PARENTESI DI POISSON & MOMENTO ANGOLARE

Prendiamo un corpo che si muove in $\mathbb{R}^3$, con coord. cartesiane $\bar{q}$, allora $\bar{p}$ è la quantità di moto.

$$\bar{M} = \bar{q} \times \bar{p}$$ ← funz. delle p_h e delle q_h

$$M_i = \sum_{m,k=1}^{3} \epsilon_{imk} q_m p_k$$

$$\{p_\ell, M_i\} = -\frac{\partial M_i}{\partial q_\ell} = -\frac{\partial}{\partial q_\ell} \sum_{mk} \epsilon_{imk} q_m p_k =$$

$$= -\sum_{mk} \epsilon_{imk} \delta_{m\ell} p_k = -\sum_k \epsilon_{i\ell k} p_k$$

$$= \sum_k \epsilon_{\ell i k} p_k$$

$$\{p_\ell, \sum_{mk} \epsilon_{imk} q_m p_k\} \overset{\text{BILIN.}}{=} \sum_{mk} \epsilon_{imk} \{p_\ell, q_m p_k\} =$$

$$\overset{d}{=} \sum_{mk} \epsilon_{imk} \Big[\underbrace{\{p_\ell, q_m\}}_{= -\delta_{\ell m}} p_k + \underbrace{\{p_\ell, p_k\}}_{= 0} q_m \Big] \overset{\text{P.P. fondam.}}{=}$$

$$= -\sum_k \epsilon_{i\ell k} p_k$$

$$\alpha\{p_\ell, M_z\} = \alpha \epsilon_{\ell 3 k} p_k$$

$$\hookrightarrow \{p_3, M_3\} = 0$$

$$\left\{\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, M_3\right\} = \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

↑ ROTAZIONE INFINITESIMA ATTORNO ASSE z

$$\{q_\ell, M_i\} = \sum_m \epsilon_{\ell i m} q_m$$

$$\boxed{\{ M_i , M_j \} = \sum_{k=1}^{3} \epsilon_{ijk} M_k}$$

[M_i soddisfano l'algebra di Lie di $SU(2)$]

$$M_i = \sum_{m,s=1}^{3} \epsilon_{ims}\, q_m P_s$$

c)

$$\{M_i, M_j\} = \Big\{ \sum_{ms} \epsilon_{ims}\, q_m P_s , M_j \Big\} \overset{\text{BILIN.}}{=} \sum_{ms} \epsilon_{ims} \{ q_m P_s , M_j \} =$$

$$= \sum_{ms} \epsilon_{ims} \Big[q_m \underbrace{\{ P_s , M_j \}}_{=\sum_r \epsilon_{sjr} P_r} + \underbrace{\{ q_m , M_j \}}_{=\sum_r \epsilon_{mjr}\, q_r} P_s \Big]$$

$$= \sum_{msr} \epsilon_{ims} \left(\epsilon_{sjr} P_r q_m + \epsilon_{mjr} P_s q_r \right) =$$

$$= \sum_{msr} \epsilon_{ims}\, \epsilon_{sjr}\, P_r q_m + \sum_{msr} \epsilon_{ims}\, \epsilon_{mjr}\, P_s q_r$$

$$\sum_{smr} \epsilon_{ism}\, \epsilon_{sjr}\, P_m q_r = - \sum_{smr} \epsilon_{ims}\, \epsilon_{sjr}\, P_m q_r$$

$$= \sum_{msr} \underbrace{\epsilon_{ims}\, \epsilon_{sjr}}_{\delta_{ij}\delta_{mr} - \delta_{ir}\delta_{mj}} \left(P_r q_m - P_m q_r \right)$$

ϵ_{jrs}

$$= \sum_{mr} \left(\delta_{ij}\, \delta_{mr} - \delta_{ir}\, \delta_{mj} \right) \left(P_r q_m - P_m q_r \right)$$

$$= \sum_m \delta_{ij} (\cancel{p_m q_m} - \cancel{p_m q_m}) - (p_i q_j - p_j q_i)$$

