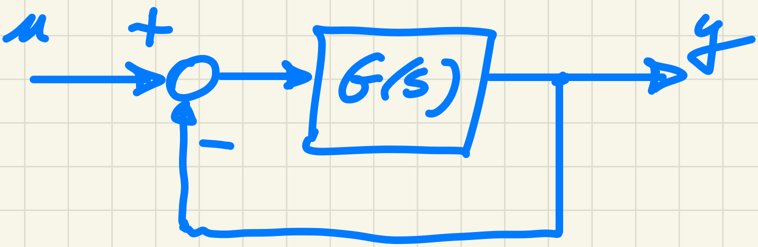
Fondamenti di Automatica

Margini di fase e guadagno

Progetto di regolatore

$$G(s) = 50 \frac{(1+20s)}{(1+200s)(1+10s)\left(1+\frac{s}{200}\right)}$$



$\varphi_m = ?$

$k_M = ?$

$\omega_{p_1} = \frac{1}{200} = 5 \cdot 10^{-3}$ $\quad \omega_{z_1} = \frac{1}{20} = 5 \cdot 10^{-2}$ $\quad \omega_{p_2} = 10^{-1}$

$\omega_{p_3} = 200 = 2 \cdot 10^{2}$

$g = 0$

$\mu_G = 50 \longrightarrow \mu_G|_{dB} = 20 \log_{10} 50 \simeq 34 \, dB$

$L \equiv G$

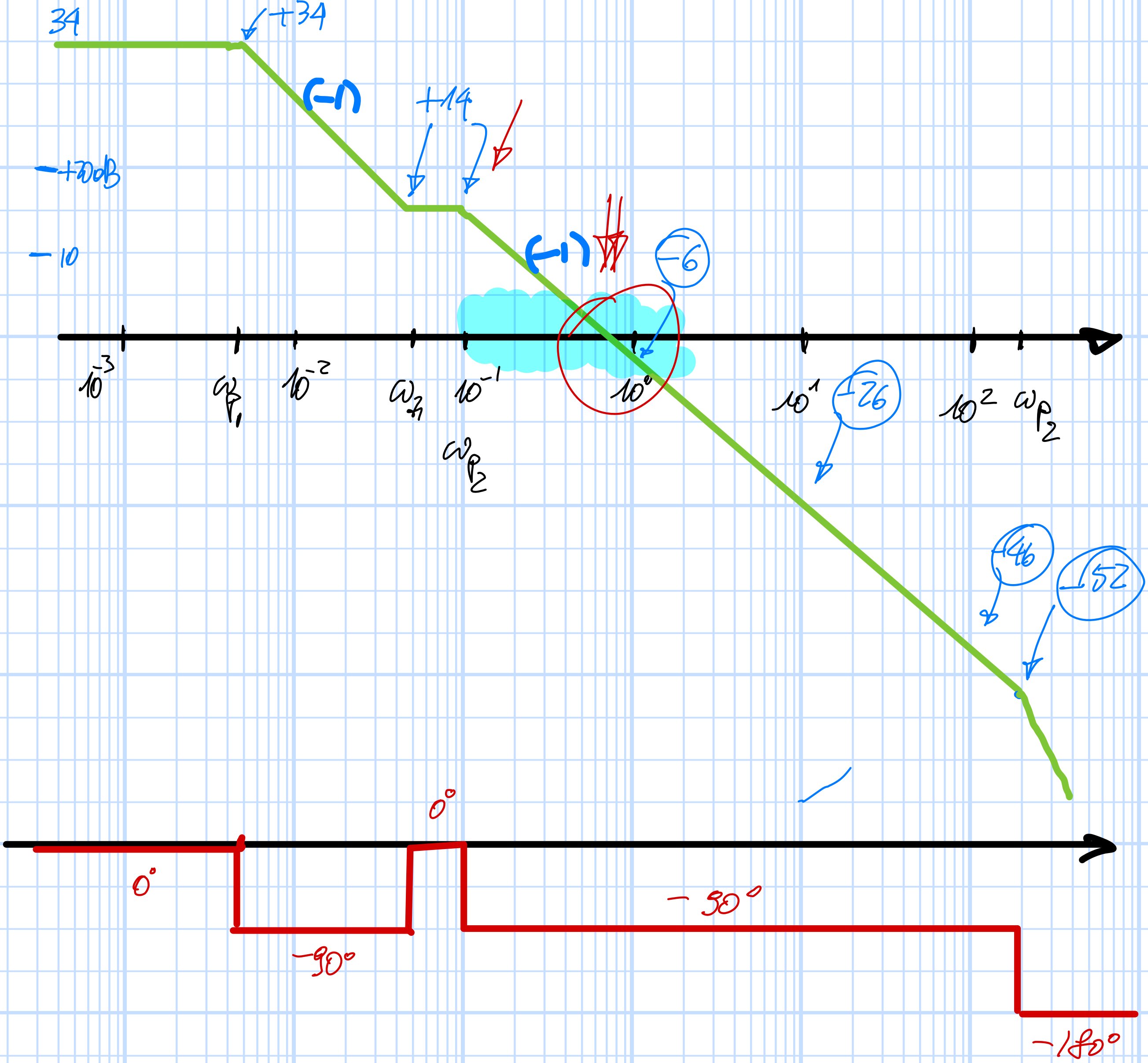
$\varphi_m \longrightarrow \varphi_c = \arg L(j\omega_c)$

$\omega_c : |L(j\omega_c)| = 1$

$\varphi_m = \varphi_c - (-180°) = \varphi_c + 180°$

$k_m = ?$ $\quad \omega_\pi \quad \arg L(j\omega_\pi) = -180°$

$X_\pi = |L(j\omega_\pi)| \longrightarrow k_m = \frac{1}{X_\pi}$



34
+34
(-1)
+14
-+20dB
- 10
(-1)
-6
10^{-3}
ω_{p_1}
10^{-2}
ω_{z_1}
10^{-1}
10^{0}
10^{1}
-26
10^{2}
ω_{p_2}
ω_{p_2}
-46
-52
0°
0°
-90°
-90°
-180°

$\omega_c \in [0{,}1 \quad 1]$

$\varphi_c \in [-90^\circ \quad 0^\circ]$

$\omega_\pi = ?$

$$\left.\begin{aligned} &\varphi = \arg L(j\omega) \xrightarrow[\omega \to +\infty]{} -180^\circ \\ &|L(j\omega)| \xrightarrow[\omega \to +\infty]{} 0 \end{aligned}\right\} k_M \to +\infty$$

