Dependence modeling with copulas

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Table of Contents

Table of Contents

Introduction

We are interested in how dependence between the components of a random vector $\mathbf{X} \in \mathbb{R}^d$, $d \geq 2$, can be investigated and modeled.

The usual Pearson linear correlation has a number of limitations, particularly when moving away from elliptical models, whereas *rank correlations* and coefficients of *tail dependence* are alternative dependence measures derived from copulas.

Tail dependence is an important concept to address the phenomenon of joint extreme values in several risk factors, which is one of the major concerns in financial risk management.

Table of Contents

Literature

The copula approach provides a convenient way of isolating the description of the *dependence structure* of individual risk factors from their marginal behaviour.

Since their introduction (Sklar (1959)), the literature on copulas has considerably grew. Major references include

- Nelsen (2006)
- Durante and Sempi (2016)
- Hofert et al. (2018)
- Kojadinovic (2010)

Background

The following two notions are fundamental:

- **Quantile transform (QT).** If $U \sim \mathrm{U}(0,1)$, then $P(F^{\leftarrow}(U) \leq x) = F(x)$, where $F^{\leftarrow}$ is the generalized inverse of the df of X, F.

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  Proof. For $x \in \mathbb{R}$, $U \sim \mathrm{U}(0,1)$: $P(F^{\leftarrow}(U) \leq x) = P(U \leq F(x)) = F(x)$.
- **Probability transform (PT).** If $X \sim F$, and F is continuous, then $F(X) \sim \mathrm{U}(0,1)$.

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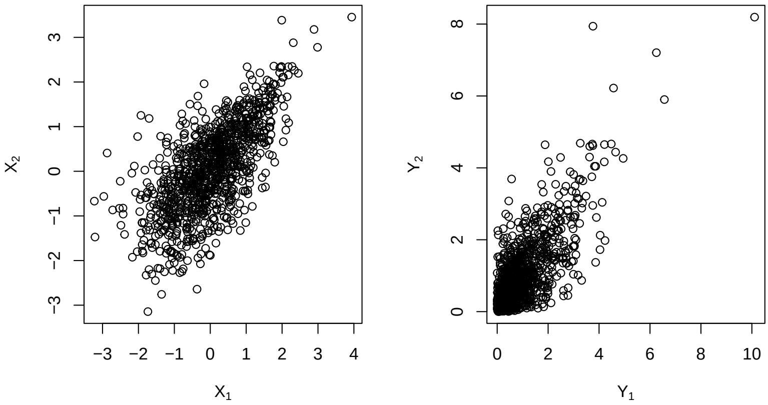
- **Probability transform (PT).** If $X \sim F$, and F is continuous, then $F(X) \sim \mathrm{U}(0,1)$.

  Proof. For $u \in (0,1)$, $P(F(X) \leq u) = P(F^{\leftarrow}(F(X)) \leq F^{\leftarrow}(u)) = P(X \leq F^{\leftarrow}(u)) = F(F^{\leftarrow}(u)) = u$.

Table of Contents

Two bivariate data sets



For which data is the dependence between the two variables *larger*?

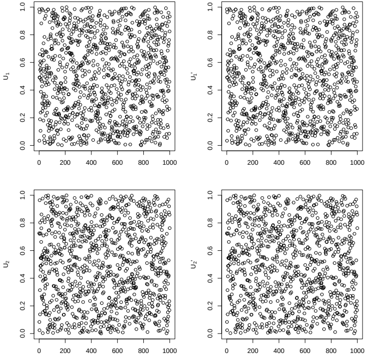
Data transformation

The *empirical cdf* $\hat{F}_{n,j}$ of the j-th margin is applied to X_{ij}, $i \in \{1, \ldots, n\}$:

$$U_{ij} = \hat{F}_{n,j}(X_{ij}) = (1/n) \sum_{k=1}^{n} \mathbf{I}_{\{X_{kj} \leq X_{ij}\}} = R_{ij}/n,$$

