1) Si calcoli la lunghezza della curva espressa in coordinate polari di

$$\gamma(\theta) = \big((1+\cos\theta)\cos\theta,\ (1+\cos\theta)\sin\theta\big) \qquad \theta\in[0,2\pi]$$

<u>Soluzione</u>

applicando la formula

$$L = \int_0^{2\pi} \sqrt{(\rho(\theta))^2 + (\rho'(\theta))^2}\, d\theta$$

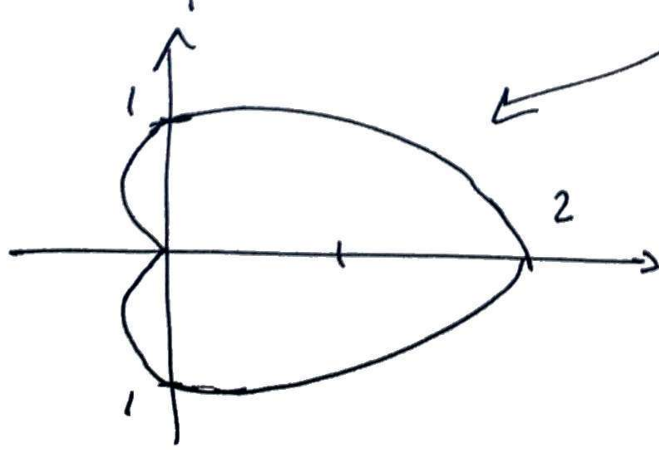
con $\rho(\theta) = 1+\cos\theta$ si ottiene

$$L(\gamma) = \int_0^{2\pi} \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{2+2\cos\theta}\, d\theta = \int_0^{2\pi} 2\left|\cos\left(\frac{\theta}{2}\right)\right| d\theta$$

$$= 8$$

La curva in questione è detta <u>CARDIOIDE</u>



2) Calcolare

$$\int_\varphi xy\,dx + xy\,dy$$

$$\varphi : [0, \tfrac{\pi}{2}] \to \mathbb{R}^2$$
$$t \longmapsto \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

SOLUZIONE

$$\int_\varphi xy\,dx + xy\,dy = \int_0^{\frac{\pi}{2}} \Big(\underbrace{-\cos t \sin^2 t}_{-\frac{\sin^3 t}{3}} + \underbrace{\cos^2 t \sin t}_{-\frac{\cos^3 t}{3}}\Big)\,dt$$

← queste sono le primitive dei due addendi.

$$= -\left(\frac{\sin^3 t + \cos^3 t}{3}\right)\Big]_0^{\frac{\pi}{2}}$$

$$= -\frac{1}{3}(1+0) + \frac{1}{3}(0+1) = 0.$$

3) Calcolare
$$\oint_\Gamma (x^3 - xy^3)\,dx + (y^2 - 2xy)\,dy$$

dove Γ è il perimetro del quadrato $\overset{Q}{\overset{\shortparallel}{[0,2]\times[0,2]}}$

SOLUZIONE

Non conviene calcolare direttamente l'integrale curvilineo. Dovrei spezzarlo in 4 parti, ovvero lungo i bordi del quadrato Q.
Usiamo Gauss-Green "al contrario":

$$\oint_\Gamma \underbrace{(x^3 - xy^3)}_{A(x,y)}dx + \underbrace{(y^2 - 2xy)}_{B(x,y)}\,dy = \iint_Q \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dx\,dy$$

$$= \iint_Q \left(\frac{\partial (y^2 - 2xy)}{\partial x} - \frac{\partial (x^2 - xy^3)}{\partial y}\right) dx\,dy$$

$$= \iint_Q (-2y + 3xy^2)\,dx\,dy = \int_0^2 \left(\int_0^2 -2y + 3xy^2\,dy\right) dx$$

$$= \int_0^2 \left[-y^2 + xy^3\right]_0^2 dx = \int_0^2 -4 + 8x\,dx = 8$$

4) Calcolare

$$\iint_D \frac{xy \exp\left(\frac{y^2}{x^2+y^2}\right)}{(x^2+y^2)^{3/2}} \, dx\, dy$$