$\omega_c = ?$

dal diagr. asintotico

$$y = +14 - 20\left[\log_{10}\omega_c - \log_{10}\frac{1}{10}\right]$$

$$0 = 14 - 20\log_{10}\omega_c + \underbrace{20\log_{10}\frac{1}{10}}_{-20\log_{10}10}$$

$$+6 = -20\log_{10}\omega_c$$

$$\omega_c = 10^{\left(-\frac{3}{10}\right)} \simeq 0{,}5012 \text{ rad/s}$$

$$|L(j\omega_c)| = 1$$

$$\frac{50\sqrt{\overset{t}{1+400\,\omega_c^2}}}{\sqrt{1+40000\,\omega_c^2}\cdot\sqrt{1+100\,\omega_c^2}\cdot\sqrt{1+\frac{\omega_c^2}{40000}}}=1$$

$\omega_c^2 \triangleq t$

$$2500\left(1+4\cdot10^2 t\right)=\left(1+4\cdot10^4 t\right)\left(1+10^2 t\right)\cdot\left(1+0{,}75\cdot10^{-4} t\right)$$

$\omega_c \simeq 0{,}492$ rad/s

$$\varphi_c = \arg L(j\omega_c) = \arg 50 + \arg\left(1+20j\omega_c\right) + -\arg\left(1+200j\omega_c\right) + -\arg\left(1+10j\omega_c\right) + -\arg\left(1+\frac{1}{200}j\omega_c\right) \simeq -84^\circ$$

$$\varphi_M = -84^\circ + 180^\circ = +96^\circ$$

$$L(s) = G(s)\, e^{-s\tau}$$

$\tau_{max} = ?$

$$\varphi_m = \varphi_m' - \tau\omega_c \cdot \frac{180}{\pi} = 0$$

$+56 \qquad 0{,}5$

$\tau_{max} \simeq 3{,}35\ s$

$$R_1(s) = \frac{\mu_R}{s}$$

$$G(s) = 50 \frac{(1+20s)}{(1+200s)(1+10s)(1+\frac{1}{200}s)}$$

μ_G

$g=1$

$+34\,dB$ per $\omega = 1$ $\qquad [0 ; \frac{1}{200}]$

M_{dB} $\quad \omega = 10^{-3}$ $\Leftarrow$ 3 decadi a sx

retta $-20\,dB/decade$

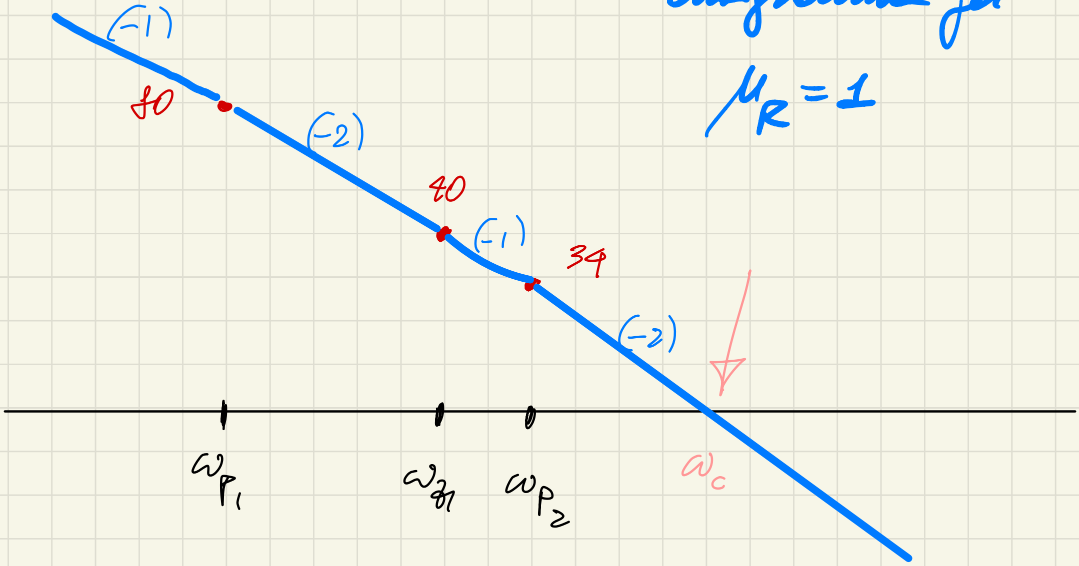
$M_{dB} = +34 + 3 \cdot 20 = +94\,dB$

$\omega_{p_1} \rightarrow \quad y = +94 - 20\left[\log_{10}(\omega_{p_1}) - \log_{10}(10^{-3})\right]$

$= +80$

$\omega_{z_1} \rightarrow \quad y = +40\,dB \qquad \omega_{p_2} \rightarrow y = 34\,dB$

diagramma per $\mu_R = 1$



ω_c:

