Corso di Laurea in Fisica - UNITS

Istituzioni di Fisica per il Sistema Terra

Solutions to Navier Stokes Equation

FABIO ROMANELLI

Department of Mathematics & Geosciences
University of Trieste
romanel@units.it

Differential Analysis of Fluid Flow Problems

- Now that we have a set of governing partial differential equations, there are 2 problems we can solve:
  - Calculate pressure (*P*) for a known velocity field
  - Calculate velocity (*U*,*V*,*W*) and pressure (*P*) for known geometry, boundary conditions (BC), and initial conditions (IC)

Calculating the Pressure Field in Cart. coord.

- Consider the steady, two-dimensional, incompressible velocity field, namely, $\vec{V} = (u, v) = (ax + b)\vec{i} + (-ay + cx)\vec{j}$. Calculate the pressure as a function of *x* and *y*.
- Solution: Check continuity equation,

$$\underbrace{\frac{\partial u}{\partial x}}_{a} + \underbrace{\frac{\partial v}{\partial y}}_{-a} + \underbrace{\cancel{\frac{\partial w}{\partial z}}}_{0\text{ (2-D)}} = a - a = 0$$

Consider the *y*-component of the Navier–Stokes equation:

$$\rho\left(\underbrace{\cancel{\frac{\partial v}{\partial t}}}_{0\text{ (steady)}} + \underbrace{u\frac{\partial v}{\partial x}}_{(ax+b)c} + \underbrace{v\frac{\partial v}{\partial y}}_{(-ay+cx)(-a)} + \underbrace{\cancel{w\frac{\partial v}{\partial z}}}_{0\text{ (2-D)}}\right) = -\frac{\partial P}{\partial y} + \underbrace{\cancel{\rho g_y}}_{0} + \mu\left(\underbrace{\cancel{\frac{\partial^2 v}{\partial x^2}}}_{0} + \underbrace{\cancel{\frac{\partial^2 v}{\partial y^2}}}_{0} + \underbrace{\cancel{\frac{\partial^2 v}{\partial z^2}}}_{0\text{ (2-D)}}\right)$$

Calculating the Pressure Field in Cart. coord.

The *y*-momentum equation reduces to

$$\frac{\partial P}{\partial y} = \rho(-acx - bc - a^2y + acx) = \rho(-bc - a^2y)$$

In similar fashion, the *x*-momentum equation reduces to

$$\frac{\partial P}{\partial x} = \rho(-a^2x - ab)$$

Pressure field from y-momentum:

$$P(x, y) = \rho\left(-bcy - \frac{a^2y^2}{2}\right) + g(x)$$

$$\Rightarrow \quad \frac{\partial P}{\partial x} = g'(x) = \rho(-a^2x - ab)$$

Calculating the Pressure Field in Cart. coord.

Then we can get

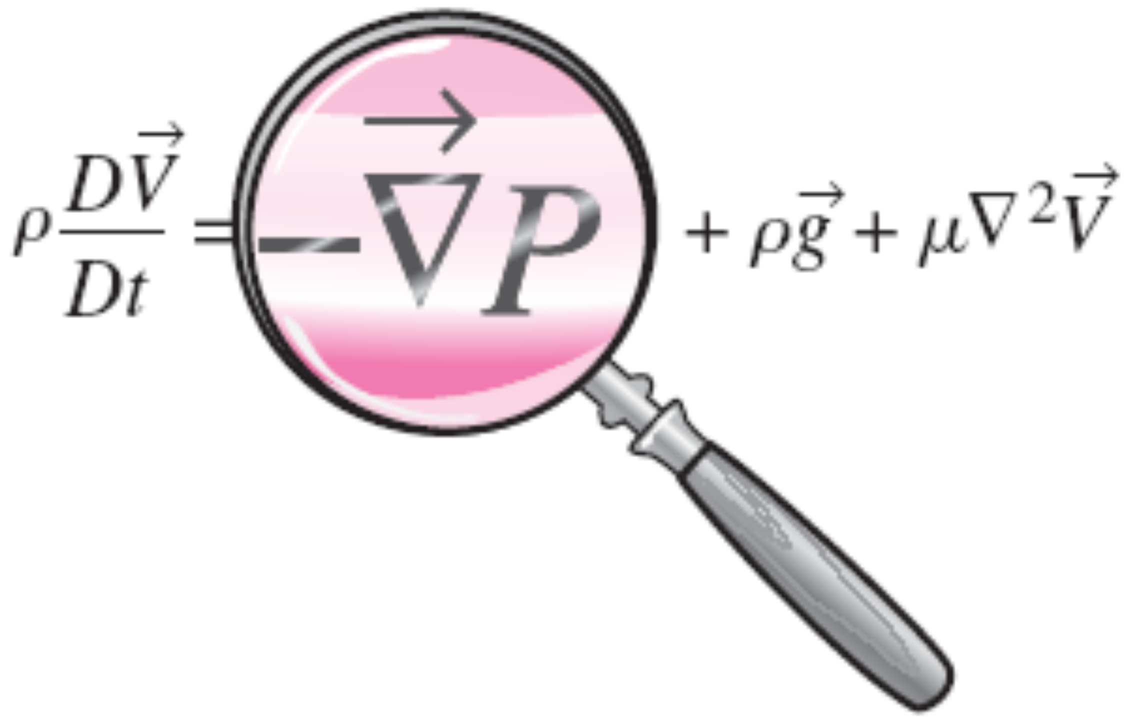
$$g(x) = \rho\left(-\frac{a^2x^2}{2} - abx\right) + C_1$$

Such that

$$P(x, y) = \rho\left(-\frac{a^2x^2}{2} - \frac{a^2y^2}{2} - abx - bcy\right) + C_1$$

