Corso di Laurea in Fisica - UNITS

Istituzioni di Fisica per il Sistema Terra

Differential Analysis of Fluid Flow

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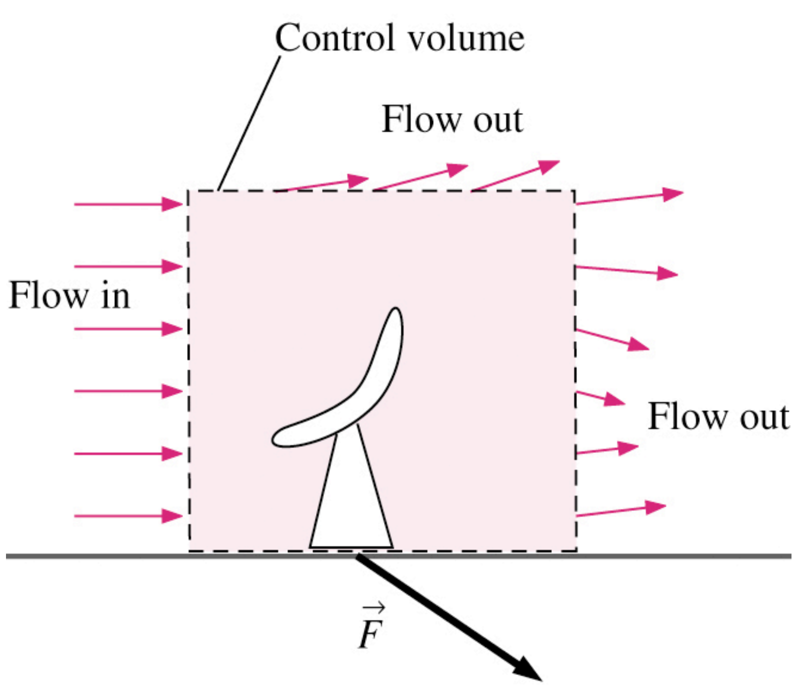
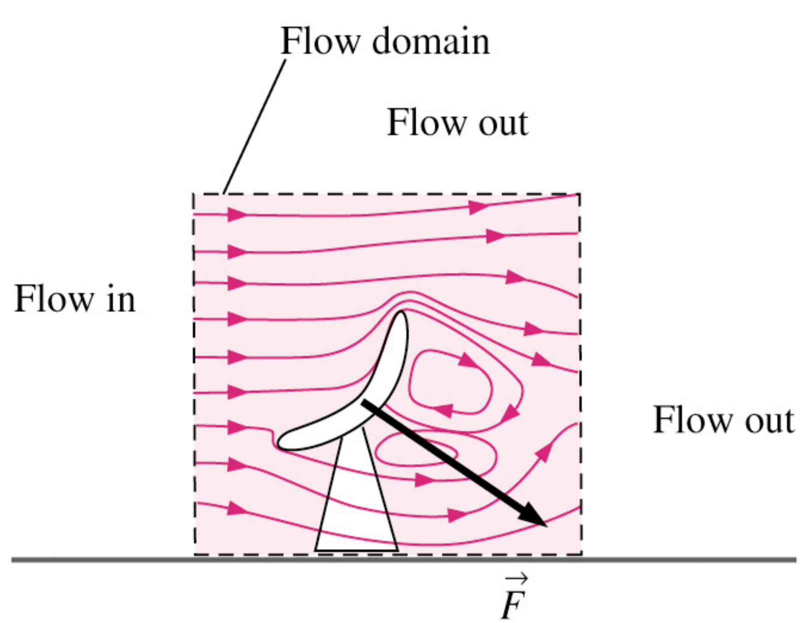
Objectives

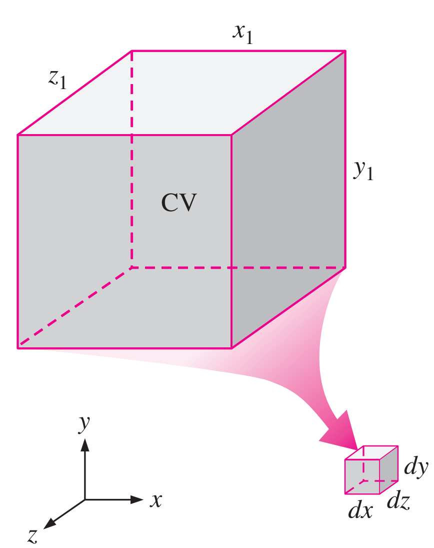
- Understand how the differential equations of mass and momentum conservation are derived.
- Calculate the stream function and pressure field, and plot streamlines for a known velocity field.
- Obtain analytical solutions of the equations of motion for simple flows.

Introduction

- Recall
  - Control volume (CV) versions of the laws of conservation of mass and energy
  - CV version of the conservation of momentum
- CV, or integral, forms of equations are useful for determining overall effects
- However, we cannot obtain detailed knowledge about the flow field inside the CV ⇒ motivation for differential analysis
  - differential equations of fluid motion to any and every point in the flow field over a region called the flow domain







Introduction

- Example: incompressible Navier-Stokes equations

$$\rho \frac{D\vec{V}}{Dt} = -\nabla P + \rho\vec{g} + \mu\nabla^2\vec{V}$$

$$\nabla \cdot \vec{V} = 0$$

- We will learn:
  - Physical meaning of each term
  - How to derive
  - How to solve

Introduction

- For example, how to solve?

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<tr><td>1</td><td colspan="2">Setup Problem and geometry, identify all dimensions and parameters</td></tr>
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<tr><td>3</td><td>Simplify PDE’s</td><td>Build grid / discretize PDE’s</td></tr>
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Conservation of Mass

- Recall CV form from Reynolds Transport Theorem (RTT)

$$0 = \int_{CV} \frac{\partial \rho}{\partial t} \, d\mathcal{V} + \int_{CS} \rho \left(\vec{V} \cdot \vec{n} \right) \, dA$$

- We'll examine two methods to derive differential form of conservation of mass
  - Divergence (Gauss's) Theorem
  - Differential CV and Taylor series expansions

Conservation of Mass - Divergence Theorem

- Divergence theorem allows us to transform a volume integral of the divergence of a vector into an area integral over the surface that defines the volume.

$$\int_{\mathcal{V}} \nabla \cdot \vec{G}\, d\mathcal{V} = \oint_A \vec{G} \cdot \vec{n}\, dA$$

Conservation of Mass - Divergence Theorem

- Rewrite conservation of mass

$$\int_{\mathcal{V}} \frac{\partial \rho}{\partial t}\,\mathrm{d}\mathcal{V} + \oint_A \rho\left(\vec{V}\cdot\vec{n}\right)\,\mathrm{d}A = 0$$

- Using divergence theorem, replace area integral with volume integral and collect terms