$$= q_i p_j - q_j p_i$$

$$= \sum_k \epsilon_{ijk} M_k$$

$$\sum_k \epsilon_{ijk} \sum_{rs} \epsilon_{krs} q_r p_s =$$

$$= \sum_{rs} (\delta_{ir}\delta_{js} - \delta_{is}\delta_{jr}) q_r p_s =$$

$$= q_i p_j - q_j p_i \;//$$

$$\{M_i, M_j\} = \sum_k \epsilon_{ijk} M_k$$

$$\{M_1, M_2\} = \sum_k \epsilon_{12k} M_k = M_3$$

$$\{M_2, M_3\} = M_1 \qquad \{M_3, M_1\} = M_2$$

$$\{\bar{M}^2, M_i\} = 0$$

$$\{\sum_e M_e^2, M_i\} \overset{b, c)}{=} \sum_e 2 M_e \{M_e, M_i\} =$$

$$= 2 \sum_e M_e \sum_k \epsilon_{eik} M_k = -2 \sum_{ek} \underbrace{\epsilon_{iek}}_{\substack{\text{antisim.} \\ \text{in } ek}} \underbrace{M_e M_k}_{\substack{\text{sim.} \\ \text{in } ek}}$$

$$\Big[\sum_{ek} \underset{\text{antisim}}{a_{ek}} \underset{\text{sim.}}{S_{ek}} = \sum_{ek} (-a_{ke}) S_{ke} = -\sum_{ij} a_{ij} S_{ij} = -\sum_{ek} a_{ek} S_{ek}$$

$$\Rightarrow \sum_{ek} a_{ek} S_{ek} = 0 \Big]$$

Questa relazione è un caso particolare del fatto che

$$\{M_1, f(\bar p, \bar q)\} = 0 \quad \text{se } f \text{ è invariante sotto rotazioni } SO(3).$$

Vediamo due casi particolari

$$\{M_h, \bar p^2\} = \{M_h, \sum_j p_j^2\} = \sum_j 2p_j \{M_h, p_j\} = 2\sum_j p_j \sum_k \epsilon_{hjk} p_k = 0$$

$$\{M_h, \bar q^2\} = \{M_h, \sum_j q_j^2\} = 2\sum_j q_j \sum_k \epsilon_{hjk} q_k = 0$$

$$\{M_h, \bar p\cdot\bar q\} = \{M_h, \sum_j p_j q_j\} = \sum_j \left(p_j\{M_h, q_j\} + q_j\{M_h, p_j\}\right)$$
$$= \sum_{jk}\left(p_j \epsilon_{hjk} q_k + q_j \epsilon_{hjk} p_k\right) =$$
$$= \sum_{jk} \underbrace{\epsilon_{hjk}}_{\text{antisim.}} \underbrace{(p_j q_k + p_k q_j)}_{\text{simm.}} = 0$$

in $j \leftrightarrow k$

Vediamo come agisce M_h su una funz. INVARIANTE sotto ROTAZIONI:

in qto caso $f(\bar p, \bar q) = g(\underbrace{\bar p^2, \bar q^2, \bar p\cdot\bar q}_{})$

unici invarianti sotto rotaz. che posso definire usando i vett. $\bar p$ e $\bar q$

$$\mathcal{L}_{\bar f}\, g(\alpha_1(x), \dots, \alpha_N(x)) = \sum_{i=1}^{2n} f_i \sum_{r=1}^{N} \frac{\partial g}{\partial \alpha_r}\frac{\partial \alpha_r}{\partial x_i} = \sum_{r=1}^{N} \frac{\partial g}{\partial \alpha_r}(\mathcal{L}_{\bar f}\alpha_r)$$

$$\{M_h, g(\bar p^2, \bar q^2, \bar p\cdot\bar q)\} = \mathcal{L}_{E\nabla M_h}\, g(\bar p^2, \bar q^2, \bar p\cdot\bar q) =$$

$$= \sum_{r=1}^{3} \frac{\partial g}{\partial \alpha_r}\left(\mathcal{L}_{E\nabla M_h}\alpha_r\right) = \sum_{r=1}^{3} \frac{\partial g}{\partial \alpha_r}\underbrace{\{M_h, \alpha_r\}}_{=0} = 0$$

dove $\alpha_1 = \bar p^2 \quad \alpha_2 = \bar q^2$

$\alpha_3 = \bar p\cdot\bar q$

0 come dimostrato sopra

Verifichiamolo su una funz. inv. semplice:

$$\{M_h, \frac{1}{r}\} = \sum_{rs} \epsilon_{rsh} \{q_r p_s, \frac{1}{r(q)}\} \overset{c)}{=} \qquad \{f(\bar q), g(\bar q)\} = 0$$