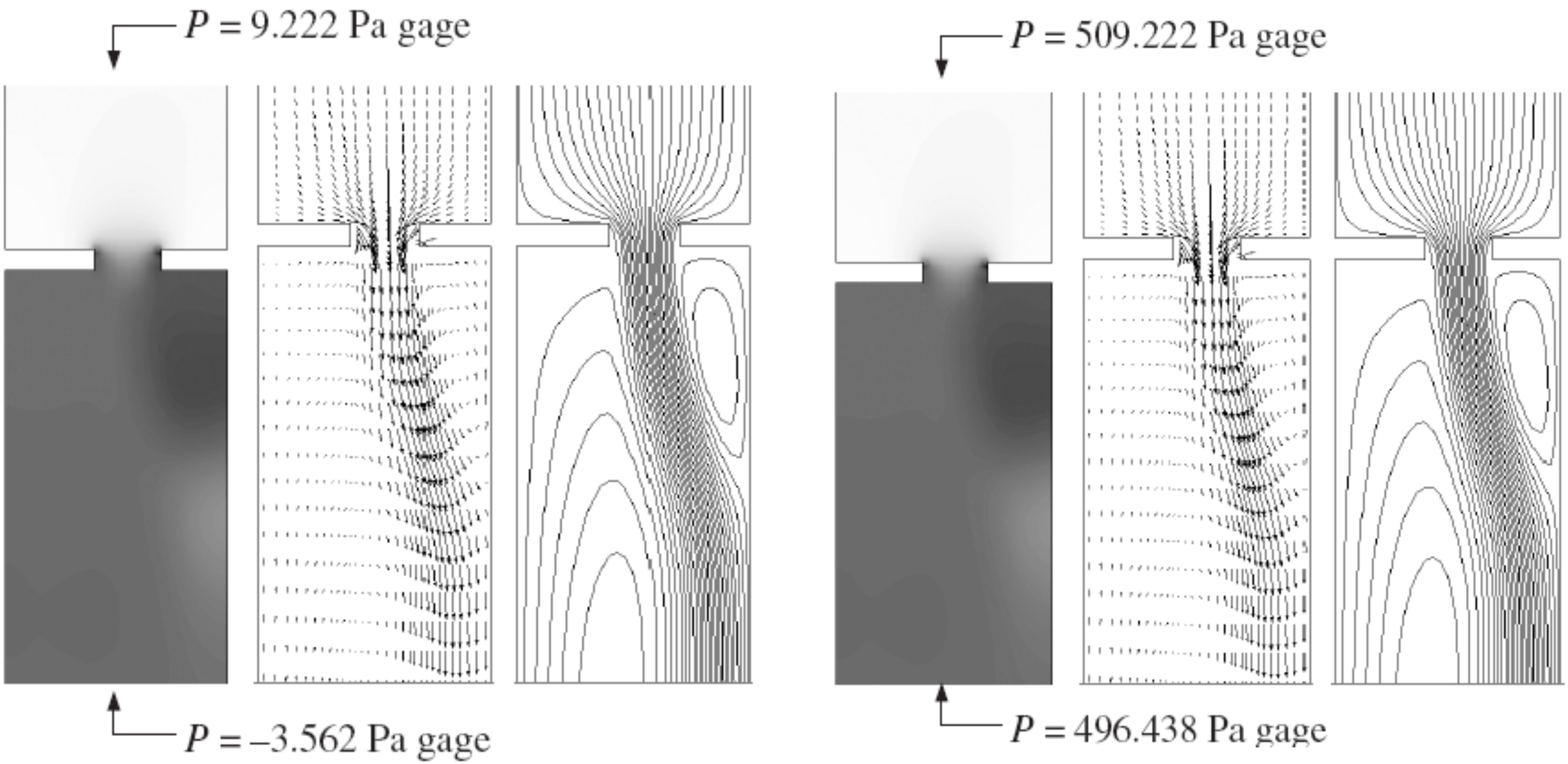
- Will the C_1 in the equation affect the velocity field? No. The velocity field in an incompressible flow is not affected by the absolute magnitude of pressure, but only by pressure differences.

Calculating the Pressure Field in Cart. coord.



- From the Navier-Stokes equation (NSE), we know the velocity field is affected by pressure gradient.
- In order to determine that constant (C_1 in Example), we must measure (or otherwise obtain) P somewhere in the flow field. In other words, we require a pressure boundary condition. Please see the CFD results on the next page.

Calculating the Pressure Field in Cart. coord.



Filled pressure contour plot, velocity vector plot, and streamlines for downward flow of air through a channel with blockage: (a) case 1; (b) case 2—identical to case 1, except P is everywhere increased by 500 Pa. On the gray-scale contour plots, dark is low pressure and light is high pressure.

Exact Solutions of the NSE

- There are about 80 known exact solutions to the NSE
- The can be classified as:
  - Linear solutions where the convective term is zero

$$\left(\vec{V} \cdot \nabla\right) \vec{V}$$

  - Nonlinear solutions where convective term is not zero

- Solutions can also be classified by type or geometry
  - Couette shear flows
  - Steady duct/pipe flows
  - Unsteady duct/pipe flows
  - Flows with moving boundaries
  - Similarity solutions
  - Asymptotic suction flows
  - Wind-driven Ekman flows

Exact Solutions of the NSE

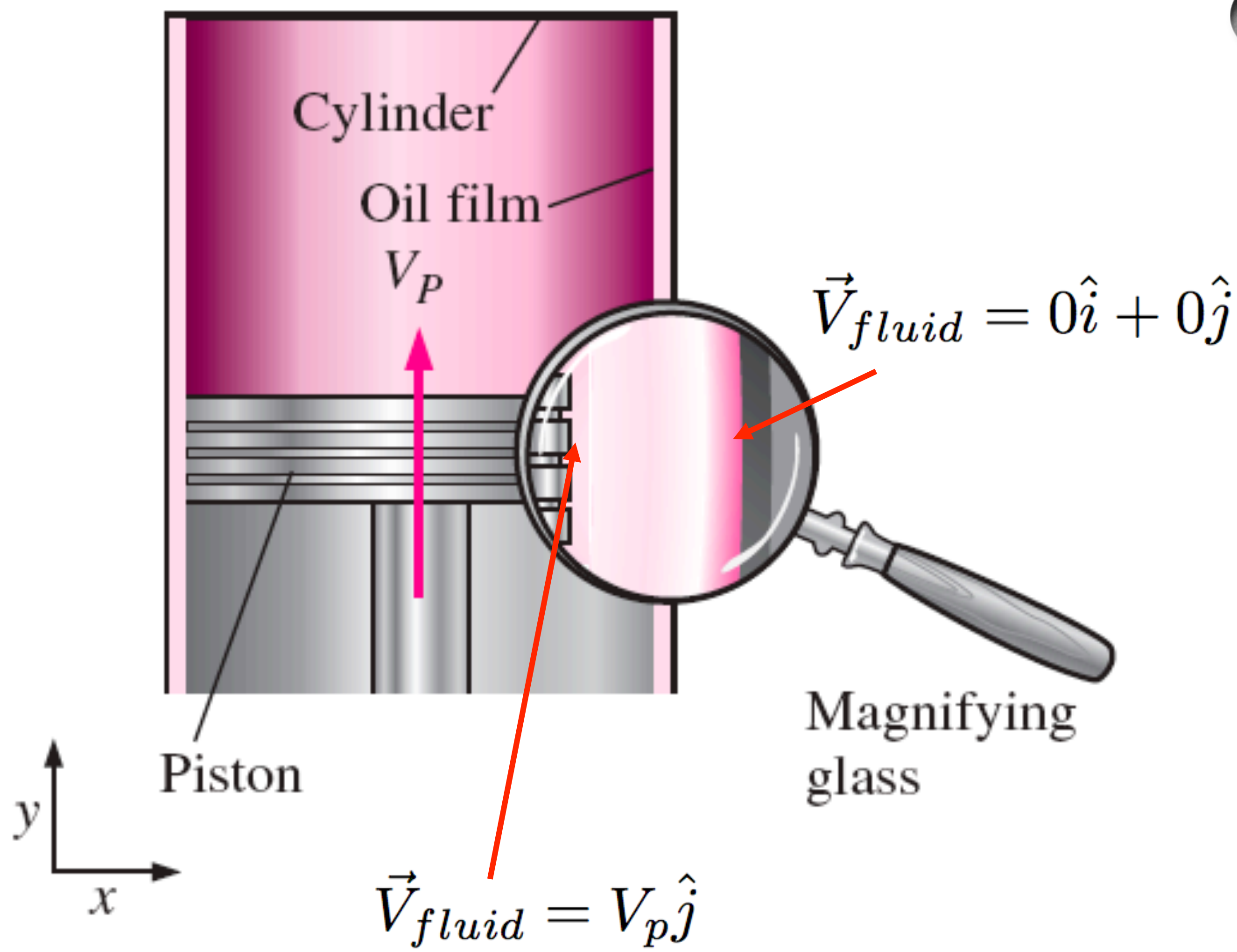
Procedure for solving continuity and NSE