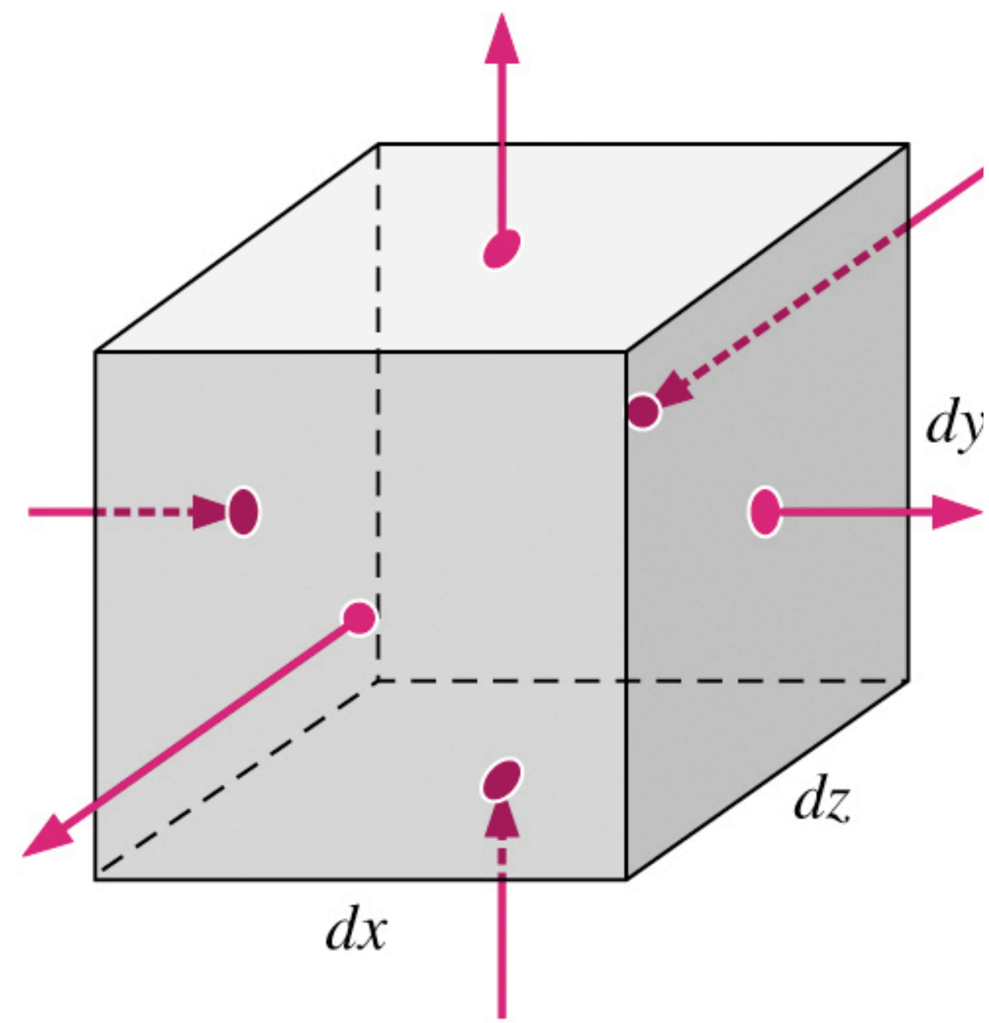
$$\int_{\mathcal{V}} \frac{\partial \rho}{\partial t}\,\mathrm{d}\mathcal{V} + \int_{\mathcal{V}} \nabla\cdot\rho\vec{V}\,\mathrm{d}\mathcal{V} = 0 \Longrightarrow \int_{\mathcal{V}} \left[\frac{\partial \rho}{\partial t} + \nabla\cdot\left(\rho\vec{V}\right)\right]\mathrm{d}\mathcal{V} = 0$$

- Integral holds for ANY CV, therefore:

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot\left(\rho\vec{V}\right) = 0}$$

Conservation of Mass - Differential CV and Taylor series

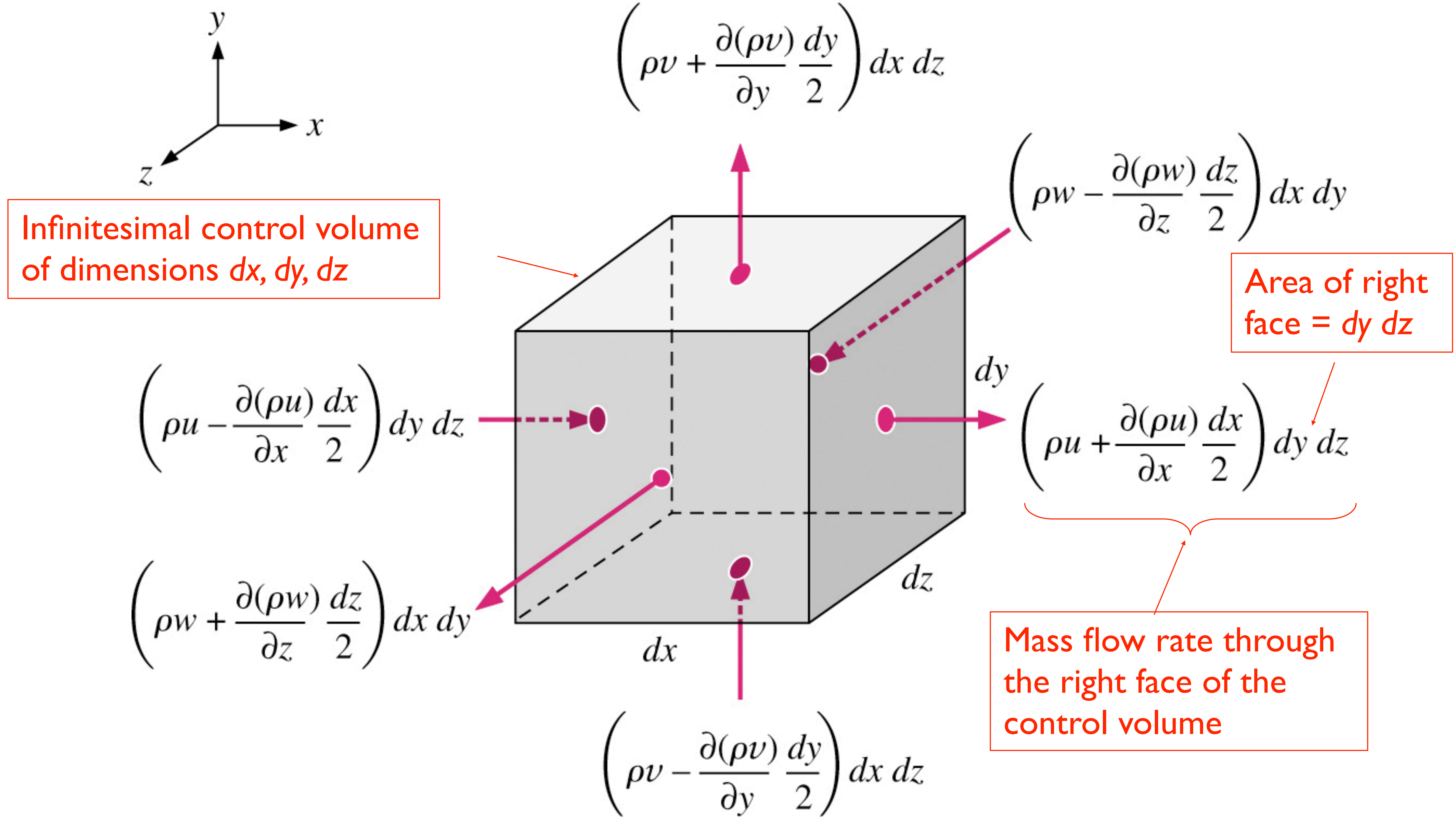
- First, define an infinitesimal control volume $dx \cdot dy \cdot dz$
- Next, we approximate the mass flow rate into or out of each of the 6 faces using Taylor series expansions around the center point, e.g., at the right face



Ignore terms higher than order *dx*

$$(\rho u)_{\text{center of right face}} = \rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2} + \frac{1}{2!}\frac{\partial^2(\rho u)}{\partial x^2}\left(\frac{dx}{2}\right)^2 + \ldots$$

Conservation of Mass - Differential CV and Taylor series



Conservation of Mass - Differential CV and Taylor series

- Now, sum up the mass flow rates into and out of the 6 faces of the CV

Net mass flow rate into CV:

$$\sum_{in} \dot{m} \approx \left(\rho u - \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\right) dy\, dz + \left(\rho v - \frac{\partial(\rho v)}{\partial y}\frac{dy}{2}\right) dx\, dz + \left(\rho w - \frac{\partial(\rho w)}{\partial z}\frac{dx}{2}\right) dx\, dy$$

Net mass flow rate out of CV:

$$\sum_{out} \dot{m} \approx \left(\rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\right) dy\, dz + \left(\rho v + \frac{\partial(\rho v)}{\partial y}\frac{dy}{2}\right) dx\, dz + \left(\rho w + \frac{\partial(\rho w)}{\partial z}\frac{dx}{2}\right) dx\, dy$$

- Plug into integral conservation of mass equation

$$\int_{CV} \frac{\partial \rho}{\partial t}\, d\mathcal{V} = \sum_{in} \dot{m} - \sum_{out} \dot{m}$$

Conservation of Mass - Differential CV and Taylor series

- After substitution,

$$\frac{\partial \rho}{\partial t} dx\, dy\, dz = -\frac{\partial(\rho u)}{\partial x} dx\, dy\, dz - \frac{\partial(\rho v)}{\partial y} dx\, dy\, dz - \frac{\partial(\rho w)}{\partial z} dx\, dy\, dz$$