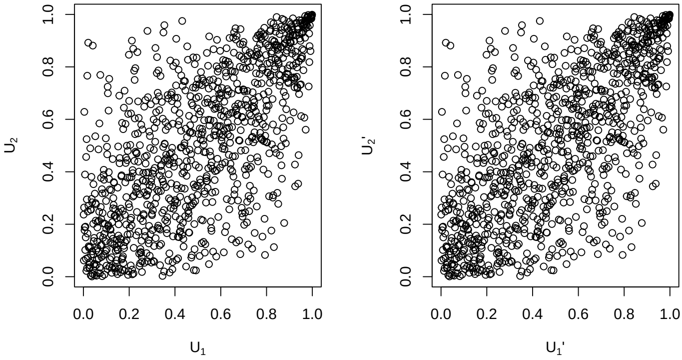
where R_{ij} denotes the rank of X_{ij} among $X_{1j}, \ldots, X_{nj}$

Function `pobs()` uses the scaled version $R_{ij}/(n+1)$. As shown in the plots, new margins are approximately U(0,1):



Pseudo-observations

Pseudo-observations for $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$



The pseudo observations give us insight in the actual dependence structure (copula) underlying our data sets **x** and **y**.

Table of Contents

Table of Contents

Definition: d-dimensional copula

Definition (Copula)

A d-dimensional copula is a distribution function on $[0,1]^d$ with standard uniform marginal distributions.

Hence, the copula

$$C(\mathbf{u}) = C(u_1, \dots, u_d)$$

is a mapping of the unit hypercube into the unit interval

$$C : [0,1]^d \to [0,1].$$

Example: Independence Copula

One of the simplest copulas is the *independence copula*

$$\Pi_d(\mathbf{u}) = \prod_{j=1}^{d} u_j, \quad \mathbf{u} \in [0,1]^d$$

Π_d is the df which is the df of a random vector $\mathbf{U} = (U_1, \ldots, U_d)$ with independent components $U_1, \ldots, U_d \sim \mathrm{U}(0,1)$:

For any $\mathbf{u} \in [0,1]^d$,

$$P(\mathbf{U} \le \mathbf{u}) = P(U_1 \le u_1, \ldots, U_d \le u_d) = \prod_{j=1}^{d} P(U_j \le u_j) = \prod_{j=1}^{d} u_j = \Pi_d(\mathbf{u})$$

Example: Independence Copula/ 2

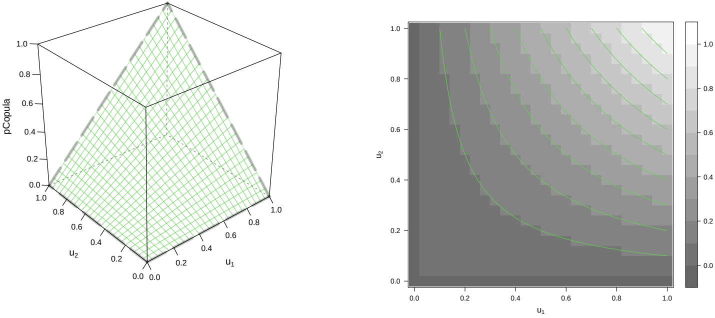


Figure: (Left) Surface (or perspective) plot and (right) contour plot of the independence copula for $d = 2$.

Remark: Π_2 is zero on all edges of the unit square which start at $(0, 0)$, $\Pi_2(u_1, 1) = u_1$ and $\Pi_2(1, u_2) = u_2 \, \forall u_1, u_2 \in [0, 1]$; this means that the copula is *grounded* ($C(\mathbf{u}) = 0$ if $u_j = 0$ for at least one j) and has *standard uniform univariate margins* ($C(1, \ldots, 1, u_j, 1, \ldots, 1) = u_j, \, \forall u_j$)

C-volumes

In order to obtain a characterization of copulas we need the following additional definitions.

C-volume. For any $\mathbf{a}, \mathbf{b} \in [0,1]^d$, $\mathbf{a} \leq \mathbf{b}$, let $(\mathbf{a}, \mathbf{b}]$ denote the *hyperrectangle* defined by $\mathbf{u} \in [0,1]^d : \mathbf{a} < \mathbf{u} \leq \mathbf{b}$. Then, for any hyperrectangle $(\mathbf{a}, \mathbf{b}]$, define its *C-volume* as

$$\Delta_{(\mathbf{a},\mathbf{b}]} C = \sum_{i \in \{0,1\}^d} (-1)^{\sum_{j=1}^d i_j} C(a_1^{i_1} b_1^{1-i_1}, \ldots, a_d^{i_d} b_d^{1-i_d}) \quad (1)$$

where the summation is taken over all 2^d vectors $(i_1, \ldots, i_d)$, $i_j \in 0, 1$. If

$$\Delta_{(\mathbf{a},\mathbf{b}]} C \geq 0 \text{ for all } \mathbf{a}, \mathbf{b} \in [0,1]^d, \mathbf{a} \leq \mathbf{b}$$

then C is called *d-increasing*. When $d = 2$, (1) becomes

$$\Delta_{(\mathbf{a},\mathbf{b}]} C = C(b_1, b_2) - C(b_1, a_2) - C(a_1, b_2) + C(a_1, a_2)$$