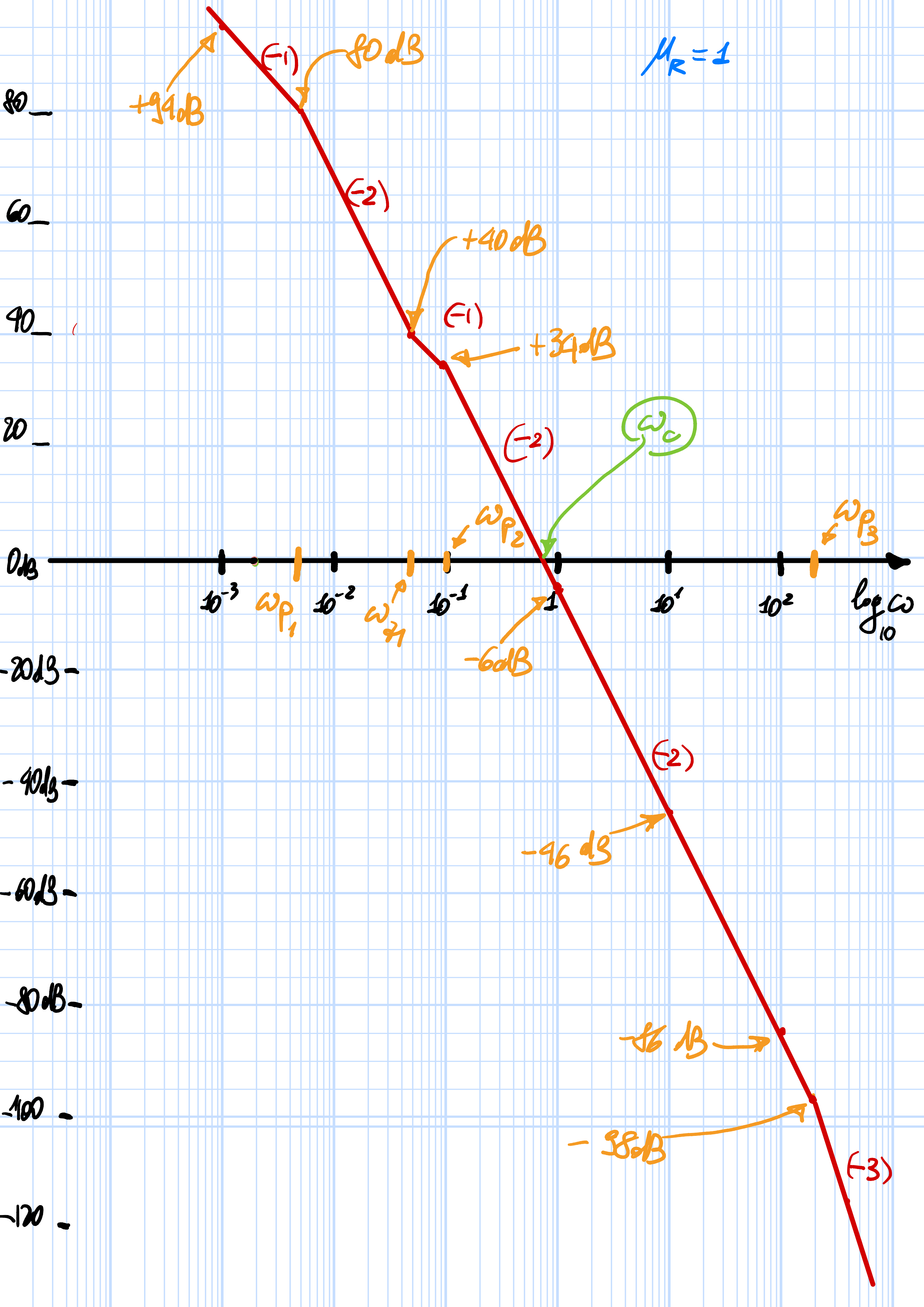
$$y = +34 - 40\left[\log_{10}(\omega_c) - \log_{10}\frac{1}{10}\right]$$

$$= +34 - 40\log_{10}\omega_c - 40$$

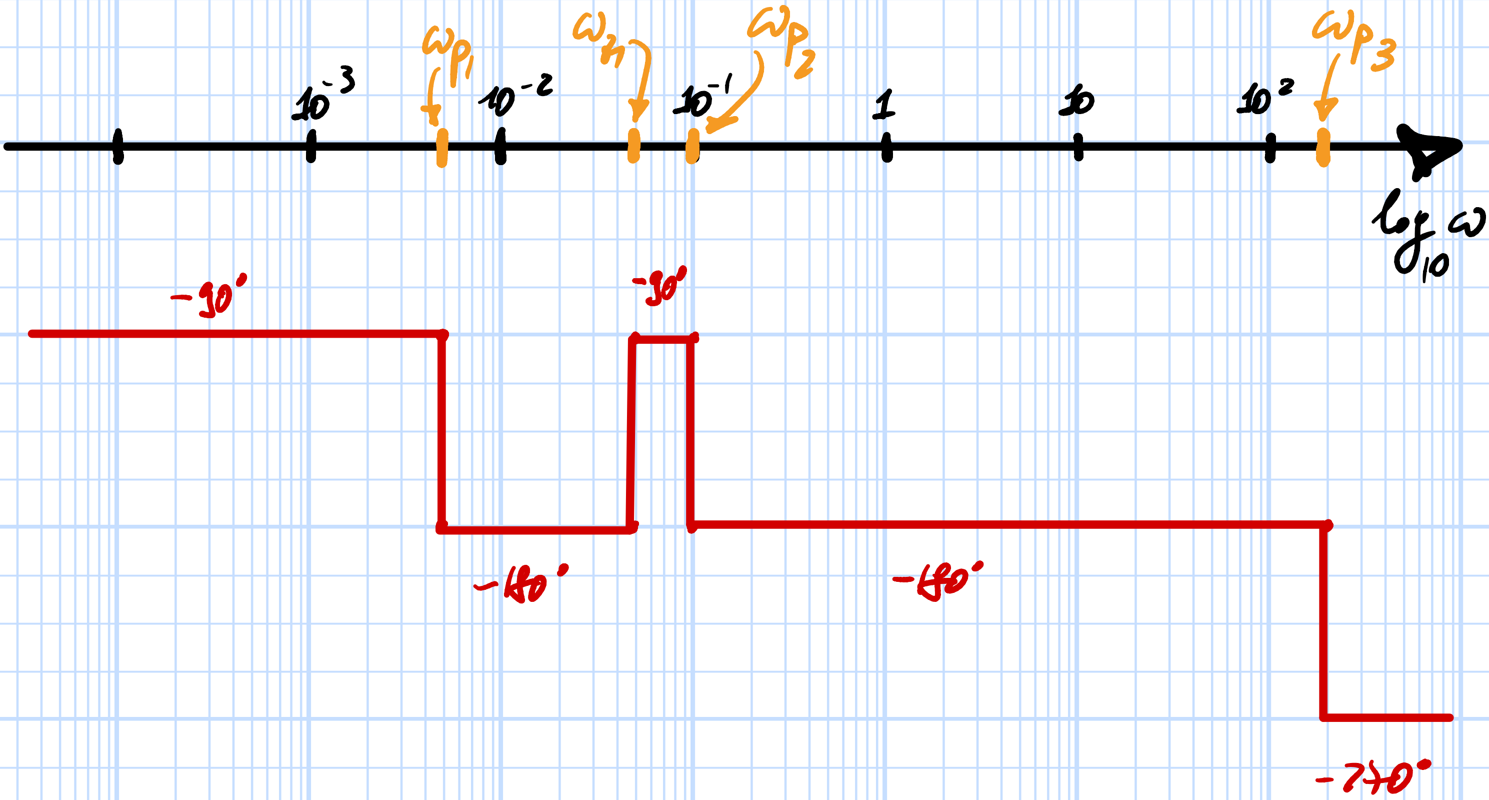
$$\log_{10}\omega_c = -\frac{6}{40} = -\frac{3}{20}$$

