Algorithms with Untrusted Predictions

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Fundamentals of algorithms
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Dealing with Uncertainty

Machine-learned systems

- Good on average
- Not robust
- No guarantees

Classic (online) algorithms

- Provably good in the worst case
- Often overly pessimistic

A New Paradigm in Algorithmics

Machine-learned systems

- Good on average
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Classic (online) algorithms

- Provably good in the worst case
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Learning-augmented algorithms

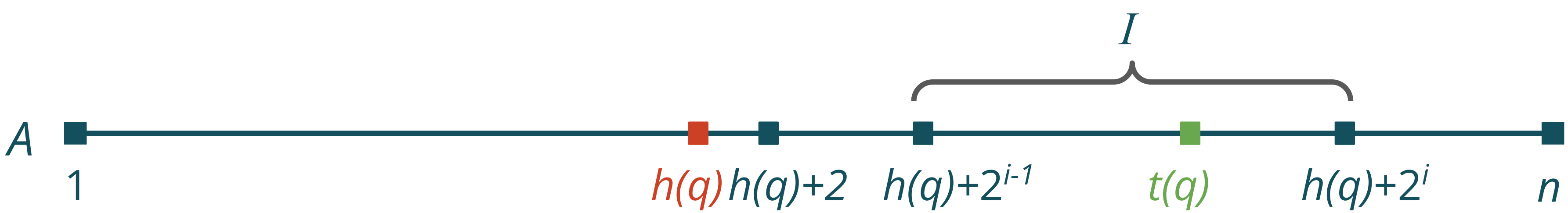
- Do not assume a worst-case scenario
- Strong performance guarantees both in theory and in practice
- No assumptions on the type of errors

Warmup: Predictive Binary Search*

A **-** array of length *n*

q **-** query

h(q) **-** predicted location of *q* in *A*



Strategy: probe $h(q)$. If $q > A[h(q)]$ probe $h(q)+2^i$ until finding the right interval I for q.

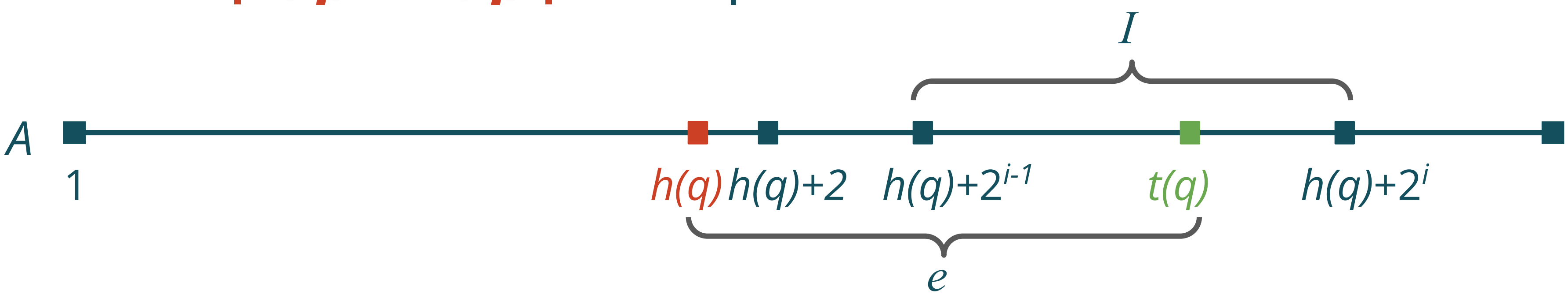
Then apply binary search to I.

*Lykouris and Vassilvitskii: *Competitive Caching with Machine Learned Advice,* ICML, 2018

Warmup: Predictive Binary Search

t(q) - the true position of *q* in *A*

***e* = |*t(q)* – *h(q)*|** - the prediction error



The cost of Predictive Binary Search is at most 2log(*e*) which is bounded by 2log(*n*).

Even relatively weak predictions make Binary Search faster!