$r = \sqrt{q_1^2+q_2^2+q_3^2}$

$$= \sum_{rs} \epsilon_{rsh} \Big[q_r \underbrace{\{p_s, \frac{1}{r}\}}_{= -\frac{\partial}{\partial q_s}\left(\frac{1}{r}\right)} + \cancel{\underbrace{\{q_r, \frac{1}{r(q)}\}}_{= \frac{\partial}{\partial p_r}\left(\frac{1}{r(q)}\right) = 0}} p_s \Big]$$

$$= \sum_{rs} \epsilon_{rsh} \, q_r \left(\frac{1}{2r^3} \, 2 q_s \right) = \sum_{rs} \epsilon_{hrs} q_r q_s = 0$$

In qsto caso $g = \frac{1}{\sqrt{\alpha}}$ $\qquad \alpha(\bar p, \bar q) = q_1^2 + q_2^2 + q_3^2$

Vettore di Laplace - Runge - Lenz e Parentesi di Poisson

$$\bar A = \bar p \times \bar M - mK \frac{\bar q}{r} \qquad r = \sqrt{q_1^2 + q_2^2 + q_3^2}$$

indip. da t

$$A_i = \sum_{mh} \epsilon_{imh} p_m M_h - mK \frac{q_i}{r}$$

$$H = \frac{\bar p^2}{2m} - \frac{K}{r}$$

Kepl.

→ verifichiamo che $\bar A$ è cost. del moto, cioè $\{A_i, H\} = 0$

$$\{A_i, H\} = \sum_{mn} \epsilon_{imn} \left\{ p_m M_n, \frac{1}{2m}\sum_j p_j^2 - \frac{k}{r} \right\}$$

$$- mk \left\{ \frac{q_i}{r}, \frac{1}{2m}\sum_j p_j^2 - \frac{k}{r} \right\} =$$

$\{F(\bar{p}), G(\bar{p})\} = 0$

$$= \sum_{mn} \epsilon_{imn} \Big[p_m \cancel{\{ M_n, \frac{1}{2m}\sum_j p_j^2 \}} + \cancel{\{ p_m, \frac{1}{2m}\sum_j p_j^2 \}} M_n$$

inv. rotaz.

$$- p_m \cancel{\{ M_n, \frac{k}{r} \}} - \left\{ p_m, \frac{k}{r} \right\} M_n \Big]$$

$\{f(\bar{q}), g(\bar{q})\} = 0$

$$- mk \left[\left\{ \frac{q_i}{r}, \frac{1}{2m}\sum_j p_j^2 \right\} - \cancel{\left\{ \frac{q_i}{r}, \frac{k}{r} \right\}} \right]$$

$$= \sum_{mn} \epsilon_{imn} \underbrace{\left\{ p_m, -\frac{k}{r} \right\}}_{-\frac{q_m}{r^3}} M_n - \frac{k}{2} \sum_j \underbrace{\left\{ \frac{q_i}{r}, p_j^2 \right\}}_{2\left\{ p_j, -\frac{q_i}{r} \right\} p_j}$$

$$= \underbrace{-k \sum_{mn} \epsilon_{imn} \frac{q_m M_n}{r^3}}_{(A)} + \underbrace{k \sum_j p_j}_{(B)} \underbrace{\left\{ p_j, \frac{q_i}{r} \right\}}_{\frac{q_i q_j}{r^3} - \frac{\delta_{ij}}{r}}$$

$$-\frac{\partial}{\partial q_j}\left(\frac{q_i}{r} \right)$$

$$\frac{\partial}{\partial q_m}\left(\frac{1}{r} \right) = \frac{\partial}{\partial q_m} \frac{1}{\sqrt{q_1^2 + q_2^2 + q_3^2}} = -\frac{1}{r^2} \frac{1}{2r} 2 q_m = -\frac{q_m}{r^3}$$

$$(A) = -k\sum_{mh} \epsilon_{imh} \frac{q_m}{r^3} \sum_{ab} \epsilon_{hab}\, q_a p_b = -k\sum_{m\,ab} \left(\delta_{ia}\delta_{mb} - \delta_{ib}\delta_{ma}\right) \cdot \frac{1}{r^3}\left(q_m q_a p_b\right)$$

$$= -\frac{k}{r^3}\sum_m \left(q_m q_i p_m - \underbrace{q_m q_m}^{r^2} p_i \right) =$$