1. Set up the problem and geometry, identifying all relevant dimensions and parameters
2. List all appropriate assumptions, approximations, simplifications, and boundary conditions
3. Simplify the differential equations as much as possible
4. Integrate the equations
5. Apply BC to solve for constants of integration
6. Verify results

Boundary conditions

- Boundary conditions are critical to exact, approximate, and computational solutions.
  - BC's used in analytical solutions are discussed here:
    - No-slip boundary condition
    - Interface boundary condition
  - These are used in CFD as well, plus there are some BC's which arise due to specific issues in CFD modeling:
    - Inflow and outflow boundary conditions
    - Symmetry and periodic boundary conditions

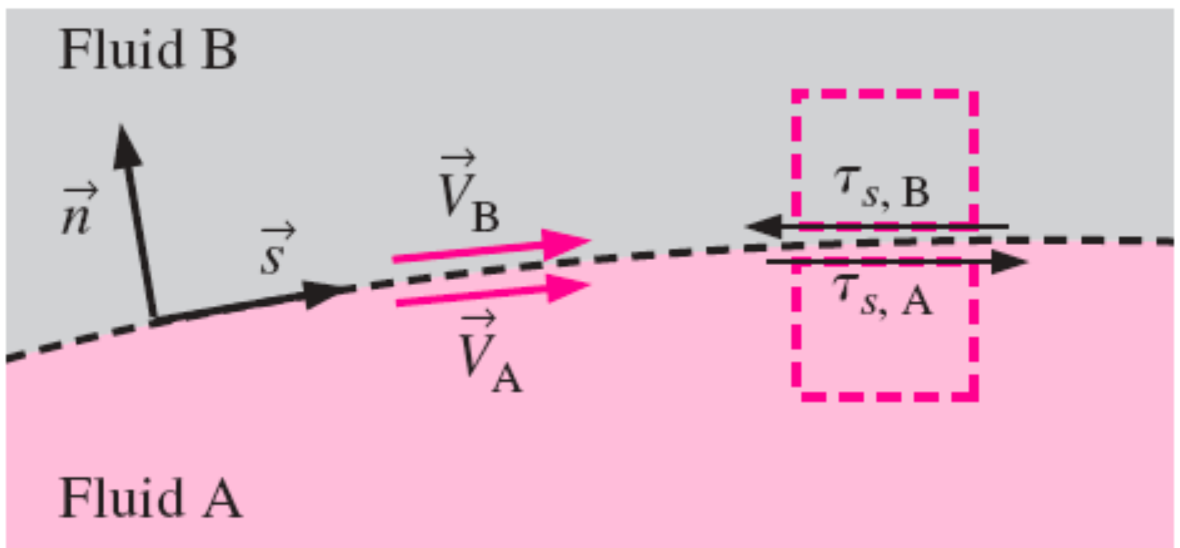
Kinematic (no-slip) boundary condition



- For a fluid in contact with a solid wall, the velocity of the fluid must equal that of the wall

$$\vec{V}_{fluid} = \vec{V}_{wall}$$

Interface boundary condition



- When two fluids meet at an interface, the velocity and shear stress must be the same on both sides

$$\vec{V}_A = \vec{V}_B \qquad \tau_{s,A} = \tau_{s,B}$$

- If surface tension effects are negligible and the surface is nearly flat

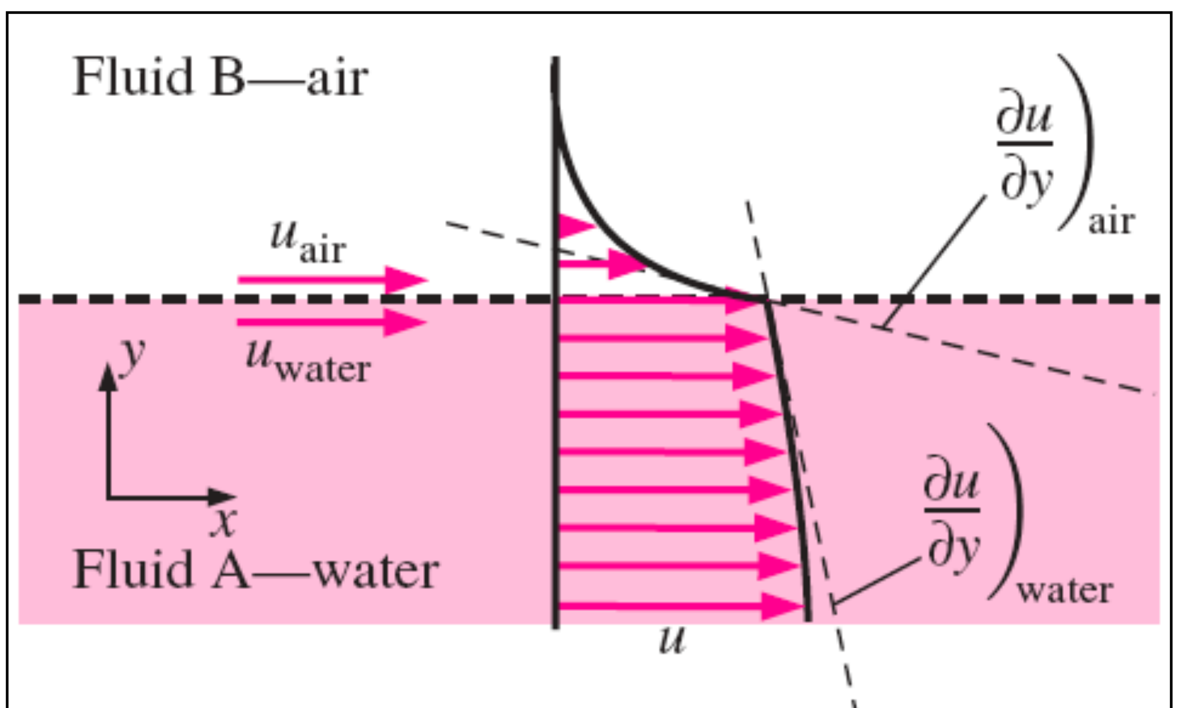
$$P_A = P_B$$

Interface boundary condition

- Degenerate case of the interface BC occurs at the free surface of a liquid.
- Same conditions hold

$$u_{air} = u_{water}$$

$$\tau_{s,water} = \mu_{water}\left(\frac{\partial u}{\partial y}\right)_{water} = \tau_{s,air} = \mu_{air}\left(\frac{\partial u}{\partial y}\right)_{air}$$