- Dividing through by volume *dxdydz*

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

Or, if we apply the definition of the divergence of a vector

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \left(\rho \vec{V}\right) = 0$$

Conservation of Mass - Alternative form

- Use product rule on divergence term

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \left(\rho \vec{V} \right) = \frac{\partial \rho}{\partial t} + \vec{V} \cdot \nabla \rho + \rho \nabla \cdot \vec{V} = 0$$

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{V} = 0$$

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \cdot \vec{V} = 0$$

Conservation of Mass - Special Cases

- Steady compressible flow

$$\cancel{\frac{\partial \rho}{\partial t}} + \vec{\nabla} \cdot \left(\rho \vec{V}\right) = 0$$

$$\boxed{\vec{\nabla} \cdot \left(\rho \vec{V}\right) = 0}$$

Cartesian

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

Cylindrical

$$\frac{1}{r}\frac{\partial(r\rho U_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho U_\theta)}{\partial \theta} + \frac{\partial(\rho U_z)}{\partial z} = 0$$

Conservation of Mass - Special Cases

- Incompressible flow

$$\frac{\partial \rho}{\partial t} = 0 \qquad \text{and } \rho = \text{constant}$$

$$\vec{\nabla} \cdot \vec{V} = 0$$

Cartesian

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Cylindrical

$$\frac{1}{r}\frac{\partial (rU_r)}{\partial r} + \frac{1}{r}\frac{\partial (U_\theta)}{\partial \theta} + \frac{\partial (U_z)}{\partial z} = 0$$

Conservation of Mass

- In general, continuity equation cannot be used by itself to solve for flow field, however it can be used to
  - Determine if velocity field is incompressible
  - Find missing velocity component

Finding a Missing Velocity Component

- Two velocity components of a steady, incompressible, three-dimensional flow field are known, namely, $u = ax^2 + by^2 + cz^2$ and $w = axz + byz^2$, where a, b, and c are constants. The y velocity component is missing. Generate an expression for v as a function of x, y, and z.

- Solution:

Condition for incompressibility:

$$\frac{\partial v}{\partial y} = -\underbrace{\frac{\partial u}{\partial x}}_{2ax} - \underbrace{\frac{\partial w}{\partial z}}_{ax + 2byz} \quad \rightarrow \quad \frac{\partial v}{\partial y} = -3ax - 2byz$$

- Therefore,

$$v = -3axy - by^2z + f(x,z)$$

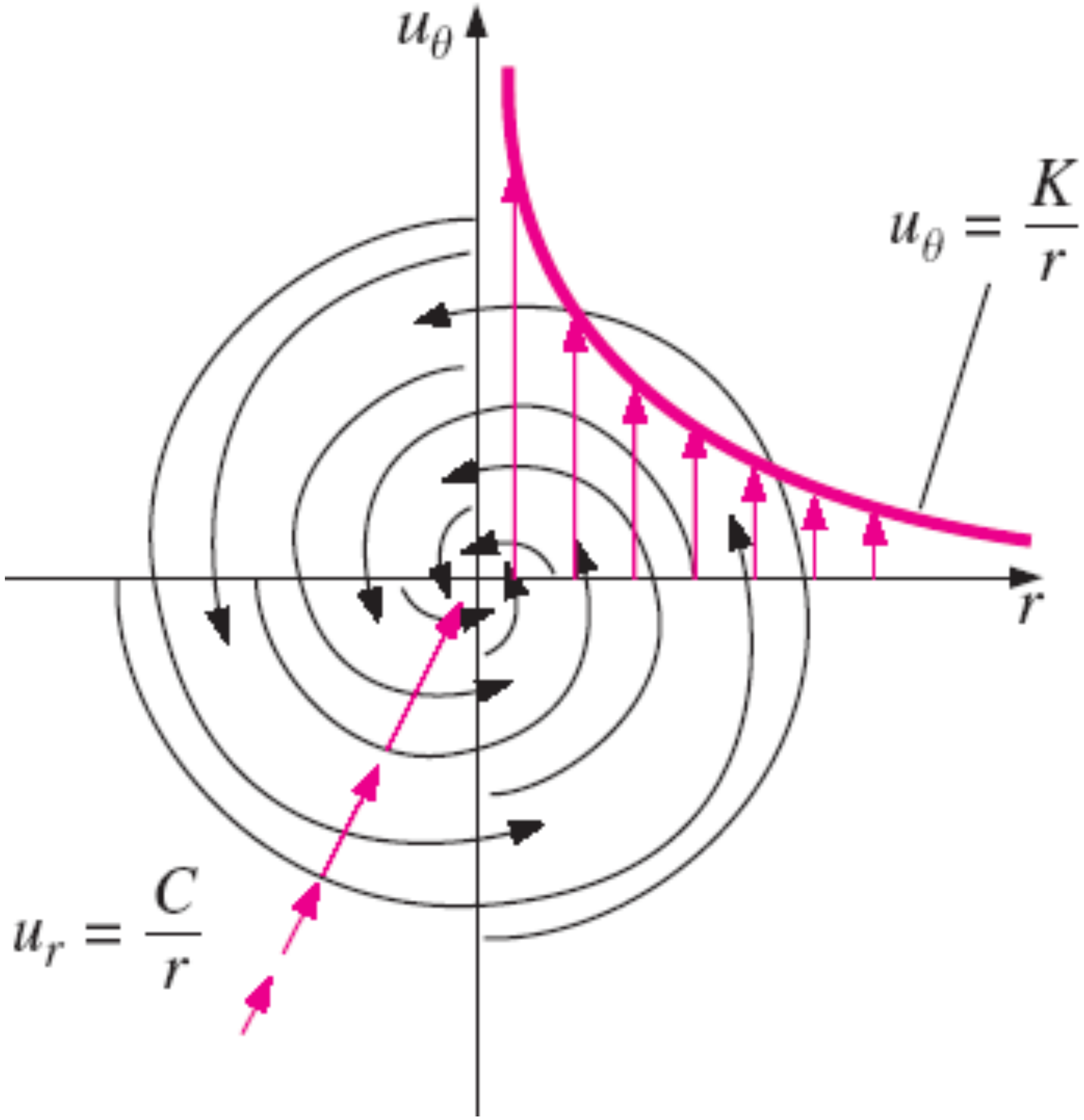
2D Incompressible Vortical Flow

- Consider a two-dimensional, incompressible flow in cylindrical coordinates; the tangential velocity component is $u_\theta = K/r$, where K is a constant. This represents a class of vortical flows. Generate an expression for the other velocity component, u_r.
- Solution: The incompressible continuity equation for this two dimensional case simplifies to