$$\omega_c = 10^{-\frac{3}{20}}$$

$\varphi_c \simeq -176^\circ$ $\qquad$ $\varphi_M \simeq 4^\circ$



$\mu_R = 1$
80
60
40
20
0dB
-20dB
-40dB
-60dB
-80dB
-100
-120
10^{-3}
10^{-2}
10^{-1}
1
10^{1}
10^{2}
$\log_{10}\omega$
ω_{p_1}
ω_{z_1}
ω_{p_2}
ω_{p_3}
ω_c
(-1)
(-2)
(-1)
(-2)
(-2)
(-3)
+94dB
80 dB
+40dB
+34dB
-6dB
-46 dB
-86 dB
- 98dB



10^{-3}
10^{-2}
10^{-1}
1
10
10^{2}
$\log_{10}\omega$
ω_{p_1}
ω_{z_1}
ω_{p_2}
ω_{p_3}
-90°
-180°
-90°
-180°
-270°

Calcoli svolti per ottenere i punti evidenziato nel diagramma asintotico del modulo:

1° tratto → retta a pendenza $-20\,dB/decade$

che passerebbe per $[\omega = 1,\ M = 34\,dB]$

per $\omega = 10^{-3} \rightarrow M = +34 + 20 \cdot 3 = +94\,dB$

per $\omega = \omega_{p_1} \rightarrow M = +94 - 20\left[\log_{10} \omega_{p_1} - \log_{10}(10^{-3})\right]$

$$= +94 - 20\left[\log_{10}\left(\frac{1}{200}\right) - \log_{10}\left(\frac{1}{1000}\right)\right]$$

$$= +94 - 20 \log_{10}\left(\frac{\cancel{1000}^{\,5}}{\cancel{200}_{\,1}}\right) =$$

$$= 80\,dB$$

per $\omega = \omega_{z_1} \rightarrow M = 80 - 40\left[\log_{10} \omega_{z_1} - \log_{10} \omega_{p_1}\right]$

$\downarrow$

$$M = 80 - 40\left[\log_{10}\left(\frac{1}{20}\right) - \log_{10}\left(\frac{1}{200}\right)\right]$$

$$= 80 - 40\log_{10}\left(\frac{\cancel{200}^{\,10}}{\cancel{20}_{\,1}}\right) = 40\,dB$$

per $\omega = \omega_{p_2} \rightarrow M = +40 - 20\left[\log_{10}\omega_{p_2} - \log_{10}\omega_{p_1}\right]$

$$M = 40 - 20\left[\log_{10}\left(\frac{1}{10}\right) - \log_{10}\left(\frac{1}{20}\right)\right]$$

$$= 40 - 20\log_{10}(2) \simeq 34\,dB$$

per $\omega = \omega_{p_3} \rightarrow M = 34 - 40\left[\log_{10}(\omega_{p_3}) - \log_{10}(\omega_{p_2})\right]$

$$= 34 - 40\left[\log_{10}(200) - \log_{10}\left(\frac{1}{10}\right)\right]$$

$$= 34 - 40\log_{10}(2 \cdot 10^3) =$$

$$= 34 - 120 - 12 = -98\,dB$$

Dal grafico del modulo

$$\frac{1}{10} < \omega_c < 1$$

stimo ω_c dal diagramma asintotico:

$$\omega_c \longrightarrow 0 = +34 - 40\left[\log_{10}\omega_c - \log_{10}\omega_{p_2}\right]$$

$$= +34 - 40\log_{10}\omega_c + 40\log_{10}\frac{1}{10} =$$

$$= -6 - 40\log_{10}\omega_c$$

$$\log_{10}\omega_c = -\frac{\cancel{6}^{\,3}}{\cancel{40}_{\,20}}$$

$$\omega_c = 10^{\left(-\frac{3}{20}\right)} \simeq 0{,}708\ rad/s$$

$$\varphi_c = \arg L(j\omega_c) = ?$$

$$L(j\omega_c) = 50\mu_R \frac{(1+20j\omega_c)}{j\omega_c(1+200j\omega_c)(1+10j\omega_c)\left(1+j\frac{\omega_c}{200}\right)}$$

$$\begin{aligned}
\arg L(j\omega_c) = \ & \arg 50 + \arg \mu_R + \\
& + \arg(1+20j\omega_c) + \\
& - \arg(+j\omega_c) + \\
& - \arg(1+200j\omega_c) + \\
& - \arg(1+10j\omega_c) + \\
& - \arg\left(1+j\frac{1}{200}\omega_c\right) =
\end{aligned}$$

$$\begin{aligned}
& = 0^\circ + \arctan(20\omega_c) - 90^\circ - \arctan(200\omega_c) + \\
& - \arctan(10\omega_c) - \arctan\left(\frac{\omega_c}{200}\right) =
\end{aligned}$$

$\omega_c \simeq 0{,}708 \ rad/s$

$$\cong -3{,}068 \cdot 180/\pi \simeq -176^\circ$$

$\varphi_c \simeq -176° \Longrightarrow \varphi_m = \varphi_c + 180° = +4°$

AB nel caso
del 1° progetto era $\varphi_m \simeq +56°$
$k_m \to +\infty$

Ora $k_m = ?$

ω_π esiste certamente (dal grafico della fase della risposta in frequenza in questo caso).

