The Cobb-Douglas Production Function

- The Cobb-Douglas production function: \( Y = K^\beta AN^{1-\beta} \)
  - \( \beta \) is a number between 0 and 1
  - Assume that there is no technological progress: \( A=1; \) and \( N \) constant
- This function gave a good description of the relation between output, physical capital, and labor in the United States from 1899 to 1922, and even today.
- Does it imply constant returns to scale? \( Y(cK, cL) = cY \)?
  - \( Y(cK, cL) = (cK)^\beta (cN)^{1-\beta} = c^\beta K^\beta c^{1-\beta} N^{1-\beta} = c^{\beta+1-\beta} K^\beta N^{1-\beta} \)
  - \( Y(cK, cL) = c^{\beta+1-\beta} K^\beta N^{1-\beta} = cK^\beta N^{1-\beta} = cY \)
- Does it imply diminishing marginal returns to capital, and to labour?
  - \( \frac{\partial Y}{\partial K} = \beta K^{\beta-1} N^{1-\beta} > 0 \) so \( Y \) is increasing in \( K \)
  - \( \frac{\partial^2 Y}{\partial K^2} = \beta (\beta - 1) K^{\beta-2} N^{1-\beta} < 0 \) so \( Y \) is concave in \( K \): diminishing marginal returns
Output per worker?

\[
\frac{Y}{N} = K^\beta \frac{N^{1-\beta}}{N} = K^\beta N^{1-\beta} N^{-1} = K^\beta N^{-\beta} = \frac{K^\beta}{N^{\beta}} = (\frac{K}{N})^\beta
\]

In steady state, saving per worker must be equal to depreciation of capital per worker.

Saving per worker is Investment per worker, that is

\[
\frac{I_t}{N} = s \frac{Y_t}{N} = s \left(\frac{K}{N}\right)^\beta
\]

Depreciation per worker

\[
\frac{D}{N} = \delta \frac{K_t}{N}
\]

What is the level of \(\frac{K}{N}\) which makes depreciation equal to saving?

Saving equal to depreciation implies:

\[
s \left(\frac{K_t}{N}\right)^\beta = \delta \frac{K_t}{N}
\]
What is the level of $\frac{K}{N}$ which makes depreciation equal to saving?

$s \left( \frac{K_t}{N} \right)^\beta = \delta \frac{K_t}{N} \rightarrow \text{divide by } \left( \frac{K_t}{N} \right)^\beta \rightarrow s = \delta \frac{K_t}{N} \left( \frac{K_t}{N} \right)^{-\beta}$

$\rightarrow s = \delta \left( \frac{K_t}{N} \right)^{1-\beta} \rightarrow \frac{s}{\delta} = \left( \frac{K_t}{N} \right)^{1-\beta} \rightarrow \text{raise to } \frac{1}{1-\beta} \rightarrow \left( \frac{s}{\delta} \right)^{\frac{1}{1-\beta}} = \frac{K_t}{N}$

Hence capital p.w. will stop growing once it reaches $\frac{K_t}{N} = \left( \frac{s}{\delta} \right)^{\frac{1}{1-\beta}}$

What will be the level of output p.w. in steady state?

$\frac{Y}{N} = \left( \frac{K}{N} \right)^\beta \rightarrow \frac{Y}{N} = \left( \left( \frac{s}{\delta} \right)^{\frac{1}{1-\beta}} \right)^\beta \rightarrow \frac{Y}{N} = \left( \frac{s}{\delta} \right)^{\frac{\beta}{1-\beta}}$
Final comments on steady state if technological progress exists (ch.13)
Where this ends: tech drives growth, not capital accumulation

- In the long run, capital per effective worker $\frac{K}{AN}$ reaches a constant level, and so does output per effective worker $\frac{Y}{AN}$.
  - Steady state of this economy: $\frac{K}{AN}$ and $\frac{Y}{AN}$ are constant over time. They do not grow anymore.