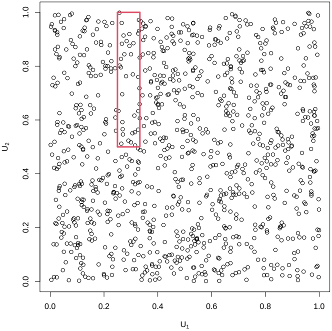
C-volume: Example

Let $C = \Pi_2 = u_1 u_2$. We will verify that $\Delta_{(\mathbf{a},\mathbf{b}]} C = P(\mathbf{U} \in (\mathbf{a}, \mathbf{b}])$. Using (1),

$$\begin{aligned}
&\Delta_{(a_1,a_2),(b_1,b_2)]} C \\
&= b_1 b_2 - b_1 a_2 - a_1 b_2 + a_1 a_2 \\
&\quad = (b_1 - a_1)(b_2 - a_2)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&P(\mathbf{U} \in (\mathbf{a}, \mathbf{b}]) \\
&= P(a_1 < U_1 \leq b_1) P(a_2 < U_2 \leq b_2) \\
&= (b_1 - a_1)(b_2 - a_2)
\end{aligned}$$



Approximation of the Π_2-volume of the hyperrectangle with lower end point $\mathbf{a} = (1/4, 1/2)$ and upper end point $\mathbf{b} = (1/3, 1)$ based on 1000 independent observations of $\mathbf{U} \sim \Pi_2$.

Characterization

The function $C : [0,1]^d \to [0,1]$ is a copula if and only if

1. C is grounded, that is,

$$C(u_1, \ldots, u_d) = 0 \text{ if } u_j = 0 \text{ for at least one } j \in \{1, \ldots, d\}$$

2. C has standard uniform univariate margins, that is,

$$C(1, \ldots, 1, u_j, 1, \ldots, 1) = u_j \text{ for all } u_j \in [0,1] \text{ and } j \in \{1, \ldots, d\}$$

3. C is d-increasing, that is, any C-volume $\Delta_{(\mathbf{a},\mathbf{b}]} C$ is nonnegative, for all $\mathbf{a} = (a_1, \ldots, a_d)$, $\mathbf{b} = (b_1, \ldots, b_d) \in [0,1]^d$, $a_i \leq b_i$

Note that, for $2 \leq k < d$, the k-dimensional margins of a d-dimensional copula are themselves copulas.

Copula density

A copula C is called *absolutely continuous* if it admits a density, that is, if

$$c(\mathbf{u}) = c(u_1, \ldots, u_d) = \frac{\partial^d}{\partial u_d \ldots \partial u_1} C(u_1, \ldots, u_d), \quad \mathbf{u} \in (0,1)^d$$

exists and is integrable.

Remark: If the density c is nonnegative for all $\mathbf{u} \in (0,1)^d$ then C is d-increasing.
Example: the independence copula Π_d is absolutely continuous with constant density $c(\mathbf{u}) = 1, \mathbf{u} \in (0,1)^d$.

Table of Contents

Theorem: Fréchet-Hoeffding Bounds

Any d-dimensional copula C is pointwise bounded from below by the lower Fréchet-Hoeffding bound W and from above by the upper Fréchet-Hoeffding bound M

$$W(\mathbf{u}) \leq C(\mathbf{u}) \leq M(\mathbf{u}), \quad \mathbf{u} \in [0,1]^d$$

where

$$W(\mathbf{u}) = \max\left\{\sum_{j=1}^{d} u_j - d + 1, 0\right\} \quad \text{and} \quad M(\mathbf{u}) = \min_{1 \leq j \leq d}(u_j)$$

Note that W is a copula only if $d = 2$ whereas M is a copula for all $d \geq 2$.

