISSON

variabile dinamica

Cost. del Moto : $f(\bar{p}, \bar{q}, t)$ t.c. se la valutiamo sulle soluzioni $(\bar{p}(t), \bar{q}(t))$ delle eq. di Hamilton

$$\frac{d}{dt} f(\bar{p}(t), \bar{q}(t), t) = 0.$$

$$\begin{aligned} \frac{d}{dt} f(\bar{p}(t), \bar{q}(t), t) &= \frac{\partial f}{\partial t} + \sum_{e=1}^n \left(\frac{\partial f}{\partial p_e} \dot{p}_e + \frac{\partial f}{\partial q_e} \dot{q}_e \right) = \\ &= \frac{\partial f}{\partial t} + \sum_{e=1}^n \left[\frac{\partial f}{\partial p_e} \left(-\frac{\partial H}{\partial q_e} \right) + \frac{\partial f}{\partial q_e} \left(\frac{\partial H}{\partial p_e} \right) \right] \end{aligned}$$

prendiamo $\bar{p}(t), \bar{q}(t)$
che soddisfanno eq. Ham.

$$\begin{aligned} &= \frac{\partial f}{\partial t} + \underbrace{\sum_{e=1}^n \left(\frac{\partial f}{\partial q_e} \frac{\partial H}{\partial p_e} - \frac{\partial f}{\partial p_e} \frac{\partial H}{\partial q_e} \right)}_{= \{f, H\}} \\ &= \{f, H\} \quad \text{PARENTESI DI POISSON} \end{aligned}$$

Def. PARENTESI DI POISSON è un'operazione bilineare

$$f(\bar{p}, \bar{q}, t), g(\bar{p}, \bar{q}, t) \mapsto \{f, g\}(\bar{p}, \bar{q}, t) = \sum_{e=1}^n \left(\frac{\partial f}{\partial q_e} \frac{\partial g}{\partial p_e} - \frac{\partial f}{\partial p_e} \frac{\partial g}{\partial q_e} \right)$$

soddisfanno
eq. Ham.

$$\frac{d}{dt} f(\bar{p}(t), \bar{q}(t), t) = \frac{\partial f}{\partial t}(\bar{p}(t), \bar{q}(t), t) + \{f, H\}(\bar{p}(t), \bar{q}(t), t)$$

f è una cost.
del Moto
(f indep. da t)

$$\iff \frac{\partial f}{\partial t} + \{f, H\} = 0$$

($\{f, H\} = 0$)

Formalismo completo per sistemi Hamiltoniani

Partiamo da un sist. Ham. a 1 grado di lib. ($n=1$)

→ spazio delle fasi è parametrizzato da due coord p, q .

→ Eq. Ham.
$$\begin{cases} \dot{p} = -\frac{\partial H}{\partial q} \\ \dot{q} = \frac{\partial H}{\partial p} \end{cases}$$

Def. $\bar{x} = \begin{pmatrix} p \\ q \end{pmatrix} \rightsquigarrow \nabla_{\bar{x}} H = \begin{pmatrix} \frac{\partial H}{\partial p} \\ \frac{\partial H}{\partial q} \end{pmatrix}$

Consideriamo la matrice

$$E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Eq. Ham.
$$\dot{\bar{x}} = \begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} -\frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial p} \\ \frac{\partial H}{\partial q} \end{pmatrix} = E \nabla_{\bar{x}} H$$

$$\dot{\bar{x}} = E \nabla_{\bar{x}} H \quad \leftarrow \text{è della forma } \dot{\bar{x}} = \bar{f}(\bar{x}, t)$$

Prendiamo ora un sist. a n gradi di lib.

$$\bar{x} = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \\ q_1 \\ \vdots \\ q_m \end{pmatrix} = \begin{pmatrix} \bar{p} \\ \bar{q} \end{pmatrix} \quad \bar{\nabla}_{\bar{x}} f = \begin{pmatrix} \partial f / \partial p_1 \\ \partial f / \partial p_2 \\ \vdots \\ \partial f / \partial p_m \\ \partial f / \partial q_1 \\ \vdots \\ \partial f / \partial q_m \end{pmatrix} = \begin{pmatrix} \bar{\nabla}_p f \\ \bar{\nabla}_q f \end{pmatrix}$$

$$\rightarrow F = \begin{pmatrix} 0 & -\mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix} = \begin{pmatrix} \overbrace{0 \dots 0}^n & \overbrace{-1 \ 0 \dots 0}^n \\ \vdots & \vdots \\ 1 & 0 \dots 0 \\ \vdots & \vdots \\ 0 & \overbrace{0 \dots 0}^n \\ \vdots & \vdots \\ 0 & 1 \dots 0 \end{pmatrix}$$

matrice
 $2n \times 2n$

$$\text{Eq. Ham.:} \quad \dot{\bar{x}} = E \bar{\nabla}_{\bar{x}} H$$

$$\begin{pmatrix} \dot{\bar{p}} \\ \dot{\bar{q}} \end{pmatrix} = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} \bar{\nabla}_p H \\ \bar{\nabla}_q H \end{pmatrix} = \begin{pmatrix} -\bar{\nabla}_q H \\ \bar{\nabla}_p H \end{pmatrix}$$

$$\dot{p}_n = - \frac{\partial H}{\partial q_n} \quad \dot{q}_n = \frac{\partial H}{\partial p_n}$$