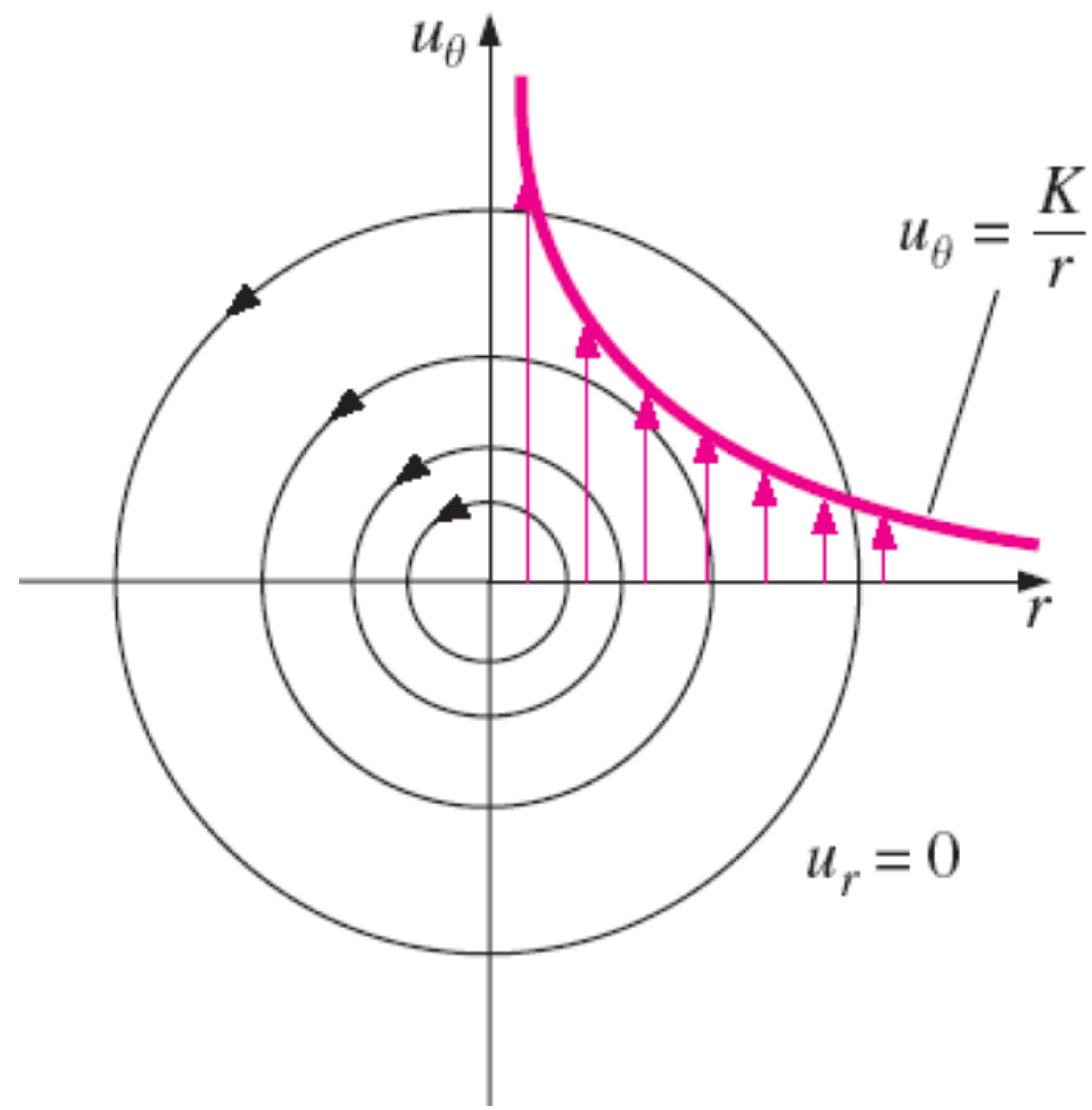
$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \underbrace{\frac{\partial u_z}{\partial z}}_{0\ (2\text{-D})} = 0 \quad \rightarrow \quad \frac{\partial(ru_r)}{\partial r} = -\frac{\partial u_\theta}{\partial \theta}$$

$$\frac{\partial(ru_r)}{\partial r} = 0 \quad \rightarrow \quad ru_r = f(\theta, t) \quad \Rightarrow \quad u_r = \frac{f(\theta, t)}{r}$$

2D Incompressible Vortical Flow



A spiraling line vortex/sink flow



Line Vortex

The Stream Function

- Consider the continuity equation for an incompressible 2D flow

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

- Substituting the clever transformation

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

- Gives

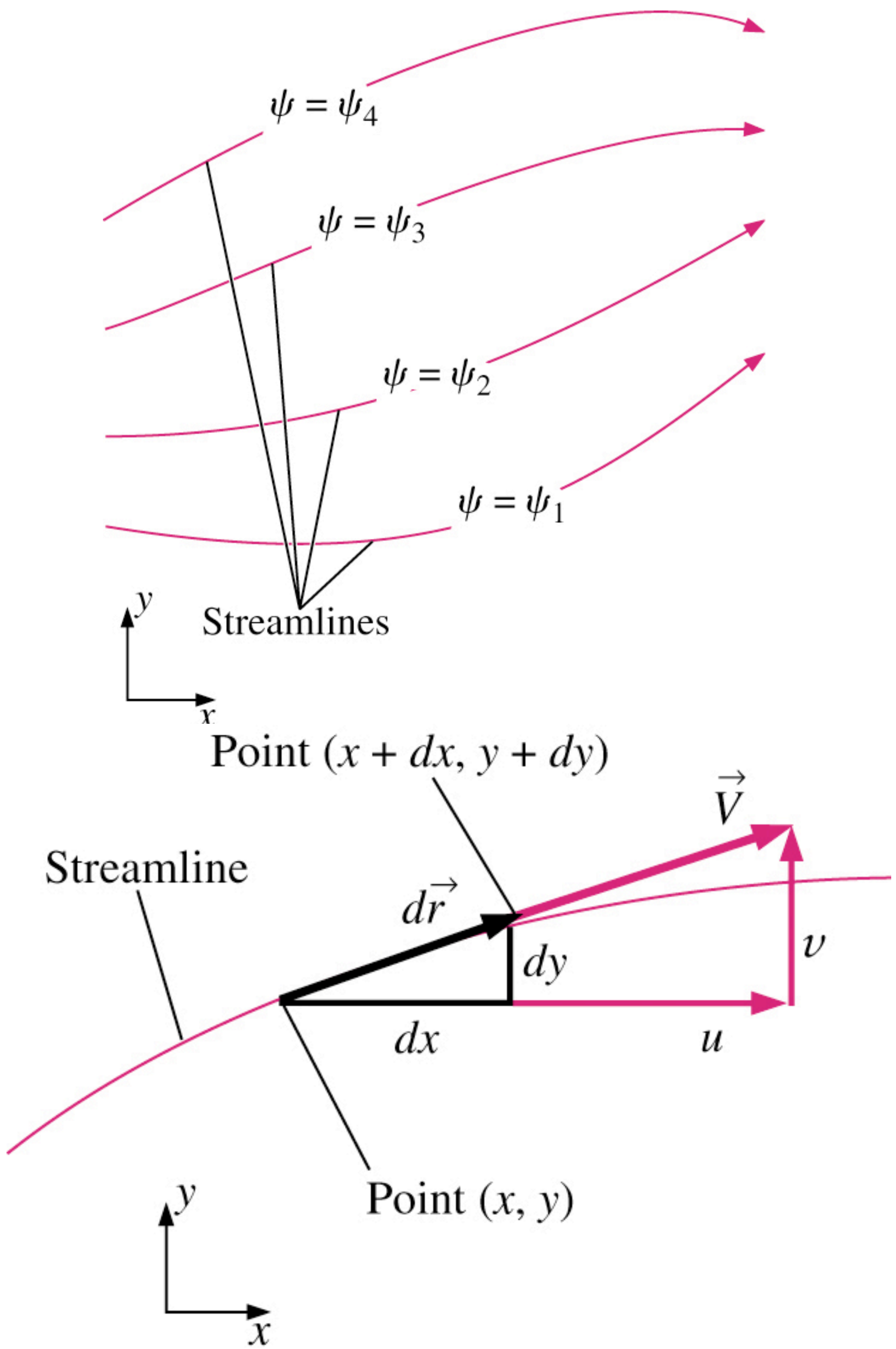
$$\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} \equiv 0$$

This is true for any smooth function $\psi(x,y)$

The Stream Function

- Why do this?
  - Single variable ψ replaces (u,v). Once ψ is known, (u,v) can be computed.
  - Physical significance
    - Curves of constant ψ are streamlines of the flow
    - Difference in ψ between streamlines is equal to volume flow rate between streamlines
    - The value of ψ increases to the left of the direction of flow in the xy-plane, "left-side convention."