Since $\mu_{air} << \mu_{water}$,

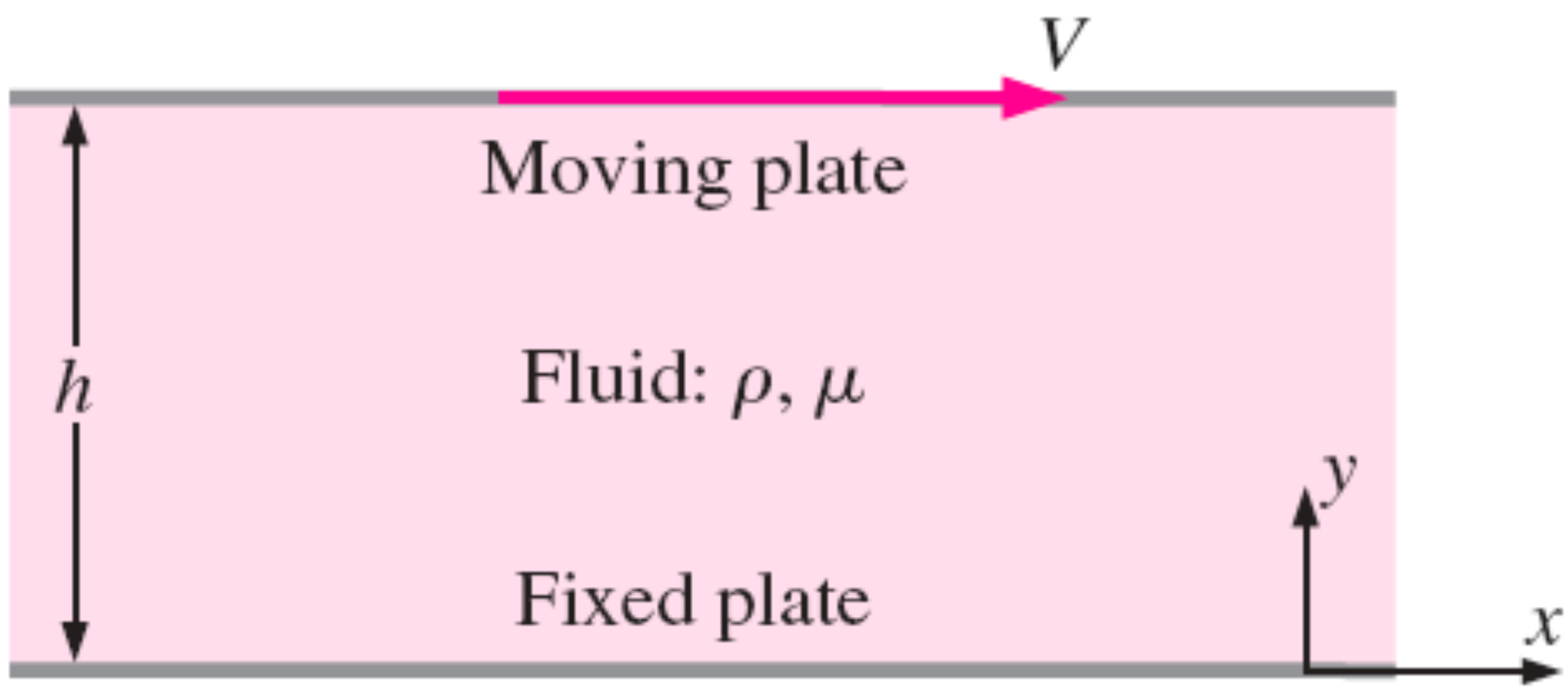
$$\left(\frac{\partial u}{\partial y}\right)_{water} \approx 0$$

As with general interfaces, if surface tension effects are negligible and the surface is nearly flat $P_{water} = P_{air}$

Fully Developed Couette Flow

- For the given geometry and BC's, calculate the velocity and pressure fields, and estimate the shear force per unit area acting on the bottom plate

- Step 1: Geometry, dimensions, and properties



Fully Developed Couette Flow

- Step 2: Assumptions and BC's
  - Assumptions
    1. Plates are infinite in x and z
    2. Flow is steady, $\partial/\partial t = 0$
    3. Parallel flow, $V=0$
    4. Incompressible, Newtonian, laminar, constant properties
    5. No pressure gradient
    6. 2D, $W=0$, $\partial/\partial z = 0$
    7. Gravity acts in the -z direction, $\vec{g} = -g\vec{k}, g_z = -g$
  - Boundary conditions
    - Bottom plate (y=0) : $u=0, v=0, w=0$
    - Top plate (y=h) : $u=V, v=0, w=0$

Fully Developed Couette Flow

- Step 3: Simplify

Note: these numbers refer to the assumptions on the previous slide

Continuity

$$\frac{\partial U}{\partial x} + \cancel{\frac{\partial V}{\partial y}}^{\,3} + \cancel{\frac{\partial W}{\partial z}}^{\,6} = 0$$

$$\boxed{\frac{\partial U}{\partial x} = 0}$$

This means the flow is "*fully developed*" or not changing in the direction of flow

X-momentum

$$\rho\left(\cancel{\frac{\partial U}{\partial t}}^{\,2} + U\cancel{\frac{\partial U}{\partial x}}^{\,\text{Cont.}} + V\cancel{\frac{\partial U}{\partial y}}^{\,3} + W\cancel{\frac{\partial U}{\partial z}}^{\,6}\right) = -\cancel{\frac{\partial P}{\partial x}}^{\,5} + \cancel{\rho g_x}^{\,7} + \mu\left(\cancel{\frac{\partial^2 U}{\partial x^2}}^{\,\text{Cont.}} + \frac{\partial^2 U}{\partial y^2} + \cancel{\frac{\partial^2 U}{\partial z^2}}^{\,6}\right)$$

$$\boxed{\frac{d^2 u}{dy^2} = 0}$$

Fully Developed Couette Flow

- Step 3: Simplify, cont.

Y-momentum

$$\rho\left(\cancelto{2,3}{\frac{\partial V}{\partial t}} + U\cancelto{3}{\frac{\partial V}{\partial x}} + V\cancelto{3}{\frac{\partial V}{\partial y}} + W\cancelto{3,6}{\frac{\partial V}{\partial z}}\right) = -\frac{\partial P}{\partial y} + \cancelto{7}{\rho g_y} + \mu\left(\cancelto{3}{\frac{\partial^2 V}{\partial x^2}} + \cancelto{3}{\frac{\partial^2 V}{\partial y^2}} + \cancelto{3}{\frac{\partial^2 V}{\partial z^2}}\right)$$

$$\frac{\partial p}{\partial y} = 0 \Longrightarrow \boxed{p = p(z)}$$

Z-momentum

$$\rho\left(\cancelto{2,6}{\frac{\partial W}{\partial t}} + U\cancelto{6}{\frac{\partial W}{\partial x}} + V\cancelto{6}{\frac{\partial W}{\partial y}} + W\cancelto{6}{\frac{\partial W}{\partial z}}\right) = -\frac{\partial P}{\partial z} + \rho g_z + \mu\left(\cancelto{6}{\frac{\partial^2 W}{\partial x^2}} + \cancelto{6}{\frac{\partial^2 W}{\partial y^2}} + \cancelto{6}{\frac{\partial^2 W}{\partial z^2}}\right)$$

$$\frac{\partial p}{\partial z} = \rho g_z \Longrightarrow \boxed{\frac{dp}{dz} = -\rho g}$$

Fully Developed Couette Flow

- Step 4: Integrate

X-momentum

$$\frac{d^2u}{dy^2} = 0 \xrightarrow{\text{integrate}} \frac{du}{dy} = C_1 \xrightarrow{\text{integrate}} \boxed{u(y) = C_1 y + C_2}$$