The Ski Rental Problem

 **1** **-** cost of renting per day

 ***b*** **-** cost of buying the skis

 ***x*** **-** number of skiing days

Problem: to buy or not to buy?

2-Competitive Algorithm

Competitive ratio: worst-case cost of ALG / best case cost

2-competitive strategy for Ski Rental:

- rent for the first $b - 1$ days
- buy on day b

No deterministic algorithm can do better!

Ski Rental with Prediction

What if we have a prediction of the number of skiing days?

y **-** predicted number of skiing days

x **-** true number of skiing days

$\eta = |x-y|$ **-** prediction error

Ski Rental with Prediction - Naïve

A naïve strategy is to blindly follow the prediction:

- if $y \geq b$ then buy on day 1;
- otherwise rent every day.

If the prediction is good, this has optimal cost OPT.

In case of large prediction error, though, the cost can be unbounded: in general the cost is at most OPT + η .

Consistency and Robustness[*]

Let c(η) be the competitive ratio of a learning-augmented algorithm A.

A is α-robust if c(η) ≤ α for all η

A is β-consistent if c(0) = β

Consistency measures how well A does with perfect predictions

Robustness measures how well A does with terrible predictions

[*]Lykouris and Vassilvitskii: *Competitive Caching with Machine Learned Advice,* 2018

Ski Rental with Untrusted Prediction

Naïve Predictive Ski Rental: 1-consistent, not robust (as $c(\eta) = (OPT + \eta) / OPT = 1 + \eta/OPT$.

A more judicious strategy* depending on $\lambda \in (0,1)$ is:

- if $y \geq b$ then buy on day $\lceil \lambda b \rceil$;
- otherwise buy on day $\lceil b/\lambda \rceil$.

The parameter λ determines how much we trust the prediction.

This algorithm is $(1+1/\lambda)$-robust and $(1+\lambda)$-consistent.

*Kumar, Purohit, Svitkina: *Improving Online Algorithms via ML Predictions*, NeurIPS, 2018

Ski Rental with Untrusted Prediction

Theorem: the competitive ratio of the algorithm is at most $\min\{\, (1+\lambda)/\lambda \,,\, (1+\lambda)+\eta/(1-\lambda)OPT \,\}$, $\lambda \in (0,1)$.

Ski Rental with Untrusted Prediction

Theorem: the competitive ratio of the algorithm is at most min{ $(1+\lambda)/\lambda$, $(1+\lambda)+\eta/((1-\lambda)\mathrm{OPT})$ }, $\lambda \in (0,1)$.

Proof.

When y ≥ b and $x \geq \lceil \lambda b \rceil$, ALG=$b+\lceil \lambda b \rceil - 1$.

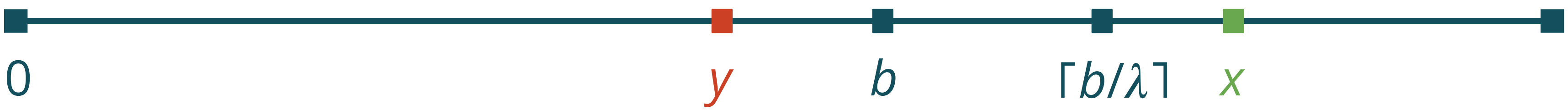
The worst competitive ratio is when $x = \lceil \lambda b \rceil$, for which OPT=$\lceil \lambda b \rceil$. In this case,

$$\mathrm{ALG} = b + \lceil \lambda b \rceil - 1 \leq b + \lambda b \leq ((1+\lambda)/\lambda)\lceil \lambda b \rceil = ((1+\lambda)/\lambda)\mathrm{OPT}$$

Ski Rental with Untrusted Prediction

Theorem: the competitive ratio of the algorithm is at most $\min\{ (1+\lambda)/\lambda, (1+\lambda)+\eta/((1-\lambda)\mathrm{OPT}) \}$, $\lambda \in (0,1)$.

Proof.