The Stream Function - Physical Significance



Recall that along a streamline

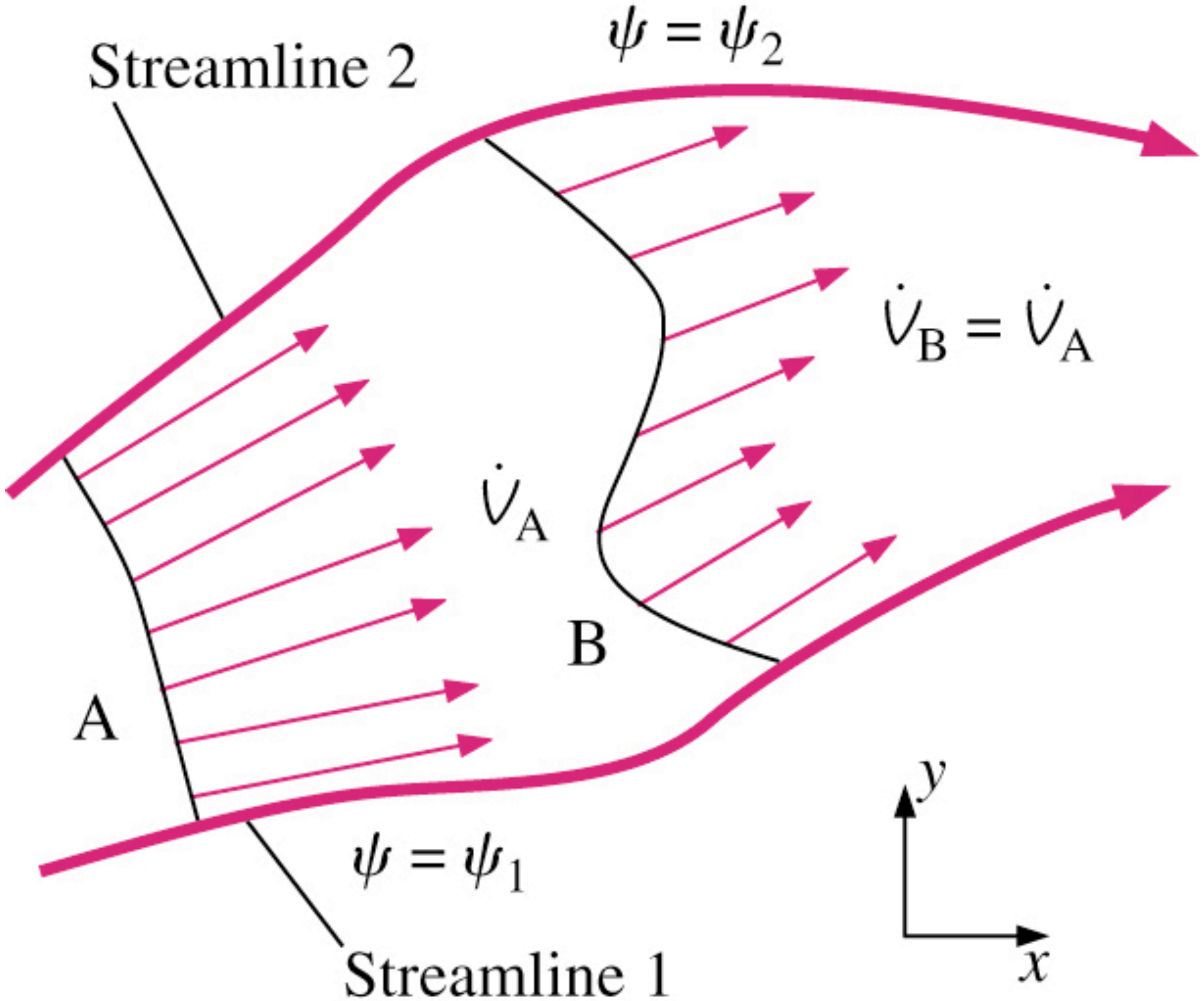
$$\frac{dy}{dx} = \frac{v}{u} \Rightarrow -v\,dx + u\,dy = 0$$

$$\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0$$

$$d\psi = 0$$

∴ Change in ψ along streamline is zero

The Stream Function - Physical Significance



Difference in ψ between streamlines is equal to volume flow rate between streamlines

$$\dot{V}_A = \dot{V}_B = \psi_2 - \psi_1$$

Stream Function in Cylindrical Coordinates

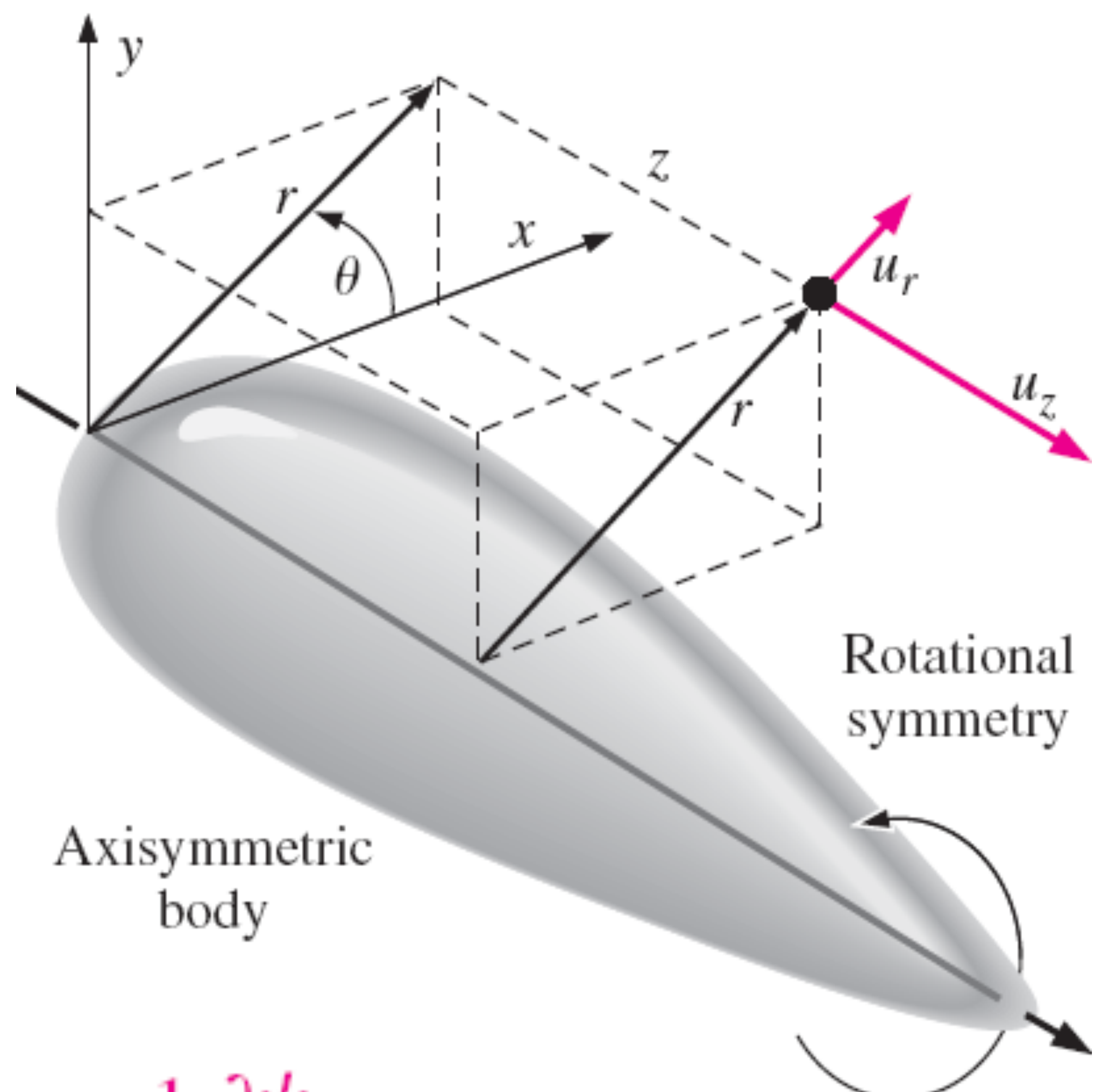
- Incompressible, planar stream function in cylindrical coordinates:

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad \text{and} \qquad u_\theta = -\frac{\partial \psi}{\partial r}$$