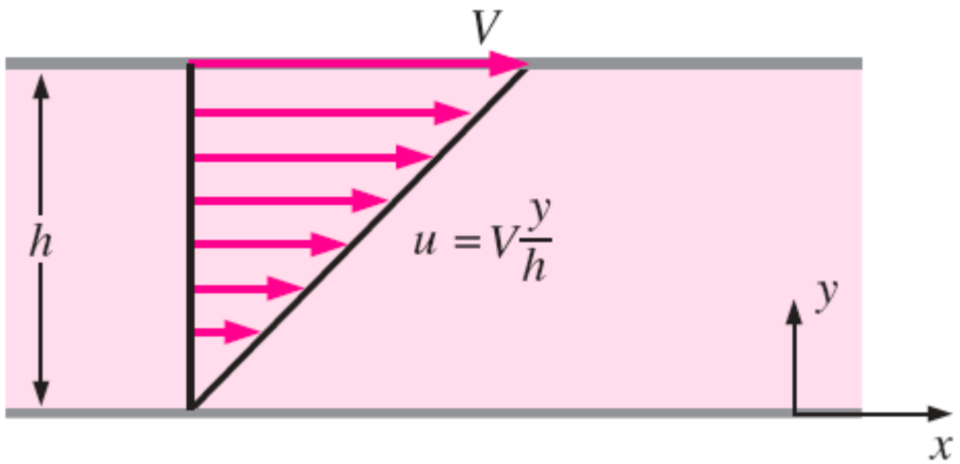
Z-momentum

$$\frac{dp}{dz} = -\rho g \xrightarrow{\text{integrate}} \boxed{p = -\rho g z + C_3}$$

Fully Developed Couette Flow

- Step 5: Apply BC's
  - $y=0$, $u=0=C_1(0) + C_2 \Rightarrow \underline{C_2 = 0}$
  - $y=h$, $u=V=C_1 h \Rightarrow \underline{C_1 = V/h}$
  - This gives

$$\boxed{u(y) = V\frac{y}{h}}$$



  - For pressure, no explicit BC, therefore C_3 can remain an arbitrary constant (recall only ∇P appears in NSE).
    - Let $p = p_0$ at $z = 0$ (C_3 renamed p_0)

$$\boxed{p(z) = p_0 - \rho g z}$$

1. Hydrostatic pressure
2. Pressure acts independently of flow

Fully Developed Couette Flow

- Step 6: Verify solution by back-substituting into differential equations
  - Given the solution (*u*,*v*,*w*)=(Vy/h, 0, 0)

$$\frac{\partial u}{\partial x} = 0, \frac{\partial v}{\partial y} = 0, \frac{\partial w}{\partial z} = 0$$

  - Continuity is satisfied

0 + 0 + 0 = 0

  - X-momentum is satisfied

$$\rho\left(\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + W\frac{\partial U}{\partial z}\right) = -\frac{\partial P}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right)$$

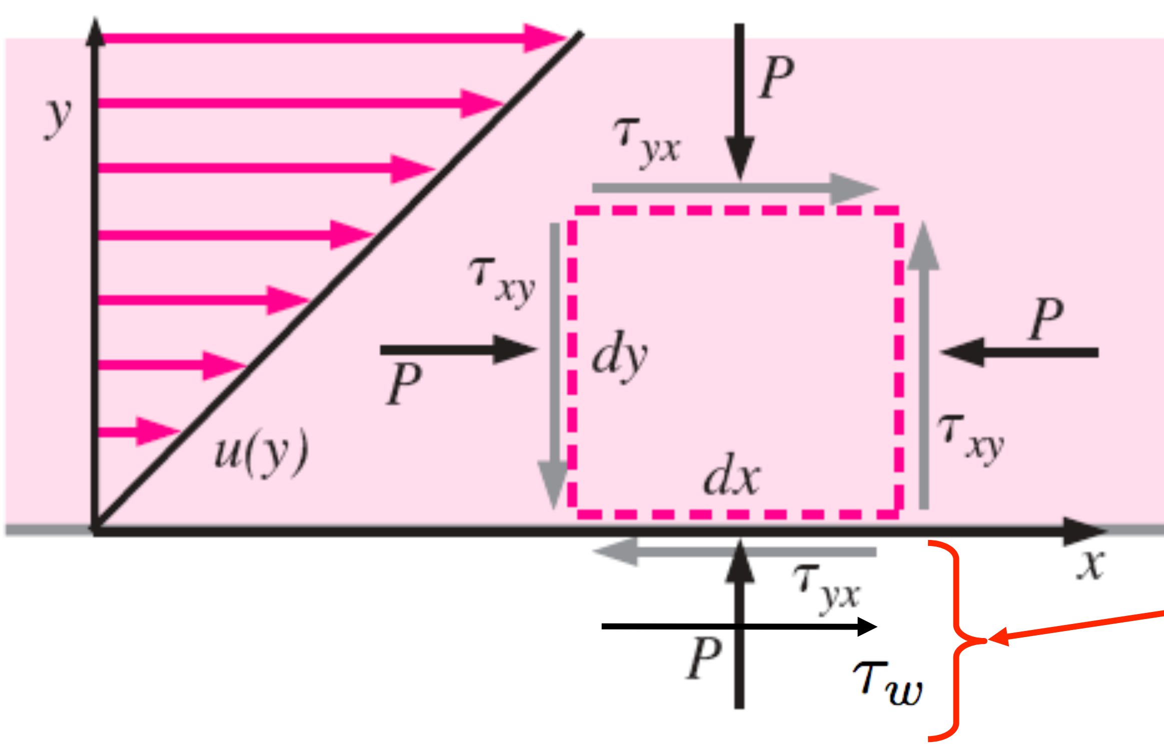
$$\rho\left(0 + V\frac{y}{h}\cdot 0 + 0\cdot V/h + 0\cdot 0\right) = -0 + \rho\cdot 0 + \mu\left(0 + 0 + 0\right)$$

$$0 = 0$$

Fully Developed Couette Flow

- Finally, calculate shear force on bottom plate

$$\tau_{ij} = \begin{pmatrix} 2\mu\frac{\partial U}{\partial x} & \mu\left(\frac{\partial U}{\partial y}+\frac{\partial V}{\partial x}\right) & \mu\left(\frac{\partial U}{\partial z}+\frac{\partial W}{\partial x}\right) \\ \mu\left(\frac{\partial V}{\partial x}+\frac{\partial U}{\partial y}\right) & 2\mu\frac{\partial V}{\partial y} & \mu\left(\frac{\partial V}{\partial z}+\frac{\partial W}{\partial y}\right) \\ \mu\left(\frac{\partial W}{\partial x}+\frac{\partial U}{\partial z}\right) & \mu\left(\frac{\partial W}{\partial y}+\frac{\partial V}{\partial z}\right) & 2\mu\frac{\partial W}{\partial z} \end{pmatrix} = \begin{pmatrix} 0 & \mu\frac{V}{h} & 0 \\ \mu\frac{V}{h} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