- If $\frac{Y}{AN}$ is constant, output ($Y$) is growing at the same rate as effective labour $AN$
  - We define growth rate of $A = g_A$; growth rate of $N = g_N$

- Because effective labour grows at rate $(g_A + g_N)$ output growth $g_Y$ in steady state must also equal $(g_A + g_N)$
  - but we care about $Y/N$, output per worker; see about that soon

- The same reasoning applies to capital. Because capital per effective worker $\frac{K}{AN}$ is constant in steady state, capital is also growing at rate $(g_A + g_N)$. 

\[ \frac{Y}{AN} = f \left( \frac{K}{AN} \right) \]
In the long run (steady state), the growth rate of output equals the rate of population growth \(g_N\) plus the rate of technological progress \(g_A\).

- By implication, the growth rate of output is independent of the saving rate.

Capital accumulation is not the key of growth in the long run, because of its diminishing marginal returns.

- Why is capital not enough? Suppose that you want the economy to grow faster than the steady state growth rate of output, \(g_A + g_N\). And you want to keep increasing capital, to do reach this aim.
- But capital has diminishing returns. So you would need capital to grow faster than output by a certain amount, to increase the growth rate of output.
- The economy would have to devote a larger and larger proportion of output to capital accumulation. At some point, there would be no more output to devote to capital accumulation. Thus, the economy cannot permanently grow faster than \((g_A + g_N)\).

Let’s now look at output per worker (not “effective worker”).

\[
\frac{Y}{AN} = f\left(\frac{K}{AN}\right)
\]
Now focus on \( \frac{Y}{N} \), almost the focus of Malthus

- in the long run output \( Y \) grows at rate \( (g_A + g_N) \)
- the number of workers \( N \) grows at rate \( g_N \).

Hence output per worker \( \frac{Y}{N} \) grows at rate \( g_A (g_A + g_N - g_N) \).

- see Proposition 8 in Appendix 2 at the end of the book

In other words, when the economy is in steady state, output per worker grows at the rate of technological progress.

An economy’s rate of growth of output per person is eventually (in the long run) determined by its rate of technological progress.
### Summary

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<thead>
<tr>
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<th>Growth rate</th>
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<tbody>
<tr>
<td>1</td>
<td>Capital per effective worker</td>
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<tr>
<td>2</td>
<td>Output per effective worker</td>
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<tr>
<td>3</td>
<td>Capital per worker</td>
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<td>4</td>
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Most technological progress is the outcome of firms’ research and development (R&D) activities.

The level of R&D spending depends on
- the fertility of research (how spending on R&D translates into new ideas and new products)
- the appropriability of research results (the extent to which firms can benefit from the results of their own R&D).

The fertility of research depends on the successful interaction between basic research (the search for general principles and results) and applied research and development (the application of these results to specific uses, and the development of new products).

It takes many years, and often many decades, for the full potential of major discoveries to be realised. The usual sequence is one in which a major discovery leads to the exploration of potential applications, then to the development of new products, and finally to the adoption of these new products.
Patents give a firm that has discovered a new product the right to exclude anyone else from the production or use of that new product for some time.

If it is widely believed that the discovery of a new product by one firm will quickly lead to the discovery of an even better product by another firm, there may be little advantage to being first.

How should governments design patent laws?
- Protection is needed to provide firms with the incentives to spend on R&D.
- But once firms have discovered new products, it would be best for society if the knowledge embodied in those new products were made available without restriction to other firms and to people.

Institutions matter, for example, the protection of property rights.
- Few individuals are going to create firms, introduce new technologies and invest in R&D if they expect that profits will be appropriated by the state, extracted in bribes by corrupt bureaucrats, or stolen by other people in the economy.

Protection of property rights means a good political system, a good judicial system, laws against insider trading in the stock market, clearly written and well-enforced patent laws, good antitrust laws, etc...
After the Korean War, South Korea has provided private ownership and legal protection of private producers, while North Korea relied on central planning with no property rights for individuals. Fifty years later, GDP per person was 10 times higher in South Korea.
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