- For incompressible axisymmetric flow, the continuity equation is

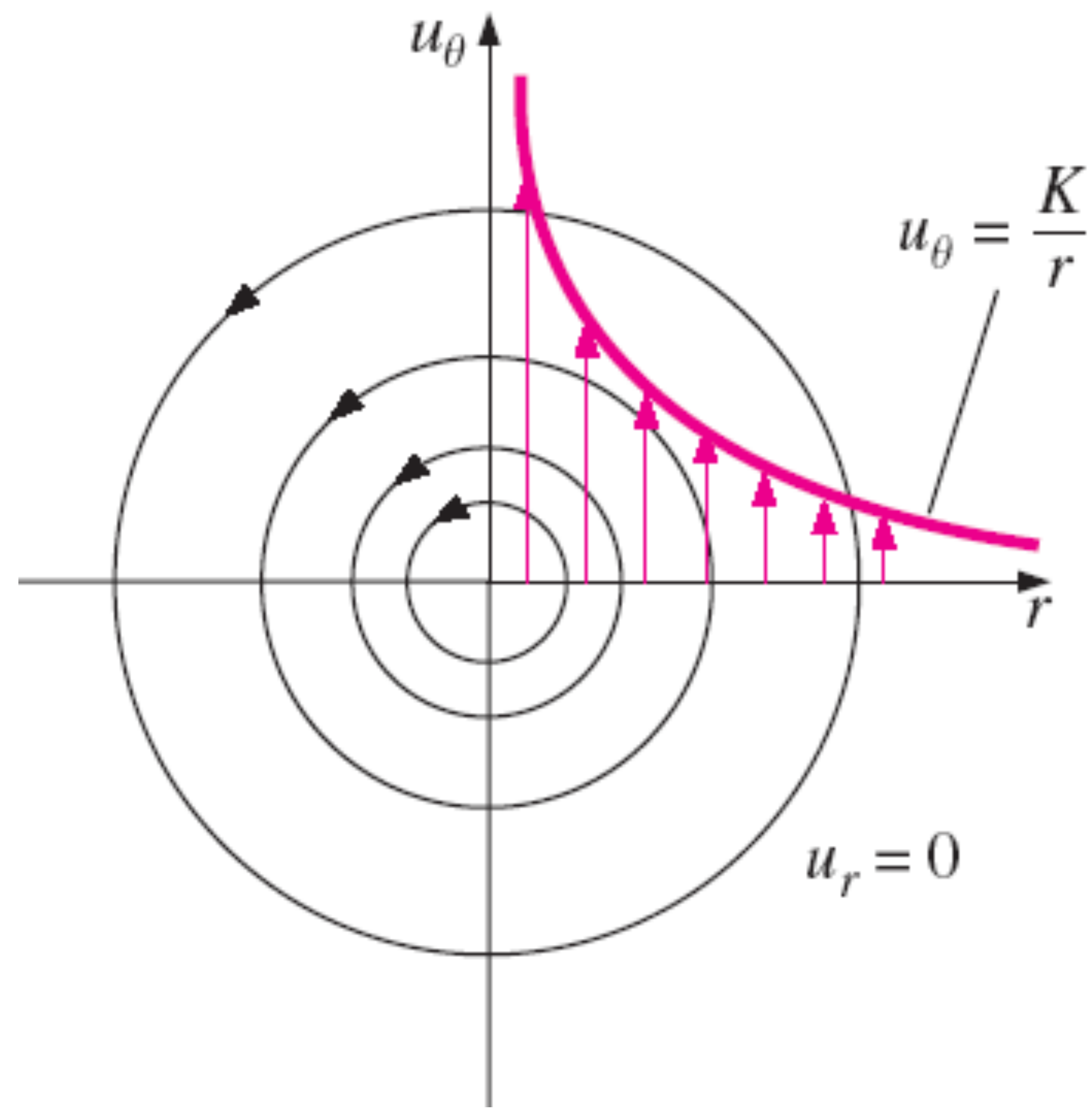
$$\frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{\partial (u_z)}{\partial z} = 0$$

$$\Rightarrow \qquad u_r = -\frac{1}{r}\frac{\partial \psi}{\partial z} \qquad \text{and} \qquad u_z = \frac{1}{r}\frac{\partial \psi}{\partial r}$$



Stream Function in Cylindrical Coordinates

- Consider a line vortex, defined as steady, planar, incompressible flow in which the velocity components are $u_r = 0$ and $u_\theta = K/r$, where K is a constant. Derive an expression for the stream function $\psi(r, \theta)$, and prove that the streamlines are circles.



Line Vortex

Stream Function in Cylindrical Coordinates

- Solution:

$$\frac{\partial \psi}{\partial r} = -u_\theta = -\frac{K}{r} \qquad \rightarrow$$

$$\psi = -K \ln r + f(\theta)$$

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{1}{r} f'(\theta)$$

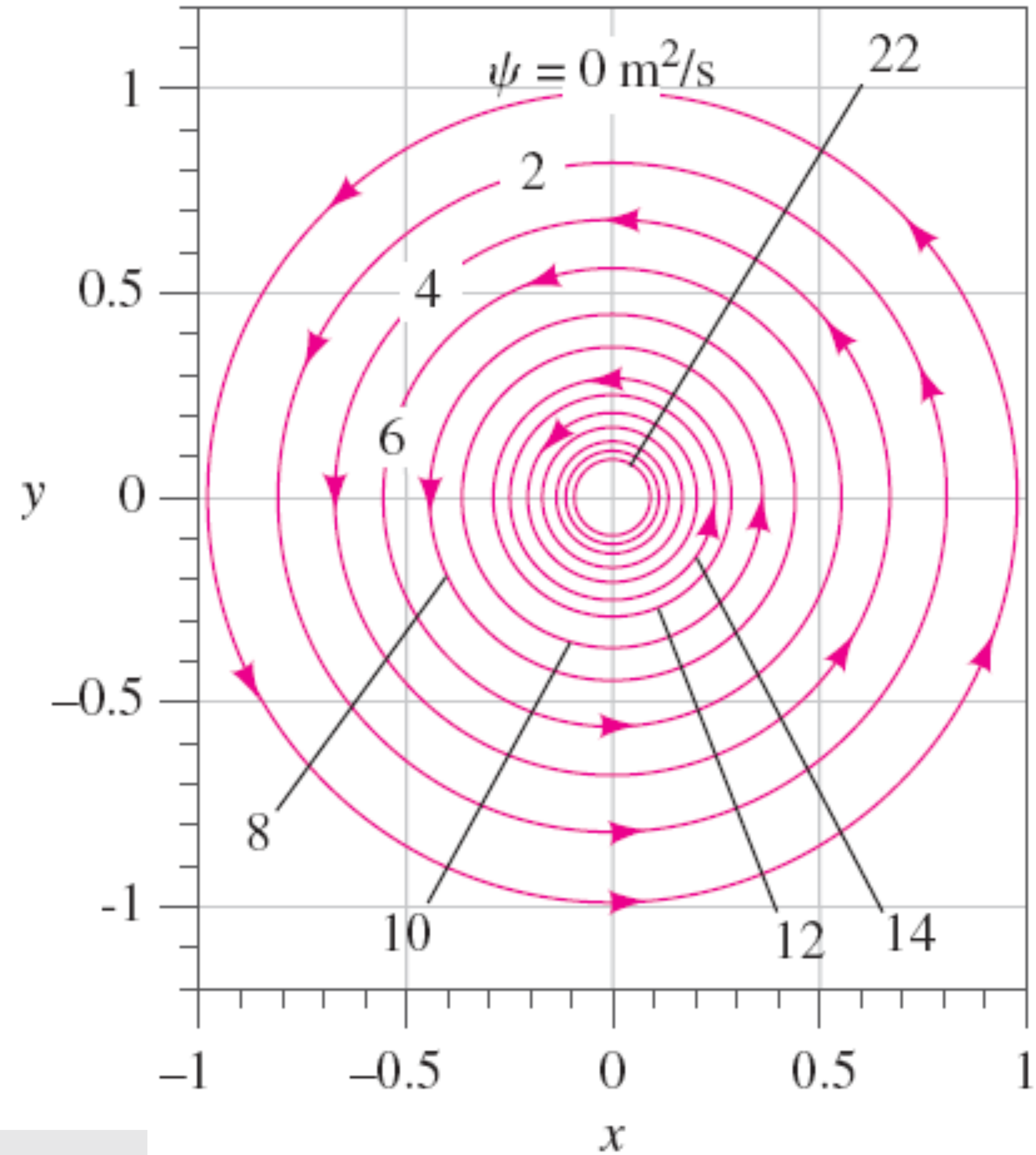
$$f'(\theta) = 0 \qquad \rightarrow$$

$$f(\theta) = C$$

$$\Rightarrow \quad \psi = -K \ln r + C$$

Equation for streamlines:

$$r = e^{-(\psi - C)/K}$$



Conservation of Linear Momentum

- Recall CV form

$$\sum \vec{F} = \underbrace{\int_{CV} \rho g \, \mathrm{d}\mathcal{V}}_{\text{Body Force}} + \underbrace{\int_{CS} \sigma_{ij} \cdot \vec{n} \, \mathrm{d}A}_{\text{Surface Force}} = \int_{CV} \frac{\partial}{\partial t} \left(\rho \vec{V}\right) \mathrm{d}\mathcal{V} + \int_{CS} \left(\rho \vec{V}\right) \vec{V} \cdot \vec{n} \, \mathrm{d}A$$