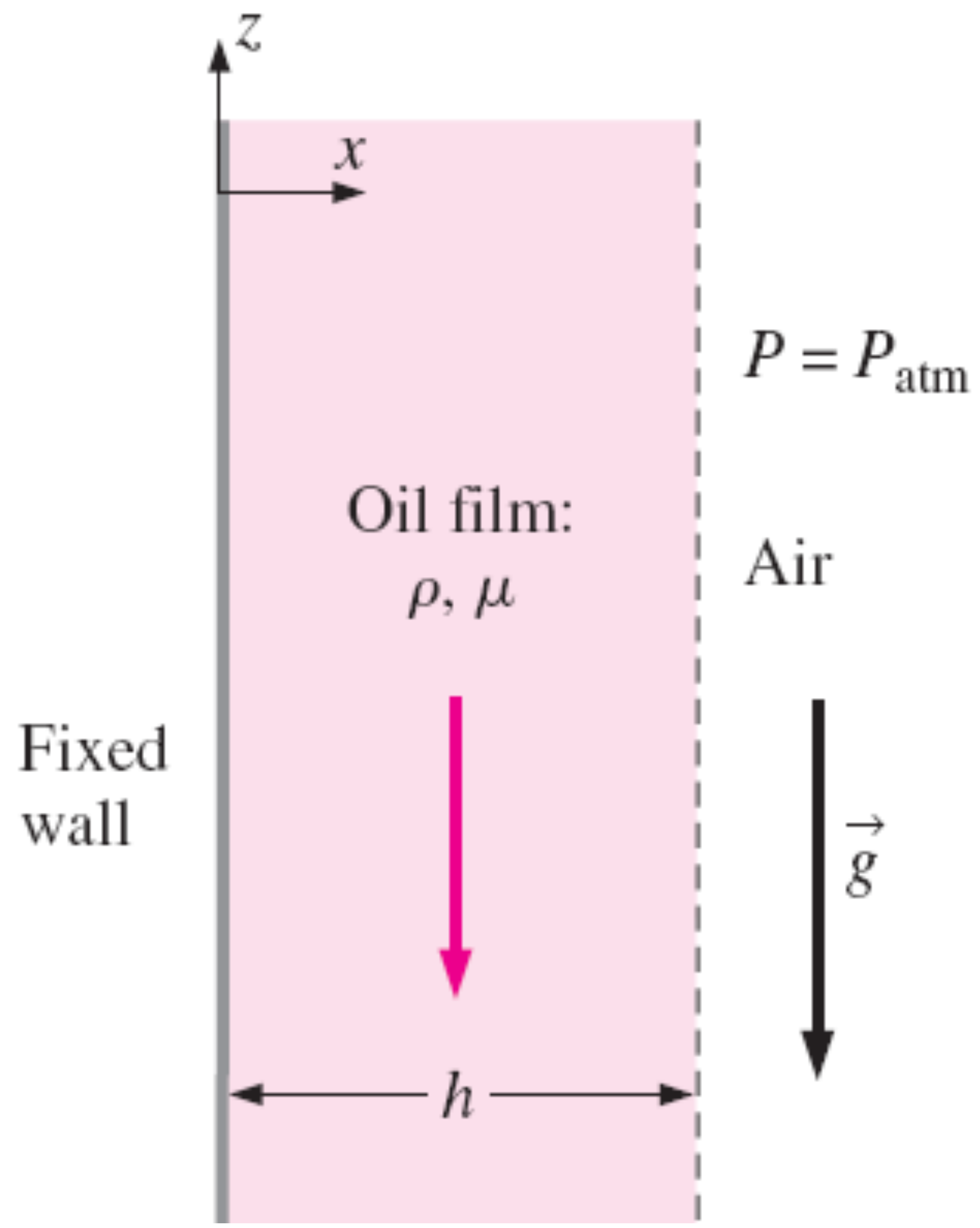
Shear force per unit area acting on the wall

$$\boxed{\frac{\vec{F}}{A} = \tau_w = \mu\frac{V}{h}\hat{i}}$$

Note that τ_w is equal and opposite to the shear stress acting on the fluid τ_{yx} (Newton's third law).

Oil Film Flowing Down a Vertical Wall

- Consider steady, incompressible, parallel, laminar flow of a film of oil falling slowly down an infinite vertical wall. The oil film thickness is *h*, and gravity acts in the negative *z*-direction. There is no applied (forced) pressure driving the flow—the oil falls by gravity alone. Calculate the velocity and pressure fields in the oil film and sketch the normalized velocity profile. You may neglect changes in the hydrostatic pressure of the surrounding air.



Oil Film Flowing Down a Vertical Wall

- Solution:
  - Assumptions
    1. Plates are infinite in y and z
    2. Flow is steady, $\partial/\partial t = 0$
    3. Parallel flow, u=0
    4. Incompressible, Newtonian, laminar, constant properties
    5. $P=P_{atm}$ = constant at free surface and no pressure gradient
    6. 2D, v=0, $\partial/\partial y = 0$
    7. Gravity acts in the -z direction
  - Boundary conditions
    - No slip at wall (x=0) : u=0, v=0, w=0
    - At the free surface ($x = h$), there is negligible shear, means $\partial w/\partial x = 0$ at $x = h$

Oil Film Flowing Down a Vertical Wall

- Step 3: Write out and simplify the differential equations.

$$\underbrace{\cancel{\frac{\partial u}{\partial x}}}_{\text{assumption 3}} + \underbrace{\cancel{\frac{\partial v}{\partial y}}}_{\text{assumption 6}} + \frac{\partial w}{\partial z} = 0 \quad \rightarrow \quad \frac{\partial w}{\partial z} = 0$$

- Therefore, $w = w(x)$ only

- Since $u = v = 0$ everywhere, and gravity does not act in the x- or y-directions, the x- and y-momentum equations are satisfied exactly (in fact all terms are zero in both equations). The z-momentum equation reduces to

Oil Film Flowing Down a Vertical Wall

$$\rho\left(\underbrace{\cancel{\frac{\partial w}{\partial t}}}_{\text{assumption 2}} + \underbrace{u\cancel{\frac{\partial w}{\partial x}}}_{\text{assumption 3}} + \underbrace{v\cancel{\frac{\partial w}{\partial y}}}_{\text{assumption 6}} + \underbrace{w\cancel{\frac{\partial w}{\partial z}}}_{\text{continuity}}\right) = \underbrace{-\cancel{\frac{\partial P}{\partial z}}}_{\text{assumption 5}} + \underbrace{\rho g_z}_{-\rho g}$$

$$+ \mu\left(\frac{\partial^2 w}{\partial x^2} + \underbrace{\cancel{\frac{\partial^2 w}{\partial y^2}}}_{\text{assumption 6}} + \underbrace{\cancel{\frac{\partial^2 w}{\partial z^2}}}_{\text{continuity}}\right) \rightarrow \frac{d^2 w}{dx^2} = \frac{\rho g}{\mu}$$

- Step 4: Solve the differential equations. (Integrating twice)

$$w = \frac{\rho g}{2\mu}x^2 + C_1 x + C_2$$

Oil Film Flowing Down a Vertical Wall

- Step 5: Apply boundary conditions.

Boundary condition (1): $\quad w = 0 + 0 + C_2 = 0 \qquad C_2 = 0$

Boundary condition (2):

$$\left.\frac{dw}{dx}\right)_{x=h} = \frac{\rho g}{\mu} h + C_1 = 0 \quad \rightarrow \quad C_1 = -\frac{\rho g h}{\mu}$$

- Velocity field:

$$w = \frac{\rho g}{2\mu} x^2 - \frac{\rho g}{\mu} h x = \frac{\rho g x}{2\mu}(x - 2h)$$

- Since x < h in the film, w is negative everywhere, as expected (flow is downward). The pressure field is trivial; namely, $P = P_{atm}$ everywhere.