For $y < b$, the worst case is when $x = \lceil b/\lambda \rceil$, as $\mathrm{OPT}=b$ while $\mathrm{ALG} = b+\lceil b/\lambda \rceil-1 \leq b+b/\lambda = ((1+\lambda)/\lambda)\mathrm{OPT}$.

In both cases, the competitive ratio is given by

$((1+\lambda)/\lambda)\mathrm{OPT} / \mathrm{OPT} = (1+\lambda)/\lambda$

Ski Rental with Untrusted Prediction

Theorem: the competitive ratio of the algorithm is at most $\min\{ (1+\lambda)/\lambda , (1+\lambda)+\eta/((1-\lambda)OPT) \}$, $\lambda \in (0,1)$.

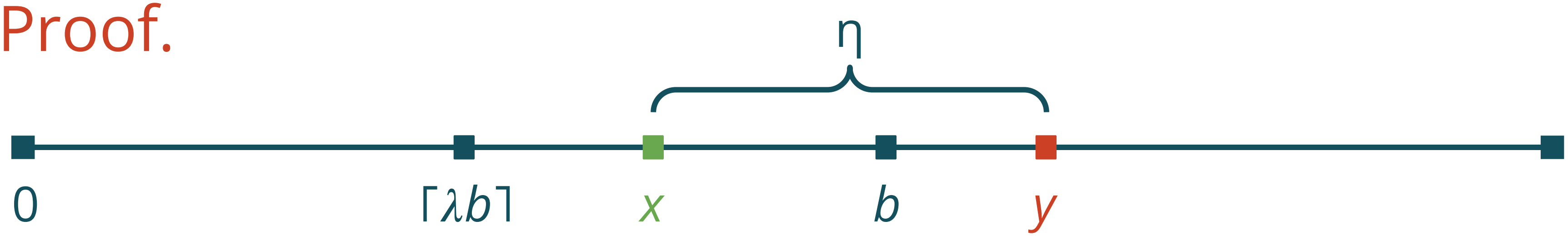
Proof.

For $y \geq b$, for all $x < \lceil \lambda b \rceil$, it holds ALG = OPT = x.

Ski Rental with Untrusted Prediction

Theorem: the competitive ratio of the algorithm is at most $\min\{ (1+\lambda)/\lambda\ ,\ (1+\lambda)+\eta/((1-\lambda)OPT)\ \}$, $\lambda \in (0,1)$.

Proof.



When $\lceil \lambda b \rceil \leq x \leq b$, it holds $b \leq y \leq OPT + \eta$

$ALG = b + \lceil \lambda b \rceil - 1 \leq (1+\lambda)b \leq (1+\lambda)(OPT+\eta)$

Ski Rental with Untrusted Prediction

Theorem: the competitive ratio of the algorithm is at most min{ $(1+\lambda)/\lambda$, $(1+\lambda)+\eta/(1-\lambda)\mathrm{OPT}$ }, $\lambda \in (0,1)$.

Proof.

When $y \geq b$ and $x \geq b$, OPT = b (buying on day 1) while

ALG = $b+\lceil \lambda b \rceil-1 \leq (1+\lambda)b = (1+\lambda)\mathrm{OPT}$

Ski Rental with Untrusted Prediction

Theorem: the competitive ratio of the algorithm is at most min{ $(1+\lambda)/\lambda$, $(1+\lambda)+\eta/((1-\lambda)OPT)$ }, $\lambda \in (0,1)$.

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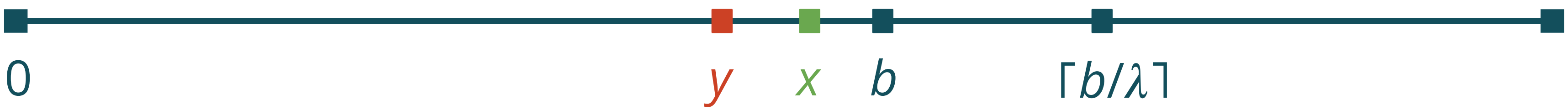