σ_{ij} = stress tensor

- Using the divergence theorem to convert area integrals

$$\int_{CS} \sigma_{ij} \cdot \vec{n} \, \mathrm{d}A = \int_{CV} \nabla \cdot \sigma_{ij} \, \mathrm{d}\mathcal{V}$$

$$\int_{CS} \left(\rho \vec{V}\right) \vec{V} \cdot \vec{n} \, \mathrm{d}A = \int_{CV} \nabla \cdot \left(\rho \vec{V} \vec{V}\right) \mathrm{d}\mathcal{V}$$

Conservation of Linear Momentum

- Substituting volume integrals gives,

$$\int_{CV} \left[\frac{\partial}{\partial t} \left(\rho \vec{V} \right) + \nabla \cdot \left(\rho \vec{V} \vec{V} \right) - \rho \vec{g} - \nabla \cdot \sigma_{ij} \right] \mathrm{d}\mathcal{V} = 0$$

- Recognizing that this holds for any CV, the integral may be dropped

$$\boxed{\frac{\partial}{\partial t} \left(\rho \vec{V} \right) + \nabla \cdot \left(\rho \vec{V} \vec{V} \right) = \rho \vec{g} + \nabla \cdot \sigma_{ij}}$$

This is Cauchy's Equation

That can also be derived using infinitesimal CV and Newton's 2nd Law

Conservation of Linear Momentum

- Alternate form of the Cauchy Equation can be derived by introducing

$$\frac{\partial\left(\rho\vec{V}\right)}{\partial t} = \rho\frac{\partial\vec{V}}{\partial t} + \vec{V}\frac{\partial\rho}{\partial t} \qquad \text{(Chain Rule)}$$

$$\nabla\cdot\left(\rho\vec{V}\vec{V}\right) = \vec{V}\nabla\cdot\left(\rho\vec{V}\right) + \rho\left(\vec{V}\cdot\nabla\right)\vec{V}$$

- Inserting these into Cauchy Equation and rearranging gives

$$\rho\left[\frac{\partial\vec{V}}{\partial t} + \left(\vec{V}\cdot\nabla\right)\vec{V}\right] = \rho\vec{g} + \nabla\cdot\sigma_{ij}$$

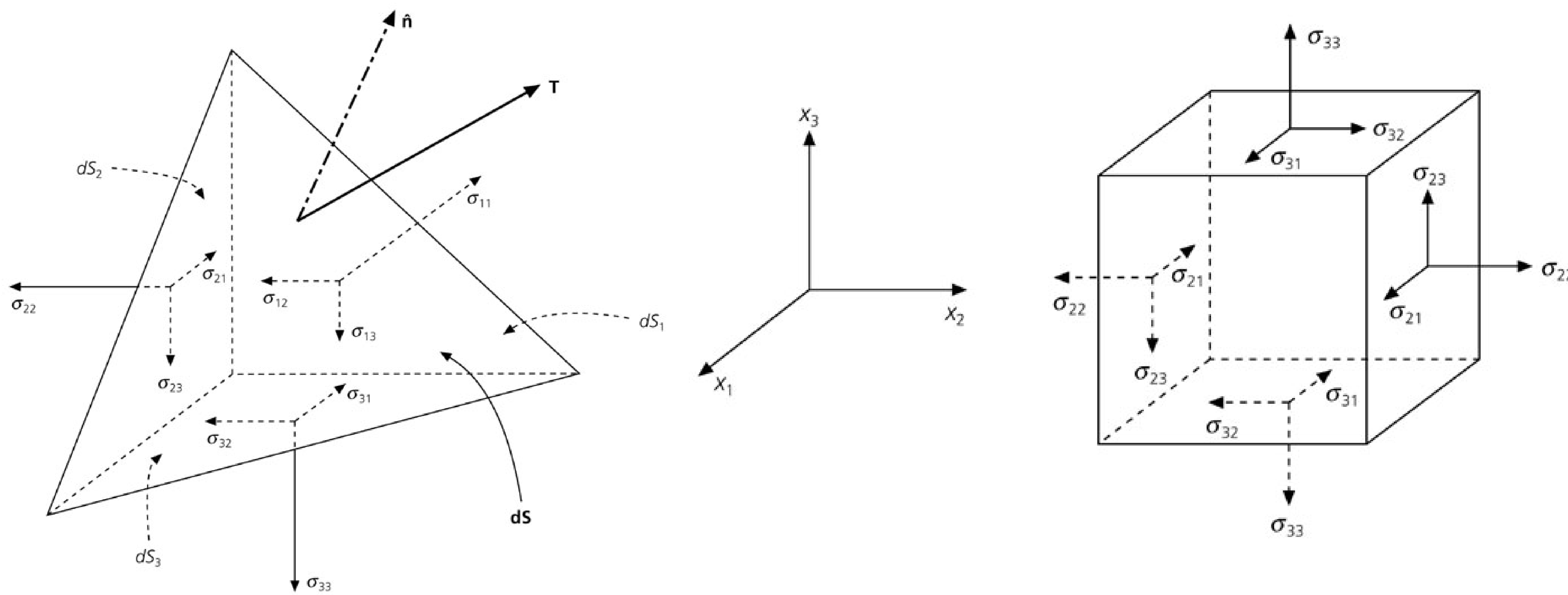
$$\rho\frac{D\vec{V}}{Dt} = \rho\vec{g} + \nabla\cdot\sigma_{ij}$$

Conservation of Linear Momentum

- Unfortunately, this equation is not very useful
  - 10 unknowns
    - Stress tensor, σ_{ij} : 6 independent components
    - Density ρ
    - Velocity, $\boldsymbol{V}$: 3 independent components
  - 4 equations (continuity + momentum)
  - 6 more equations required to close problem!

Stress Tensor

- The stress (force per unit area) at a point in a fluid needs nine components to be completely specified, since each component of the stress must be defined not only by the direction in which it acts but also the orientation of the surface upon which it is acting. The first index i specifies the direction in which the stress component acts, and the second index j identifies the orientation of the surface upon which it is acting. Therefore, the ith component of the force acting on a surface whose outward normal points in the jth direction is σ_{ij}.

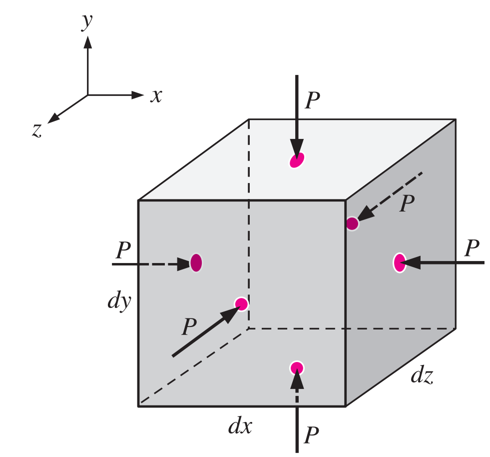


$$T^{(\mathbf{n})} = n_i T^{(i)} = n_i T_j^{(i)} \hat{x}_j = n_i \sigma_{ij} \hat{x}_j$$

Stress Tensor

- For a fluid at rest, according to Pascal's law, regardless of the orientation the stress reduces to:

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix}$$



- Hydrostatic pressure is the same as the thermodynamic pressure from study of thermodynamics. P is related to temperature and density through some type of equation of state (e.g., the ideal gas law).
  - This further complicates a compressible fluid flow analysis because we introduce yet another unknown, namely, temperature T.
  - This new unknown requires another equation—the differential form of the energy equation.