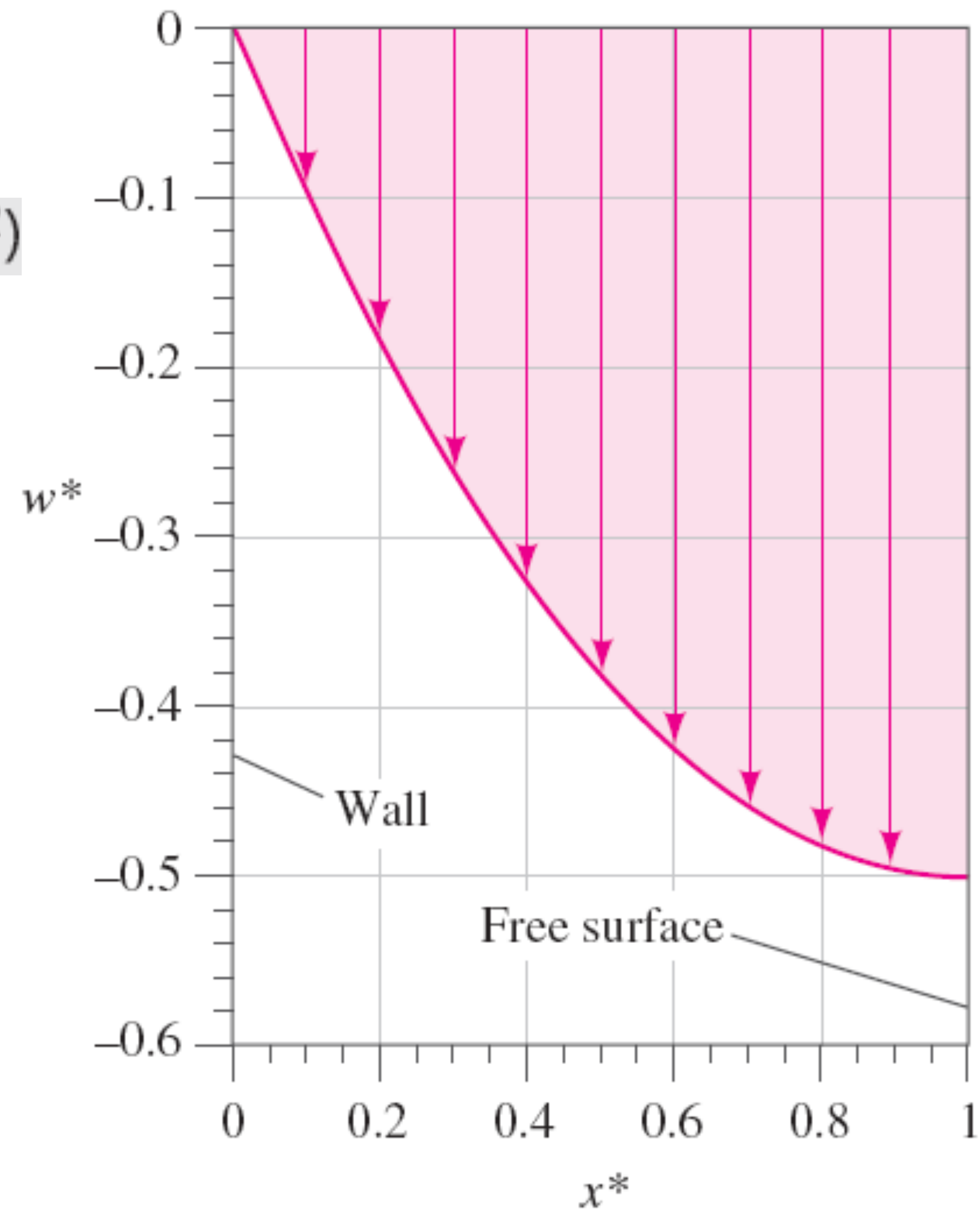
Oil Film Flowing Down a Vertical Wall

- Step 6: Verify the results.

let $x^* = x/h$ and $w^* = w\mu/(\rho g h^2)$

Normalized velocity profile:

$$w^* = \frac{x^*}{2}(x^* - 2)$$

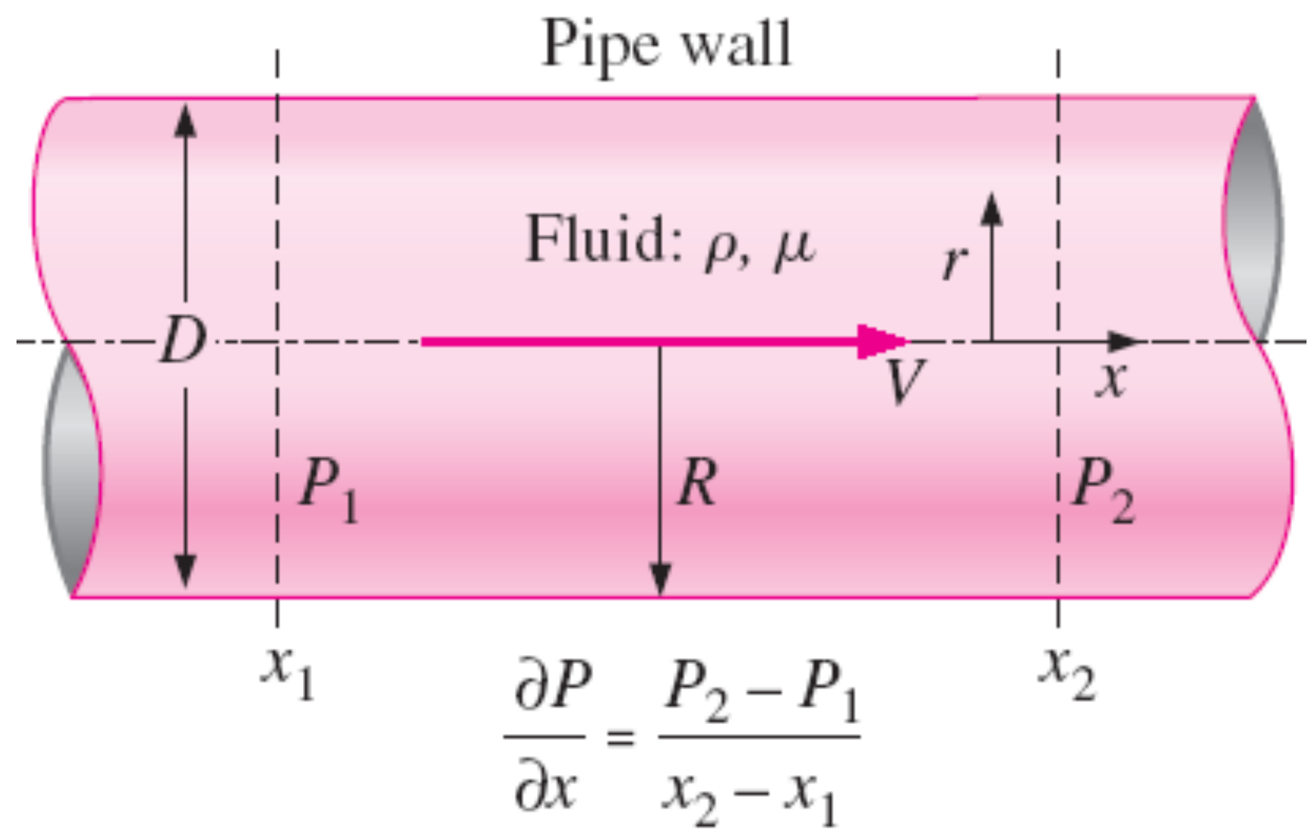


Fully Developed Flow - Poiseuille Flow

- Consider steady, incompressible, laminar flow of a Newtonian fluid in an infinitely long round pipe of radius $R = D/2$. We ignore the effects of gravity. A constant pressure gradient $\partial P/\partial x$ is applied in the x-direction,

$$\frac{\partial P}{\partial x} = \frac{P_2 - P_1}{x_2 - x_1} = \text{constant}$$

where x_1 and x_2 are two arbitrary locations along the x-axis, and P_1 and P_2 are the pressures at those two locations.



Fully Developed Flow - Poiseuille Flow

- Derive an expression for the velocity field inside the pipe and estimate the viscous shear force per unit surface area acting on the pipe wall.