Parentesi di Poisson:

$$\{f, g\} = \sum_{i,j=1}^{2n} \frac{\partial f}{\partial x_i} E_{ij} \frac{\partial g}{\partial x_j} = \bar{\nabla}_{\bar{x}} f \cdot E \bar{\nabla}_{\bar{x}} g$$

$$E_{ij} = \begin{cases} 0 & i, j = 1, \dots, m \\ -\delta_{hk} & i=h \quad j=k+m \quad h, k = 1, \dots, m \\ \delta_{hk} & i=h+m \quad j=k \quad h, k = 1, \dots, m \\ 0 & i, j = m+1, \dots, 2m \end{cases}$$

Proprietà delle parentesi di Poisson

a) $\{f, g\} = -\{g, f\} \Rightarrow \{f, f\} = 0$ ANTISIM.

b) $\{f, \alpha_1 g_1 + \alpha_2 g_2\} = \alpha_1 \{f, g_1\} + \alpha_2 \{f, g_2\}$ BILINEARE
 $\alpha_1, \alpha_2 \in \mathbb{R}$

c) $\{f, g_1 \cdot g_2\} = \{f, g_1\} g_2 + g_1 \{f, g_2\}$

d) Identità di Jacobi

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$

La parentesi di Poisson è un'applicazione BILIN. ANTISIM.

che soddisfa l'ID. DI JACOBI

(ci permette di definire un'ALGEBRA DI LIE)

Dim $\left. \begin{matrix} a) \\ b) \end{matrix} \right\}$ ovvio

$$\{f, g\} = \sum_{i,j} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_j} \right) = \sum_{i,j} \frac{\partial f}{\partial x_i} E_{ij} \frac{\partial g}{\partial x_j}$$

$$c) \sum_{i,j} \frac{\partial f}{\partial x_i} E_{ij} \frac{\partial}{\partial x_j} (g_1 \cdot g_2) = \left(\sum_{i,j} \frac{\partial f}{\partial x_i} E_{ij} \frac{\partial g_1}{\partial x_j} \right) g_2 + \left(\sum_{i,j} \frac{\partial f}{\partial x_i} E_{ij} \frac{\partial g_2}{\partial x_j} \right) g_1$$

$$d) \underbrace{\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\}} = 0$$

Notation: :

$$\partial_i f \equiv \frac{\partial f}{\partial x_i}$$

$$\begin{aligned} & \{f, \{g, h\}\} - \{g, \{f, h\}\} = \\ & = \sum_{i,j=1}^{2n} \partial_i f E_{ij} \partial_j \left[\sum_{k,m=1}^{2n} \partial_k g E_{km} \partial_m h \right] - \sum_{a,b=1}^{2n} \partial_a g E_{ab} \partial_b \left[\sum_{c,d=1}^{2n} \partial_c f E_{cd} \partial_d h \right] \end{aligned}$$

$$= \sum_{ijkm} \left[\partial_i f E_{ij} \partial_j \partial_k g E_{km} \partial_m h + \partial_i f E_{ij} \partial_k g E_{km} \partial_j \partial_m h \right]$$

$$- \sum_{abcd} \left[\partial_a g E_{ab} \partial_b \partial_c f E_{cd} \partial_d h + \partial_a g E_{ab} \partial_c f E_{cd} \partial_b \partial_d h \right]$$

$\begin{matrix} \uparrow & & \uparrow & & \uparrow & \uparrow \\ k & & i & & m & j \end{matrix}$

$$= \sum_{ijkm} \left[\partial_i f E_{ij} \partial_j \partial_k g E_{km} \partial_m h \right] + \cancel{\sum_{ijkm} \partial_i f E_{ij} \partial_k g E_{km} \partial_j \partial_m h}$$

$$- \sum_{abcd} \left[\partial_a g E_{ab} \partial_b \partial_c f E_{cd} \partial_d h \right] - \cancel{\sum_{ijkm} \partial_k g E_{km} \partial_i f E_{ij} \partial_m \partial_j h}$$