For $y < b$, for all $x \leq b$, it holds ALG = OPT = x

Ski Rental with Untrusted Prediction

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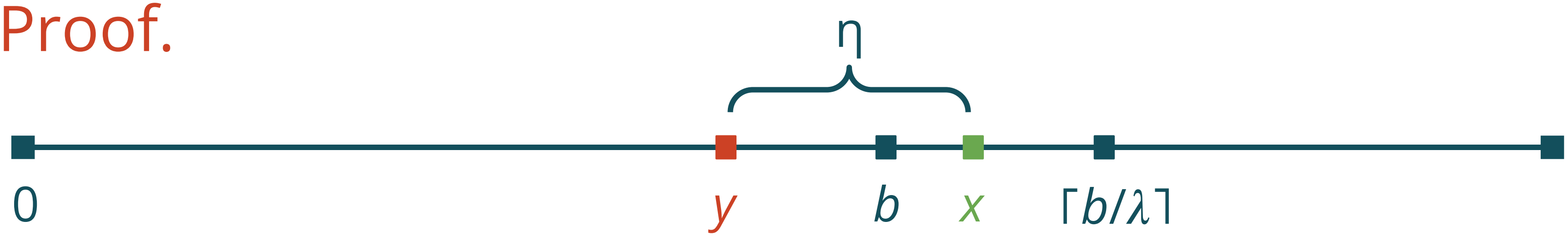


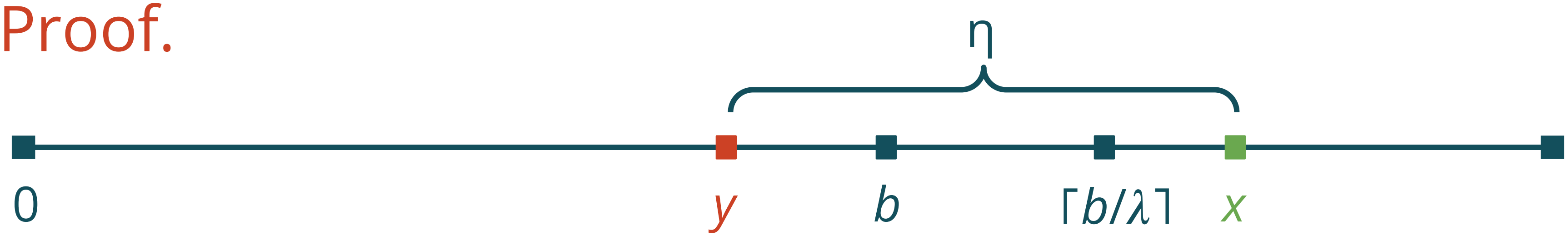
For $y < b$ and $b < x < \lceil b/\lambda \rceil$, we have

$ALG = x \leq y + \eta < b + \eta = OPT + \eta$

Ski Rental with Untrusted Prediction

Theorem: the competitive ratio of the algorithm is at most $\min\{ (1+\lambda)/\lambda \,,\, (1+\lambda)+\eta/((1-\lambda)OPT) \}$, $\lambda \in (0,1)$.

Proof.



For $y < b$ and $x \geq \lceil b/\lambda \rceil$, note that $OPT = b$,

$\eta = x - y > b/\lambda - b = (1-\lambda)b/\lambda$ and thus we have

$ALG = b + \lceil b/\lambda \rceil - 1 \leq b + b/\lambda < b + \eta/(1-\lambda) = OPT + \eta/(1-\lambda)$

From the previous cases, $ALG \leq (1+\lambda)OPT + \eta/(1-\lambda)$.

Ski Rental with Untrusted Prediction

Corollary: Ski Rental with Untrusted Predictions is $(1+\lambda)$-consistent and $(1+1/\lambda)$-robust, for any $\lambda \in (0,1)$.

Algorithms with Untrusted Predictions

- Various flavours of Rent-Or-Buy
- Various flavours of Caching
- Various flavours of Scheduling
- Frequency Estimation
- Online Facility Location
- Bloom Filters
- Online Transportation Problems (work in progress...)
- ...