$D = \{(x,y) : 1 \leq x^2+y^2 \leq 4, \quad 0 \leq y \leq x\}$

SOLUZIONE

COORDINATE POLARI

$$\tilde{D} = \{(\rho,\theta) \in [0,+\infty) \times (-\pi,\pi) : 1 \leq \rho \leq 2, \; 0 \leq \theta \leq \pi/4\}$$

$$\iint_D f(x,y)\, dx\, dy = \int_1^2 \int_0^{\pi/4} \frac{\rho^2 \cos\theta \sin\theta \, e^{\sin^2\theta}}{(\rho^2)^{3/2}} \rho \, d\rho \, d\theta$$

$$= \left(\int_1^2 d\rho\right)\left(\int_0^{\pi/4} \cos\theta \sin\theta \, e^{\sin^2\theta} \, d\theta\right)$$

$$= \left(\rho\Big|_1^2\right)\left(\frac{e^{\sin^2\theta}}{2}\Big|_0^{\pi/4}\right) = \frac{e^{1/2}-1}{2}$$

$$\frac{y^2}{x^2+y^2} = \frac{\cancel{\rho^2} \sin^2\theta}{\cancel{\rho^2}(\sin^2\theta+\cos^2\theta)} = \sin^2\theta.$$

ES 5) Si consideri la 1-forma differenziabile

$$\omega(x,y) = \frac{(3x^2-y^2)(x^2+y^2)}{x^2 y}\,dx + \frac{(3y^2-x^2)(x^2+y^2)}{x y^2}\,dy$$

e sia $A = \{(x,y) \in \mathbb{R}^2 : x>0, y>0\}$

i) Mostrare che la forma ω è chiusa in A

ii) Discutere l'esattezza di ω in A ed in caso trovarne le primitive

iii) Calcolare $\int_\gamma \omega$ dove $\gamma(t) = (t+\cos^2 t, 1+\sin^2 t)$

$t \in [0, \frac{\pi}{2}]$.

i) CHIUSA : $d\omega = 0$

$$\frac{\partial}{\partial y}\left(\frac{(3x^2-y^2)(x^2+y^2)}{x^2 y}\right) = \dots = \frac{2x^2y^2 - 3y^4 - 3x^4}{x^2y^2}$$

$$\frac{\partial}{\partial x}\left(\frac{(3y^2-x^2)(x^2+y^2)}{x y^2}\right) = \dots = \frac{-3x^4+2x^2y^2-3y^4}{x^2y^2}$$

$\Rightarrow \omega$ è chiusa.

ii) A è semplicemente connesso, (senza buchi)
ω chiusa $\Rightarrow \omega$ esatta.

Trovare una primitiva, i.e. $f: \omega = df$.

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$\Rightarrow \frac{\partial f}{\partial x} = \frac{(3x^2-y^2)(x^2+y^2)}{x^2 y} = \underbrace{3\frac{x^2}{y} - \frac{y^3}{x^2} + 2y}$$

$$\Rightarrow f(x,y) = \int \quad dx = \frac{x^3}{y} + \frac{y^3}{x} + 2xy + h(y)$$

Deriviamo $\frac{\partial}{\partial y}$

$$\Rightarrow -\frac{x^3}{y^2} + \frac{3y^2}{x} + 2x + h'(y) = \frac{(3y^2-x)(x^2+y^2)}{xy^2}$$

$$\Longleftrightarrow h'(y) = 0 \quad \Rightarrow h = c \in \mathbb{R}$$

$$\Rightarrow f(x,y) = \frac{x^3}{y} + \frac{y^3}{x} + 2xy + c \quad , c \in \mathbb{R}.$$

$$= \frac{(x^2+y^2)^2}{xy} + c.$$

iii) $\gamma(t) = (t + \cos^2 t, 1 + \sin^2 t) \quad , t \in [0, \pi/2)$

$$\int_\gamma \omega = f(\gamma(\tfrac{\pi}{2})) - f(\gamma(0)) = f(\tfrac{\pi}{2}, 2) - f(1,1)$$

$$= \left(\tfrac{\pi^2}{4}+4\right)^2 / \pi - 4$$