$= \partial_j \partial_m h$

$$= \sum_{ijkm} \left[\partial_i f E_{ij} \partial_j \partial_k g E_{km} \partial_m h - \partial_j g \underbrace{E_{ji}}_{=-E_{ij}} \partial_i \partial_k f E_{km} \partial_m h \right]$$

$$= \sum_{km} \left(\sum_{ij} \left(\partial_i f E_{ij} \partial_k \partial_j g + \partial_k \partial_i f E_{ij} \partial_j g \right) \right) E_{km} \partial_m h$$

$$= \sum_{km} \partial_k \left(\sum_{ij} \partial_i f E_{ij} \partial_j g \right) E_{km} \partial_m h = \{ \{f, g\}, h \} = - \{ h, \{f, g\} \} //$$

Parentesi di Poisson FONDAMENTALI

Scegliamo come f e g le funzioni: $\bar{p} \mapsto p_n \leftarrow p_n$

$\bar{q} \mapsto q_n \leftarrow q_n$

$$\{p_n, p_k\} = \sum_l \left(\underset{\substack{\uparrow \\ 0}}{\frac{\partial p_n}{\partial q_l}} \frac{\partial p_k}{\partial p_l} - \frac{\partial p_n}{\partial p_l} \underset{\substack{\uparrow \\ \delta_{ke}}}{\frac{\partial p_k}{\partial q_l}} \right) = 0 \quad \forall l, k=1, \dots, m$$

$$[F(x,y)=x, \partial_x F=1, \partial_y F=0]$$

$$\{q_n, q_k\} = 0 \quad \forall l, k=1, \dots, m$$

$$\{q_n, p_k\} = \sum_l \left(\underset{\substack{\uparrow \\ \delta_{ne}}}{\frac{\partial q_n}{\partial q_l}} \frac{\partial p_k}{\partial p_l} - \frac{\partial q_n}{\partial p_l} \underset{\substack{\uparrow \\ 0}}{\frac{\partial p_k}{\partial q_l}} \right) = \sum_l \delta_{ne} \delta_{ke} = \delta_{nk}$$

$$\{x_i, x_j\} = \sum_{r,s=1}^{2m} \frac{\partial x_i}{\partial x_r} \overset{\delta_{ir}}{E_{rs}} \frac{\partial x_j}{\partial x_s} \overset{\delta_{js}}{=} E_{ij}$$

Osservazione:

$$\{x_s, g(\bar{x})\} = \sum_{ij} \frac{\partial x_s}{\partial x_i} \overset{\delta_{is}}{=} E_{ij} \frac{\partial g}{\partial x_j} = \sum_j E_{sj} \frac{\partial g}{\partial x_j}$$

$$\{\bar{x}, g\} = E \nabla g \quad \Rightarrow \quad \begin{cases} \{p_n, g(\bar{p}, \bar{q})\} = -\frac{\partial g}{\partial q_n} \\ \{q_n, g(\bar{p}, \bar{q})\} = \frac{\partial g}{\partial p_n} \end{cases}$$

Se $g=H$ $\dot{x}_i = \sum_j E_{ij} \frac{\partial H}{\partial x_j} = \{x_i, H\}$

\rightarrow Es. Ham. $\dot{x}_i = \{x_i, H\} \quad i=1, \dots, 2m$

\swarrow es. particolare di $\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}$

$$\begin{aligned} \frac{d}{dt} G(\bar{x}(t), t) &= \frac{\partial G}{\partial t} + \left[G \right]_{E\nabla H = \bar{f}} = \frac{\partial G}{\partial t} + \sum_{i=1}^2 f_i \frac{\partial G}{\partial x_i} = \\ \dot{\bar{x}} &= \bar{f}(\bar{x}, t) \\ &= \frac{\partial G}{\partial t} - \frac{\partial H}{\partial q} \frac{\partial G}{\partial p} + \frac{\partial H}{\partial p} \frac{\partial G}{\partial q} = \frac{\partial G}{\partial t} + \{G, H\} \end{aligned}$$

In generale

$$\{f, g\} = \int_{E\nabla g} f = \sum_i \left(\sum_j E_{ij} \frac{\partial g}{\partial x_j} \right) \frac{\partial f}{\partial x_i} = \bar{\nabla} f \cdot E \bar{\nabla} g$$

$$\{g, f\} = \bar{\nabla} g \cdot E \bar{\nabla} f = \sum_{ij} E_{ij} \frac{\partial g}{\partial x_i} \frac{\partial f}{\partial x_j} =$$

$$= - \sum_{ij} E_{ji} \frac{\partial g}{\partial x_i} \frac{\partial f}{\partial x_j} = - \sum_{rs} E_{rs} \frac{\partial f}{\partial x_r} \frac{\partial g}{\partial x_s} = - \{f, g\}$$

ANTISIM.