Stochastic representations

Let $U \sim \mathrm{U}(0,1)$.

- the *countermonotone copula* W (in dimension two only) is the copula of the vector $(U, 1-U)$

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Fréchet-Hoeffding Bounds

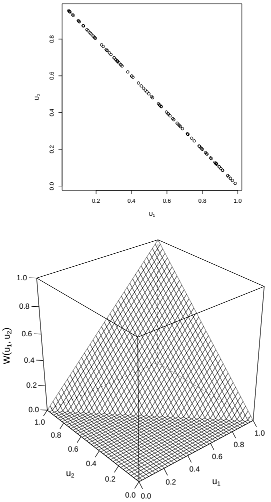

Figure: Scatter plot of $n = 100$ independent observations and perspective plot of W (left) and M (right) for $d = 2$.

Singular copulas

From the scatter plot, it can be seen that neither W nor M is absolutely continuous:

- Copulas such as W and M which put all probability mass on a set of (Lebesgue) measure 0 (the secondary and the primary diagonal for W and M, respectively) are called *singular*
- Copulas which put some probability mass in (0, 1) on a set of (Lebesgue) measure 0 have a *singular component*

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  Example: Marshall–Olkin copulas

Marshall–Olkin Copulas

The bivariate Marshall–Olkin family of copulas is given by

$$C(u_1, u_2) = \min\{u_1 u_2^{1-\alpha_2}, u_1^{1-\alpha_1} u_2\}, \quad u_1, u_2 \in [0,1] \tag{2}$$

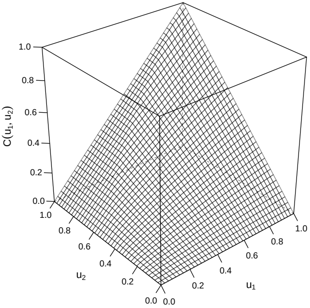
with $\alpha_1, \alpha_2 \in [0,1]$

- if $\alpha_1 = 0$ or $\alpha_2 = 0$, then $C = \Pi$
- A random vector $(U_1, U_2) \sim C$ admits the stochastic representation

$$(U_1, U_2) = \left(\max\left\{V_1^{1/(1-\alpha_1)}, V_{12}^{1/\alpha_1}\right\}, \max\left\{V_2^{1/(1-\alpha_2)}, V_{12}^{1/\alpha_2}\right\}\right)$$

  where V_1, V_2, V_{12} are independent $\mathrm{U}(0,1)$ (for any $v \in (0,1)$, $v^{1/0} = 0$ by convention)

Example: Marshall–Olkin copula



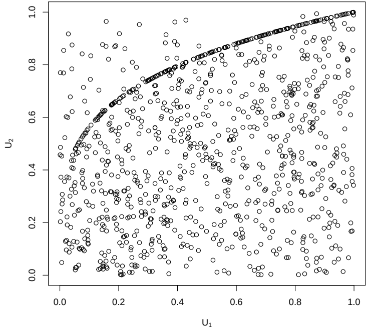


Figure: (Left) Density of a Marshall–Olkin copula. (Right) Corresponding scatter plot of a sample of size $n = 1000$. The singular component in the latter which is reflected by a kink in the former.

Table of Contents

Sklar's Theorem: preliminaries

Sklar's Theorem Sklar (1959) is the main result of copula theory: it explains how copulas determine the dependence between the components of a random vector.

Some notation:

- given a univariate df F, ran$F = \{F(x) : x \in \mathbb{R}\}$ denotes the range of F
- $F^{\leftarrow}$ denotes the quantile function associated with F (this is the ordinary inverse F^{-1} if F is continuous and strictly increasing.

Sklar's Theorem

Theorem (Sklar)

1. *For any d-dimensional df H with univariate margins* $F_1, \dots, F_d$*, there exists a d-dimensional copula C such that*

$$H(\mathbf{x}) = C(F_1(x_1), \dots, F_d(x_d)), \quad \mathbf{x} \in \mathbb{R}^d. \tag{3}$$

The copula C is uniquely defined on $\mathrm{ran}F_1 \times \cdots \times \mathrm{ran}F_d = \prod_j \mathrm{ran}F_j$*:*

$$C(\mathbf{u}) = H(F_1^{\leftarrow}(u_1), \dots, F_d^{\leftarrow}(u_d)), \quad \mathbf{u} \in \prod_{j=1}^{d} \mathrm{ran}F_j \tag{4}$$

2. *Conversely, given a d-dimensional copula C and univariate dfs* $F_1, \dots, F_d, H$ *defined by (3) is a d-dimensional df with margins* $F_1, \dots, F_d$*.*

An analytical proof can be found in Sklar (1996), a probabilistic one in Rüschendorf (2009)

Sklar's Theorem: remarks

Part [1] of Sklar's Theorem states the decomposition of any d-dimensional df H into its univariate margins $F_1, \dots, F_d$ and a copula C.