$\omega_\pi : \arg L(j\omega_\pi) = -180°$

$$\arg L(j\omega) = 0° + \operatorname{arctg}(20\omega) - 90° - \operatorname{arctg}(200\omega) - \operatorname{arctg}(10\omega) - \operatorname{arctg}\left(\frac{\omega}{200}\right) = -180°$$

$$\operatorname{arctg}(20\omega) - \operatorname{arctg}(10\omega) + 90° =$$

$$\circledast \qquad = \operatorname{arctg}(200\omega) + \operatorname{arctg}\left(\frac{\omega}{200}\right)$$

Ricordando che

$$\operatorname{tg}(\alpha + 90°) = -\frac{1}{\operatorname{tg}\alpha}$$

e che
$$\operatorname{tg}(\alpha \pm \beta) = \frac{\operatorname{tg}\alpha \pm \operatorname{tg}\beta}{1 \mp \operatorname{tg}\alpha \cdot \operatorname{tg}\beta}$$

si può rielaborare l'espressione $\circledast$

$$\operatorname{tg}\left[\left(\operatorname{arctg}(20\omega) - \operatorname{arctg}(10\omega)\right) + 90°\right] =$$

$$\operatorname{tg}\left[\operatorname{arctg}(200\omega) + \operatorname{arctg}\left(\frac{\omega}{200}\right)\right]$$

$$-\frac{1}{tg\left[\arctan(20\omega)-\arctan(10\omega)\right]} =$$

$$= \frac{200\omega + \frac{\omega}{200}}{1 - \cancel{200}^{\,1}\omega \cdot \frac{\omega}{\cancel{200}_{\,1}}}$$

$$-\frac{1 + \overbrace{20\omega \cdot 10\omega}^{200\omega^2}}{\underbrace{20\omega - 10\omega}_{10\omega}} = \frac{1}{200} \cdot \frac{40001\,\omega}{1-\omega^2}$$

$$2000\,\omega \cdot (1-\omega^2)$$

$\omega \neq 0$

$\omega \neq 1$

(solo $\omega > 0$)

$$-200\,(1-\omega^2)(1+200\,\omega^2)=400010\,\omega^2$$

È eq. biquadratica $\rightarrow \omega^2 \triangleq t$

$$200\,(1-t)(1+200t)=-400010t$$

$$-40000t^2+200+40000t-200t=-400010t$$

$$-40000t^2+400010t+39800t+200=0$$

$$40000t^2-439810t-200=0$$

$$t_{1,2}=\frac{219905\pm\sqrt{(219905)^2+8000000}}{40000}$$

$t_1 \simeq -5{,}0\cdot10^{-4}$ <0 NON accettabile

$t_2 \simeq +10{,}996$ >0 OK

$$\omega_\pi=+\sqrt{10{,}996}\simeq 3{,}316 \text{ rad/s}$$

$$k_M = \frac{1}{|L(j\omega_\pi)|} = \frac{|j\omega_\pi| \cdot |1+200j\omega_\pi| \cdot |1+10j\omega_\pi| \cdot \left|1+j\frac{\omega_\pi}{200}\right|}{50 \cdot 1 \cdot |1+20j\omega_\pi|}$$

Sostituisco ed ottengo

$$k_M = \frac{1}{4{,}545 \cdot 10^{-2}} \simeq 22$$

$$k_{M_{dB}} \simeq +26{,}8\,dB$$

// $\nexists$ ma $+\infty$ nel caso precedente!

Utilizzando il regolatore $R(s) = \frac{1}{s}$ si ottiene un sistema di "tipo 1", ma i margini di robustezza (φ_M) non sono OK!

Come fare per ottenere un sistema di tipo 1 con margini φ_m e k_m migliori ed allo stesso tempo ω_c elevata?

Per es.

$\omega_c > 10$ rad/s

$\varphi_m > 60°$

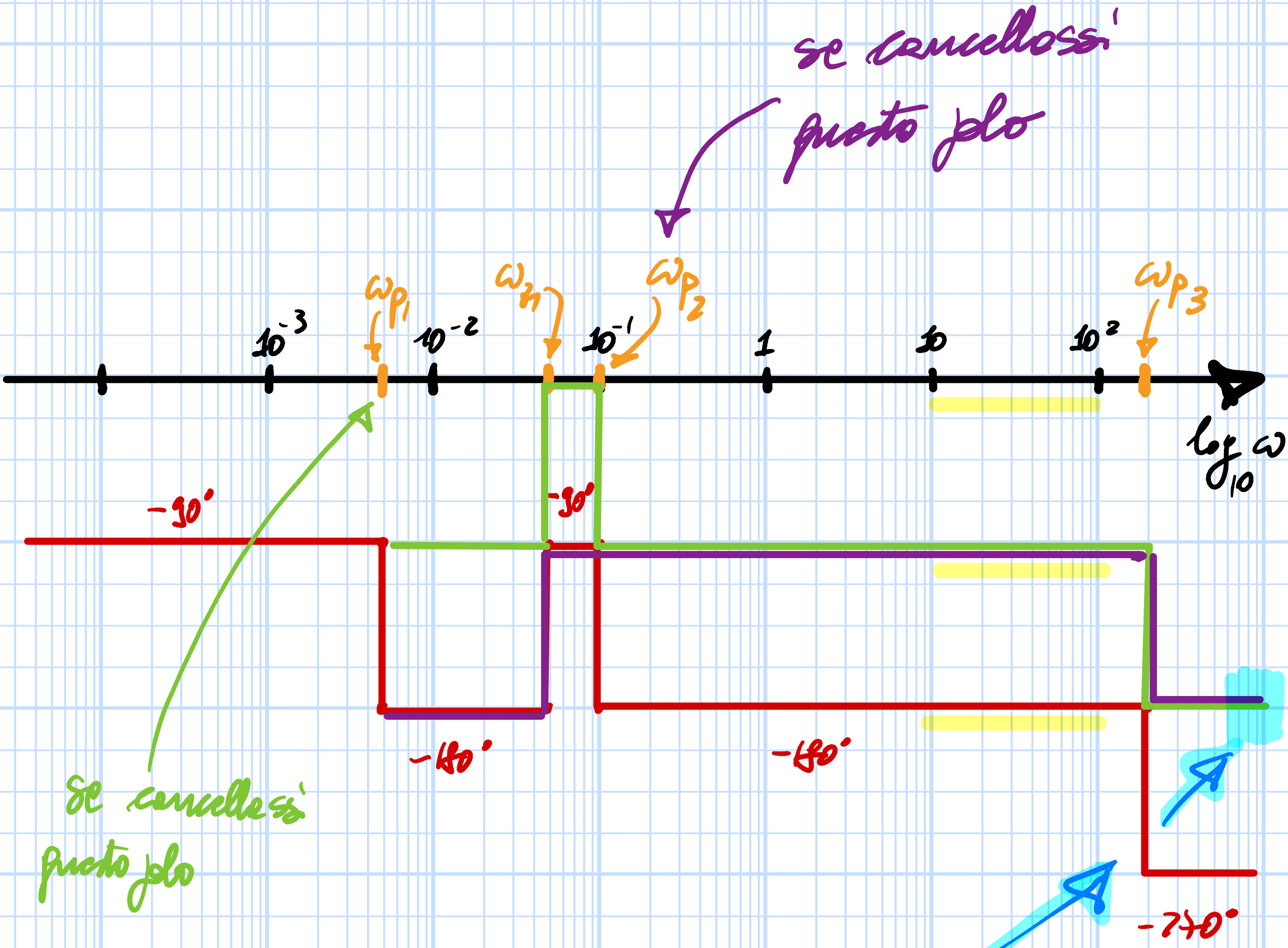
$k_m \to +\infty$

?