$$= k\left(\frac{1}{r} p_i - \frac{1}{r^3}(\bar{q}\cdot\bar{p}) q_i \right)$$

$$(B) = k \sum_j p_j \left(q_i \frac{q_j}{r^3} - \frac{\delta_{ij}}{r} \right) = k\frac{(\bar{p}\cdot\bar{q}) q_i}{r^3} - \frac{k}{r} p_i$$

$$(A) + (B) = 0 \quad //$$

$$\bar A = \frac{\bar p \times \bar M}{m} - k\frac{\bar r}{r} \quad (\circ) \qquad \bar M = \bar r \times \bar p \qquad M_i = \sum_{jk} \epsilon_{ijk}\, x_j\, p_k$$

$$H = \frac{\bar p^2}{2m} - \frac{k}{r}$$

$$\{M_i, M_j\} = \sum_l \epsilon_{ijl} M_l$$

$$\{M_i, A_j\} = \sum_l \epsilon_{ijl} A_l$$

($\bar A$ è un vett. sotto $SO(3)$)

$$\{A_i, A_j\} = -\frac{2H}{m}\sum_l \epsilon_{ijl} M_l \quad (\circ\circ)$$

$\delta_{lr}\delta_{ks} - \delta_{ls}\delta_{kr}$

$$\{H, M_i\} = \{H, A_i\} = 0$$

Inoltre da def. $(\circ)$ abbiamo

$$\bar M \cdot \bar A = 0 \qquad e \qquad \bar A^2 - k^2 = \frac{2H}{m}\bar M^2$$

$(\bar a \times \bar b)\cdot \bar c = (\bar c \times \bar a)\cdot \bar b$

$$\bar A^2 = A_i A_i = \frac{1}{m^2}\sum_{l,i,k,r,s} \epsilon_{ilk}\, p_l M_k\, \epsilon_{irs}\, p_r M_s + k^2 - \frac{2k}{mr}(\bar p \times \bar M)\cdot \bar r$$

$$= \frac{1}{m^2}\left(\bar p^2 \bar M^2 - \underbrace{\cancel{(\bar p \cdot \bar M)^2}}_{=0}\right) + k^2 - \frac{2k}{mr}\bar M^2 =$$

$$= \frac{2\bar M^2}{m}\left(\frac{\bar p^2}{2m} - \frac{k}{r}\right) + k^2 = \frac{2H\bar M^2}{m} + k^2$$

$\bar A$ introduce solo un'ulteriore cost. del moto INDIPEND. (oltre a H e M_i)

↳ 5 cost. del moto in sp. delle fasi 6d ⇒ traiettoria giace su curva di livello.

Verifichiamo $(\circ\circ)$

$$\{M_s, V_k\} = \epsilon_{skl} V_l$$

$$A_i = \frac{1}{m}\epsilon_{imn}\, p_m M_n - k\frac{x_i}{r}$$

$$\{A_i, A_j\} = \frac{1}{m^2}\epsilon_{imn}\epsilon_{jrs}\{p_m M_n, p_r M_s\} - \frac{k}{m}\epsilon_{imn}\left\{p_m M_n, \frac{x_j}{r}\right\}$$

$$- \frac{k}{m}\epsilon_{jrs}\left\{\frac{x_i}{r}, p_r M_s\right\} + k^2 \underbrace{\cancel{\left\{\frac{x_i}{r}, \frac{x_j}{r}\right\}}}_{=0}$$

$$\{p_m M_n, p_r M_s\} = p_m\{M_n, p_r\}M_s + p_m\{M_n, M_s\}p_r + M_n\{p_m, M_s\}p_r$$

$$= p_m M_s\, \epsilon_{nrk}\, p_k - p_r M_n\, \epsilon_{smq}\, p_q + p_m p_r\, \epsilon_{nsb} M_b$$

$$\frac{1}{m^2}\epsilon_{imn}\epsilon_{jrs}\{p_m M_n, p_r M_s\} = \Big\{ p_m p_k M_s\, \epsilon_{jrs}(\delta_{ri}\delta_{km} - \delta_{rm}\delta_{ki})$$

$$\epsilon_{ijm} M_m = \epsilon_{ijn}\epsilon_{nrs} x_r p_s = x_i p_j - x_j p_i$$

$$- p_r p_q M_n (\delta_{mj}\delta_{qr} - \delta_{mr}\delta_{qj})\epsilon_{imn}$$
$$+ p_m p_r M_b (\delta_{si}\delta_{bm} - \delta_{sm}\delta_{bi})\epsilon_{jrs} \Big\}\cdot 1/m^2$$