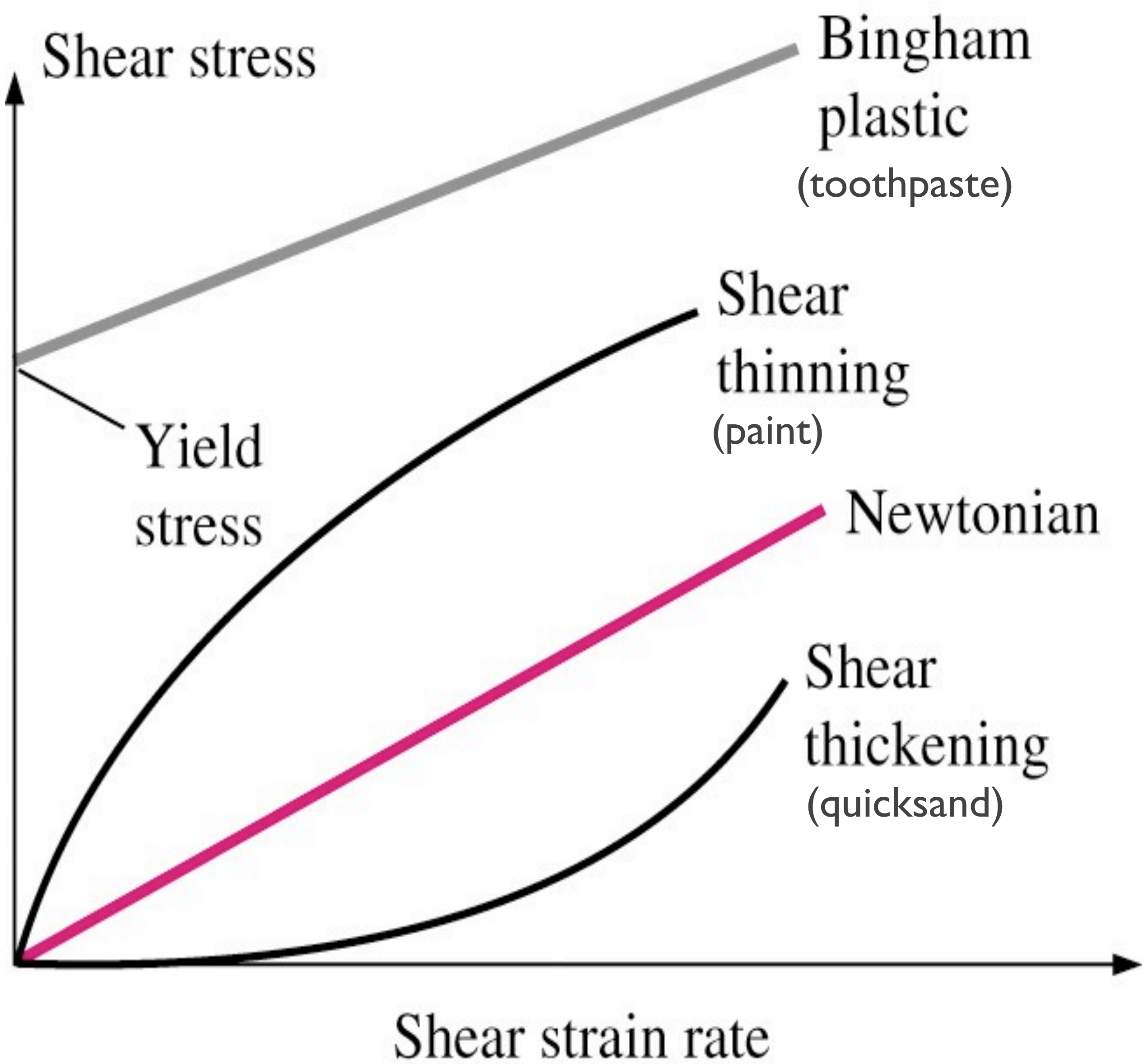
Stress Tensor

- First step is to separate σ_{ij} into pressure and viscous stresses

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix} + \underbrace{\begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}}_{\text{Viscous (Deviatoric) Stress Tensor}}$$

- Situation not yet improved
  - 6 unknowns in $\sigma_{ij} \Rightarrow$ 6 unknowns in τ_{ij} + 1 in p, which means that we've added 1!

Constitutive equation - Newtonian



Newtonian fluid includes most common fluids: air, other gases, water, gasoline

- Reduction in the number of variables is achieved by relating shear stress to strain-rate tensor.
- For Newtonian fluid with constant properties

$$\tau_{ij} = 2\mu\epsilon_{ij}$$

Newtonian closure is analogous
to Hooke's Law for elastic solids

Stresses to Strains to Velocities

- Substituting Newtonian closure into stress tensor gives

$$\sigma_{ij} = -P\delta_{ij} + 2\mu\varepsilon_{ij}$$

- Using the definition of ε_{ij}

$$\sigma_{ij} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix} + \begin{pmatrix} 2\mu\frac{\partial U}{\partial x} & \mu\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right) & \mu\left(\frac{\partial U}{\partial z} + \frac{\partial W}{\partial x}\right) \\ \mu\left(\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\right) & 2\mu\frac{\partial V}{\partial y} & \mu\left(\frac{\partial V}{\partial z} + \frac{\partial W}{\partial y}\right) \\ \mu\left(\frac{\partial W}{\partial x} + \frac{\partial U}{\partial z}\right) & \mu\left(\frac{\partial W}{\partial y} + \frac{\partial V}{\partial z}\right) & 2\mu\frac{\partial W}{\partial z} \end{pmatrix}$$

Navier-Stokes Equation

- Substituting σ_{ij} into Cauchy's equation gives the Navier-Stokes equation(s):

$$\rho \frac{D\vec{V}}{Dt} = -\nabla P + \rho \vec{g} + \mu \nabla^2 \vec{V}$$

$$\nabla \cdot \vec{V} = 0$$

Incompressible NSE written in vector form

- With Continuity Equation, this results in a closed system of equations!
  - 4 equations (continuity and momentum equations)
  - 4 unknowns (*U*, *V*, *W*, *P*)

Navier-Stokes Equation

- In addition to vector form, incompressible N-S equation can be written in several other forms:
  - Cartesian coordinates
  - Cylindrical coordinates
  - Tensor notation

Navier-Stokes Equation - Cartesian

Continuity $$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}=0$$

X-momentum

$$\rho\left(\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}+W\frac{\partial U}{\partial z}\right)=-\frac{\partial P}{\partial x}+\rho g_x+\mu\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}\right)$$

Y-momentum

$$\rho\left(\frac{\partial V}{\partial t}+U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}+W\frac{\partial V}{\partial z}\right)=-\frac{\partial P}{\partial y}+\rho g_y+\mu\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}\right)$$

Z-momentum

$$\rho\left(\frac{\partial W}{\partial t}+U\frac{\partial W}{\partial x}+V\frac{\partial W}{\partial y}+W\frac{\partial W}{\partial z}\right)=-\frac{\partial P}{\partial z}+\rho g_z+\mu\left(\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}+\frac{\partial^2 W}{\partial z^2}\right)$$

Navier-Stokes Equation - Tensor and Vector

Tensor and Vector notation offer a more compact form of the equations.

Continuity

Tensor notation

$$\frac{\partial U_i}{\partial x_i} = 0$$

Vector notation

$$\nabla \cdot \vec{V} = 0$$

Conservation of Momentum

Tensor notation

$$\rho \left(\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} \right) = -\frac{\partial P}{\partial x_i} + \rho g_{x_i} + \mu \left(\frac{\partial^2 U_i}{\partial x_j \partial x_j} \right)$$

Vector notation

$$\rho \frac{D\vec{V}}{Dt} = -\nabla P + \rho \vec{g} + \mu \nabla^2 \vec{V}$$

Repeated indices are summed over j

($x_1 = x$, $x_2 = y$, $x_3 = z$, $U_1 = U$, $U_2 = V$, $U_3 = W$)