Per tornare ad avere $k_m \to +\infty$ si deve garantire che

(•) NON esista ω_π finita

• $\begin{cases} \arg L(j\omega) \xrightarrow[\omega \to +\infty]{} -180° \\ |L(j\omega)| \xrightarrow[\omega \to +\infty]{} 0 \end{cases}$



In entrambi i casi si raggiunge $\measuredangle = -180°$ solo per $\omega \to +\infty$

In entrambi i casi per $10 < \omega < 100$ si ha
$-90° \leq \measuredangle L(j\omega) < -180°$

Scelgo di cancellare il polo più lento (P_1)

$$R_1(s) = \mu_R \frac{1+200s}{s} \quad (**)$$

Determinare un regolatore seguendo l'altra possibile via di progetto

$$\hat{R}_1(s) = \mu_R \frac{1+10s}{s}$$

Con la scelta (**) la funzione di trasferimento di ciclo aperto diventa:

$$L_2(s) = \frac{50\mu_R}{s} \cdot \frac{(1+20s)}{(1+10s)\left(1+\frac{1}{200}s\right)}$$

Diagrammi asintotici di Bode per $\mu_R = 1$

poli di $L_2(s)$

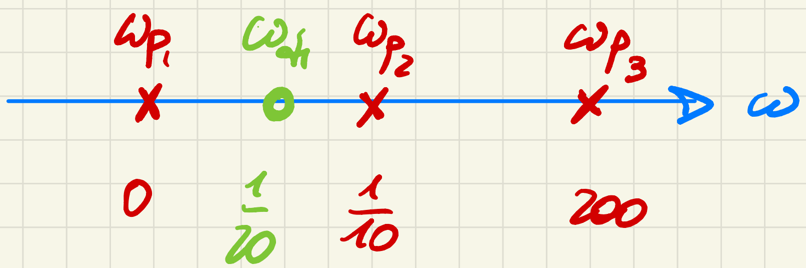
$p_1 = 0 \qquad \omega_{p_1} = 0$

$p_2 = -\frac{1}{10} \qquad \omega_{p_2} = \frac{1}{10}$

$p_3 = -200 \qquad \omega_{p_3} = 200$

zeri di $L_2(s)$

$z_1 = -\frac{1}{20} \qquad \omega_{z_1} = \frac{1}{20}$



$g = 1 \qquad \mu_{L_2} = 50 \Rightarrow$ 1° tratto della spezzata del modulo passerebbe per $\omega = 1$,

$\mu_{L_2}^{dB} \simeq 34\,dB$ $\qquad \mu_{dB} = +34\,dB$

In realtà il 1° tratto dello spettro va tracciato per $\omega \in \left[0 ; \frac{1}{20}\right]$

pendenza $-20dB/decade$

$\omega = 1 \longrightarrow M_{dB} = +34$

Valuto per $\omega = 10^{-2}$

$\omega = 5 \cdot 10^{-2} \left(= \frac{1}{20}\right)$

$$y = +34 - 20\left[\log_{10}(\omega) - \log_{10}(1)\right]$$

$\omega = 10^{-2}$

$$= +34 - 20\log_{10}(10^{-2}) + \cancel{20\log_{10} 1} \; 0 =$$

$$= +34 + 40 = +74\,dB \longleftarrow \omega = 10^{-2}$$

$M_{dB} = +74\,dB$

Ora per $\omega = \frac{1}{20} = 5 \cdot 10^{-2}$

$$y = +74 - 20\left[\log_{10}(5 \cdot 10^{-2}) - \log_{10}(10^{-2})\right] =$$

$$= +74 - \underbrace{20 \log_{10}\left[\frac{5 \cdot \cancel{10^{-2}}}{\cancel{10^{-2}}}\right]}_{\to\, 20\log_{10} 5 \,\simeq\, +14\,dB}$$

$$\simeq +74 - 14 = 60\,dB \quad \text{per } \omega = \omega_{z1}$$

Ora è possibile tracciare il primo tratto dello spettro del modulo e poi proseguire.

Per $\omega \in [\omega_{z1} ; \omega_{p2}]$ il diagramma ha pendenza nulla (tratto orizzontale)