$$= \Big\{ \epsilon_{jis}\, \bar p^2 M_s - p_i p_r M_s \epsilon_{jrs} - \bar p^2 M_n \epsilon_{ijn} + p_m p_j M_n \epsilon_{imn}$$
$$+ \bar p\cdot\bar M \epsilon_{jri} p_r - \cancel{p_s p_r M_i}\, \epsilon_{jrs} \Big\} 1/m^2$$

$$= \Big\{ - p_i \epsilon_{jrs} p_r M_s + p_j \epsilon_{irs} p_r M_s + \epsilon_{ijn}\big(\overset{=0}{\cancel{\bar p\cdot\bar M}}\, p_n - 2\bar p^2 M_n \big)\Big\} 1/m^2$$

$$\epsilon_{irs} p_j p_r M_s = \epsilon_{irs}\epsilon_{smn} p_j p_r x_m p_n =$$
$$= p_j p_r x_m p_n (\delta_{im}\delta_{rn} - \delta_{in}\delta_{rm}) =$$
$$= x_i p_j \bar p^2 - p_i p_j\, \bar x\cdot\bar p$$

$$= (x_i p_j - x_j p_i)\frac{\bar p^2}{m^2} - 2\epsilon_{ijm} M_n \frac{\bar p^2}{m^2} = -\frac{2}{m}\epsilon_{ije} M_e \frac{\bar p^2}{2m}$$

$$\{p_m, \tfrac{1}{r}\} = -\frac{\partial}{\partial x_m}\frac{1}{\sqrt{\cdots}} = \frac{1}{2r^3}2x_m = x_m/r^3$$

$$\epsilon_{imn}\cdot\{p_m M_n, \frac{x_j}{r}\} = \Big(-\delta_{mj}\frac{M_n}{r} + p_m \epsilon_{njr}\frac{x_r}{r} + M_n x_j \overbrace{\{p_m, \tfrac{1}{r}\}}^{x_m/r^3}\Big)\cdot\epsilon_{imn} =$$
$$= -\epsilon_{ijn}\frac{M_n}{r} + \frac{\delta_{ij}}{r}\bar p\cdot\bar x - p_j\frac{x_i}{r} + \epsilon_{imn} x_m M_n x_j \cdot\frac{1}{r^3}$$
$$= -\epsilon_{ijn}\frac{M_n}{r} + \delta_{ij}\frac{\bar p\cdot\bar x}{r} - p_j\frac{x_i}{r} + \frac{1}{r^3}\big(x_i x_j\, \bar x\cdot\bar p - p_i x_j r^2\big)$$
$$= -\frac{x_i}{r}p_j + \cancel{\frac{x_j}{r}} p_i + \delta_{ij}\frac{\bar p\cdot\bar x}{r} + \frac{1}{r^3}x_i x_j\, \bar x\cdot\bar p - \frac{x_i}{r}p_j - \cancel{\frac{x_j}{r}} p_i$$
$$= -\frac{2}{r}x_i p_j + \cancel{\frac{\delta_{ij}}{r}\bar p}\cdot\bar x + \frac{1}{r^3}\cancel{x_i x_j}\,\bar x\cdot\bar p$$

$$\epsilon_{jrs}\{\frac{x_i}{r}, p_r M_s\} = -\epsilon_{jmn}\{p_m M_n, \frac{x_i}{r}\} = -\uparrow \text{ con } i\leftrightarrow j$$
$$= \frac{2}{r}x_j p_i - \cancel{\frac{\delta_{ij}}{r}}\bar p\cdot\bar x - \frac{1}{r^3}x_i\cancel{x_j}\,\cancel{\bar x}\cdot\bar p$$

$$-\frac{k}{m}(\ldots + \ldots) = +\frac{2k}{mr}(x_i p_j - x_j p_i) = +\frac{2k}{mr}\epsilon_{ije} M_e$$

$$\epsilon_{imn} x_m M_n x_j = \epsilon_{imn}\epsilon_{nrs} x_m x_j x_r p_s =$$
$$= x_m x_j x_r p_s (\delta_{ir}\delta_{ms} - \delta_{is}\delta_{mr}) =$$
$$= x_i x_j\, \bar x\cdot\bar p - p_i x_j r^2$$

$$\Rightarrow \quad \{A_i, A_j\} = -\frac{2}{m}\epsilon_{ijl} M_l \left(\frac{\bar{p}^2}{2m} - \frac{k}{r} \right) \quad //$$