Solution:

Assumptions

1. The pipe is infinitely long in the *x*-direction.
2. Flow is steady, $\partial/\partial t = 0$
3. Parallel flow, u_r = zero.
4. Incompressible, Newtonian, laminar, constant properties
5. A constant-pressure gradient is applied in the *x*-direction
6. The velocity field is axisymmetric with no swirl, implying that $u_\theta = 0$ and all partial derivatives with respect to θ are zero.
7. Ignore the effects of gravity.

Fully Developed Flow - Poiseuille Flow

Solution:

- Step 2: List boundary conditions.

(1) at r = R, $\vec{V} = 0.$

(2) at r = 0, du/dr = 0.

- Step 3: Write out and simplify the differential equations.

$$\underbrace{\cancel{\frac{1}{r}\frac{\partial(ru_r)}{\partial r}}}_{\text{assumption 3}} + \underbrace{\cancel{\frac{1}{r}\frac{\partial(u_\theta)}{\partial \theta}}}_{\text{assumption 6}} + \frac{\partial u}{\partial x} = 0 \qquad \rightarrow \qquad \frac{\partial u}{\partial x} = 0$$

Fully Developed Flow - Poiseuille Flow

Solution:

Result of continuity: $u = u(r)$ only

We now simplify the axial momentum equation

$$\rho\left(\underbrace{\cancel{\frac{\partial u}{\partial t}}}_{\text{assumption 2}} + \underbrace{\cancel{u_r \frac{\partial u}{\partial r}}}_{\text{assumption 3}} + \underbrace{\cancel{\frac{u_\theta}{r}\frac{\partial u}{\partial \theta}}}_{\text{assumption 6}} + \underbrace{\cancel{u\frac{\partial u}{\partial x}}}_{\text{continuity}}\right)$$

$$= -\frac{\partial P}{\partial x} + \underbrace{\cancel{\rho g_x}}_{\text{assumption 7}} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \underbrace{\cancel{\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}}}_{\text{assumption 6}} + \underbrace{\cancel{\frac{\partial^2 u}{\partial x^2}}}_{\text{continuity}}\right)$$

Or

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = \frac{1}{\mu}\frac{\partial P}{\partial x}$$

Fully Developed Flow - Poiseuille Flow

Solution:

In similar fashion, every term in the *r*-momentum equation

r-momentum: $$\frac{\partial P}{\partial r} = 0$$

Result of r-momentum: $P = P(x)$ only

Finally, all terms of the θ-component of the Navier–Stokes equation go to zero.

- Step 4: Solve the differential equations.

After multiplying both sides of equation by *r*, we integrate once to obtain

$$r\frac{du}{dr} = \frac{r^2}{2\mu}\frac{dP}{dx} + C_1$$

Fully Developed Flow - Poiseuille Flow

Solution:
Dividing both sides by *r*, we integrate again to get

$$u = \frac{r^2}{4\mu}\frac{dP}{dx} + C_1 \ln r + C_2$$

- Step 5: Apply boundary conditions

Boundary condition (2): $0 = 0 + C_1 \quad \rightarrow \quad C_1 = 0$

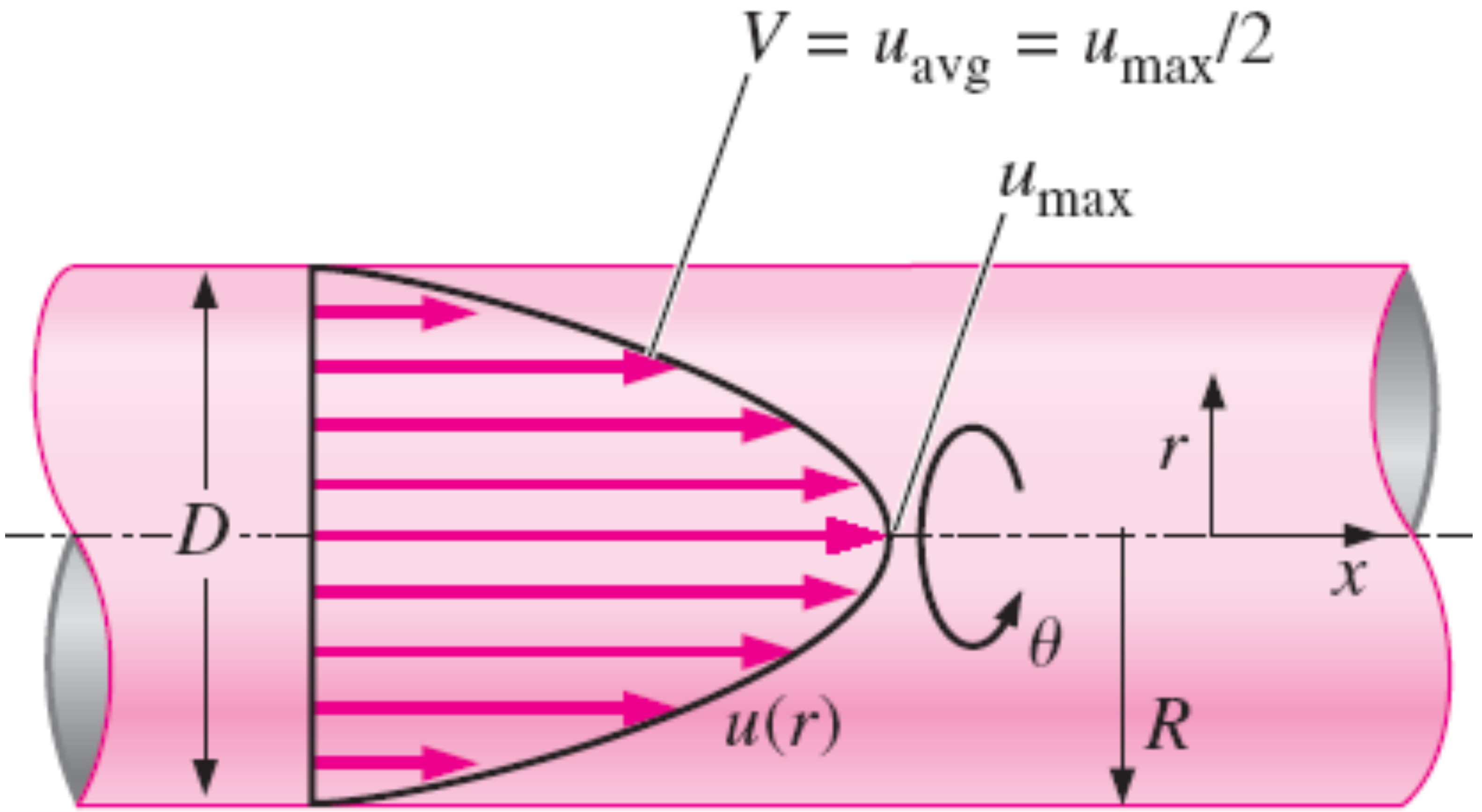
Boundary condition (1): $u = \frac{R^2}{4\mu}\frac{dP}{dx} + 0 + C_2 = 0$

$$\rightarrow \quad C_2 = -\frac{R^2}{4\mu}\frac{dP}{dx}$$

Fully Developed Flow - Poiseuille Flow

Solution:
Finally, the result becomes

$$u = \frac{1}{4\mu}\frac{dP}{dx}(r^2 - R^2)$$



- Step 6: Verify the results

  You can verify that all the differential equations and boundary conditions are satisfied.