Quindi $\omega_{p_2} \longrightarrow M_{dB} = +60\,dB$

Posso valutare il modulo per $\omega = \omega_{p_3}$

$$\omega = \omega_{p_3} = 2 \cdot 10^2$$

$$y = +60 - 20\left[\log_{10}(\omega_{p_3}) - \log_{10}(\omega_{p_2})\right]$$

$$= +60 - 20\log_{10}\left[\frac{200}{1/10}\right]$$

$$= +60 - 20\log_{10}(2 \cdot 10^3) =$$

$$= +60 - 66{,}02 = -6{,}02\,dB$$

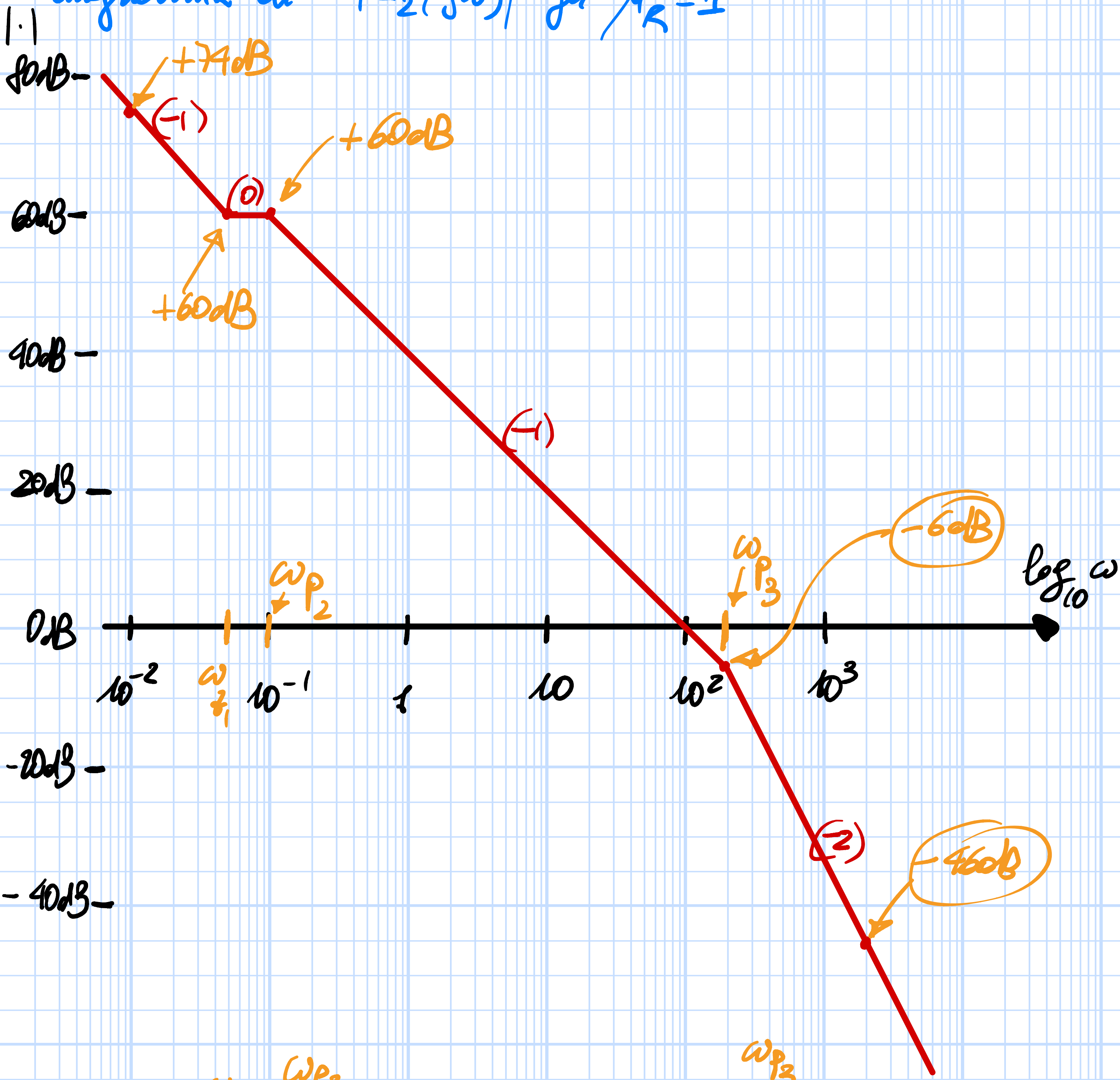
Ora per $\omega > \omega_{p_3}$, per es. $\tilde{\omega} = 2000$

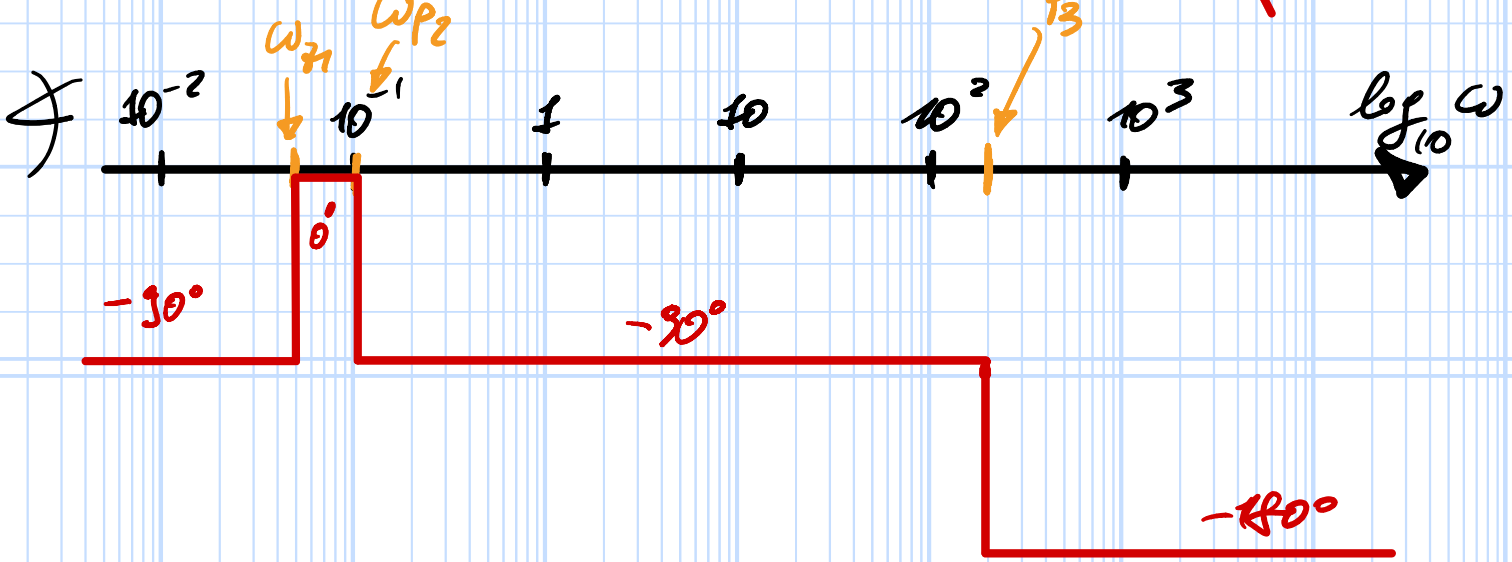
(a distanza pari ad una decade)

$$\omega = \tilde{\omega} = 2 \cdot 10^3 \quad M_{dB} = -6{,}02 - 40$$

$$= -46{,}02\,dB$$

diagramma di $|L_2(j\omega)|$ per $\mu_R = 1$





Per $\omega \to +\infty$ si ha che $[\forall\, \mu_R > 0]$

$$(\bullet) \quad |L_2(j\omega)| \xrightarrow[\omega \to +\infty]{} 0$$

$$(\bullet) \quad \measuredangle L_2(j\omega) \xrightarrow[\omega \to +\infty]{} -180^\circ$$

Allora posso concludere che

$$k_m \to +\infty$$

Il soddisfacimento della richiesta su k_m è garantito.

Dal diagramma di fase posso dedurre che esiste certamente intervallo di pulsazioni Ω_{60} tale per cui se $\omega \in \Omega_{60}$ allora $\varphi_m > 60^\circ$

Semplicemente $\Omega_{CO} = [0 \,;\, \bar{\omega}[$

con $\bar{\omega}$: $\angle L_2(j\bar{\omega}) = -120°$

Determino $\bar{\omega}$:

$$\angle L_2(j\omega) \geq -120°$$

$$L_2(s) = \frac{50\mu_R}{s} \cdot \frac{(1+20s)}{(1+10s)\left(1+\frac{1}{200}s\right)}$$

$$\begin{aligned}\angle L_2(j\omega) = {} & \overset{0°}{\cancel{\arg 50}} + \overset{0°}{\cancel{\arg \mu_R}} + \\ & + \arg(1+20j\omega) + \\ & \overset{-90°}{\cancel{- \arg(j\omega)}} + \\ & - \arg(1+10j\omega) + \\ & - \arg\left(1+\frac{1}{200}j\omega\right) \geq -120°\end{aligned}$$

$$\operatorname{arctg}(20\omega) - 90^\circ - \operatorname{arctg}(10\omega) +$$

$$- \operatorname{arctg}\left(\frac{\omega}{200}\right) \geq -120^\circ$$

$$\operatorname{arctg}(20\omega) - \operatorname{arctg}(10\omega) \geq$$

$$\operatorname{arctg}\left(\frac{\omega}{200}\right) - 30^\circ$$

$$\boxed{\operatorname{tg}(\alpha-\beta) = \frac{\operatorname{tg}\alpha - \operatorname{tg}\beta}{1+\operatorname{tg}\alpha \operatorname{tg}\beta} \quad \text{gdy} \quad \alpha+\beta \neq k\frac{\pi}{2}}$$

$$\frac{20\omega - 10\omega}{1+200\omega^2} \geq \frac{\frac{\omega}{200} - \operatorname{tg} 30^\circ}{1+\omega \cdot \frac{1}{200} \operatorname{tg} 30^\circ}$$

$$\operatorname{tg}(30^\circ) = \frac{\sqrt{3}}{3}$$

$$\underbrace{\frac{10\omega}{1+200\omega^2}}_{\text{sicuramente} > 0} \geq \frac{\frac{\omega}{200}-\frac{\sqrt{3}}{3}}{\underbrace{1+\frac{\sqrt{3}}{600}\omega}_{\text{è} > 0 \ \forall \omega > 0}}$$

Considero accettabili solo soluzioni $\omega > 0$

$$\frac{10\omega}{1+200\omega^2} \geq \frac{\frac{\omega}{200}-\frac{\sqrt{3}}{3}}{1+\frac{\sqrt{3}}{600}\omega} = \frac{\cancel{\frac{1}{600}}\left(3\omega-200\sqrt{3}\right)}{\cancel{\frac{1}{600}}\left(\sqrt{3}\omega+600\right)}$$

$$\frac{10\omega}{1+200\omega^2} \geq \frac{3\omega-200\sqrt{3}}{\sqrt{3}\omega+600}$$

$$\Big/\ (1+200\omega^2)\cdot(\sqrt{3}\omega+600)$$

certamente > 0 se $\omega > 0$

$$10\omega\left(\sqrt{3}\,\omega+600\right) \geq \left(200\omega^2+1\right)\left(3\omega-200\sqrt{3}\right)$$

$$10\sqrt{3}\,\omega^2+6000\,\omega \geq 600\omega^3-4\cdot10^4\cdot\sqrt{3}\,\omega^2+$$
$$+3\omega-200\sqrt{3}$$

$$600\omega^3-40010\cdot\sqrt{3}\,\omega^2-5997\,\omega-200\sqrt{3} \leq 0$$

$$600\omega^3-\ldots=0 \text{ per (bisezione)}$$

$$\omega_1 \simeq 115{,}58 \text{ rad/s} \quad \not\leftarrow$$

$$\omega_2,\omega_3 \in \mathbb{C} \leftarrow \text{NON ecc.}$$

Ottengo $\varphi_m > 60^\circ \;\; \forall \omega_c < 115{,}58$ rad/s

Scelgo $\bar{\omega} = 100$ rad/s come "candidata ω_c" $\longrightarrow$ DEVO scegliere valore per μ_R

in modo che risulti

$$\mu_R : \quad |L_2(j\bar{\omega})| = 1$$

$$L_2(s) = \frac{50\mu_R}{s} \cdot \frac{(1+20s)}{(1+10s)(1+\frac{1}{200}s)}$$

$$|L_2(j\bar{\omega})| = \underbrace{\frac{50\mu_R}{\bar{\omega}} \frac{\sqrt{1+400\bar{\omega}^2}}{\sqrt{1+100\bar{\omega}^2} \cdot \sqrt{1+\frac{\bar{\omega}^2}{40000}}}}_{=1}$$

$\bar{\omega} = 100$

$$\mu_R = \underbrace{\frac{\bar{\omega}}{50} \cdot \frac{\sqrt{1+100\bar{\omega}^2} \cdot \sqrt{1+\frac{\bar{\omega}^2}{40000}}}{\sqrt{1+400\bar{\omega}^2}}}_{\simeq \frac{\cancel{100}^{\,2}}{\cancel{50}_{\,1}} \cdot \frac{\cancel{1000} \cdot \sqrt{5}/2}{\cancel{2000}}} = \frac{\sqrt{5}}{2}$$

Quindi $R_1(s) = \frac{\sqrt{5}}{2} \frac{1+200s}{s}$

garantisce $\omega_c \simeq 100$ rad/s

$\varphi_m > 60°$ [$\varphi_m \simeq 63°$ verificarlo]

$k_m \to +\infty$