Thus, copulas link (or *couple*) multivariate dfs to their univariate margins.

Let $\mathbf{X} = (X_1, \dots, X_d) \sim H$ and continuous margins $F_1, \dots, F_d$. Hence, $U_i = F_i(X_i) \sim \mathrm{U}(0,1)$ (**PT**). Let C denote the df of $(U_1, \dots, U_d)$. For any $\mathbf{x} \in \bar{\mathbb{R}} = [-\infty, \infty]$ we have

$$\begin{aligned} H(x_1, \dots, x_d) &= P(X_1 \leq x_1, \dots, X_d \leq x_d) \\ &= P(F_1^{\leftarrow}(U_1) \leq x_1, \dots, F_d^{\leftarrow}(U_d) \leq x_d) \\ &= P(U_1 \leq F_1(x_1), \dots, U_d \leq F_d(x_d)) \\ &= C(F_1(x_1), \dots, F_d(x_d)) \end{aligned}$$

If the margins are continuous, then C is unique; otherwise C is uniquely determined on $\mathrm{ran} F_1 \times \cdots \times \mathrm{ran} F_d$.

Sklar's Theorem: remarks

The explicit representation of the copula of $\mathbf{X}$ can be obtained by evaluating (3) at the arguments $x_i = F_i^{\leftarrow}(u_i)$, $0 \leq u_i \leq 1, i = 1, \dots, d$

$$
\begin{aligned}
C(u_1, \dots, u_d) &= C(F_1(F_1^{\leftarrow}(u_1)), \dots, F_d(F_d^{\leftarrow}(u_d))) \\
&= H(F_1^{\leftarrow}(u_1), \dots, F_d^{\leftarrow}(u_d))
\end{aligned}
$$

For a given continuous multivariate df, part [1] of Sklar's Theorem implies that the underlying unknown copula is unique, which justifies its estimation from available data.

If $\mathbf{X} \sim H$ with margins F_j and the decomposition (3) holds, we say that $\mathbf{X}$ (or H) has copula C. Moreover, the copula expresses the dependence on a quantile scale

$$C(u_1, \dots, u_d) = P(X_1 \leq F_1^{\leftarrow}(u_1), \dots, X_d \leq F_d^{\leftarrow}(u_d))$$

Sklar's Theorem: remarks

From [1], it also follows that H is absolutely continuous if and only if C and the F_is are absolutely continuous. In that case, the density of H satisfies

$$h(\mathbf{x}) = c(F_1(x_1), \ldots, F_d(x_d)) \prod_{j=1}^{d} f_j(x_j), \quad \mathbf{x} \in \prod_{j=1}^{d} \text{ran} X_j$$

where, for any $j \in \{1, \ldots, d\}$, $\text{ran}X_j$ is the range of the rv X_j, f_j denotes the density of F_j and c denotes the density of C. Hence, c can be recovered from h via

$$c(\mathbf{u}) = h(F_1^{\leftarrow}(u_1), \ldots, F_d^{\leftarrow}(u_d)) \left(\prod_{j=1}^{d} f_j(F_j^{\leftarrow}(u_j)) \right)^{-1}, \quad \mathbf{u} \in (0,1)^d$$

and used in likelihood-based copula estimation methods.

Sklar's Theorem: remarks

Part [2] of Sklar's Theorem:

Given any copula C and univariate dfs $F_1, \dots, F_d$, a multivariate df H can be composed via (3) which then has univariate margins $F_1, \dots, F_d$ (continuous if H is continuous) and 'dependence structure' C

Let $\mathbf{U} \sim C$ and set $\mathbf{X} := (F_1^{\leftarrow}(U_1), \dots, F_d^{\leftarrow}(U_d))$. Then

$$\begin{aligned} P(\mathbf{X} \leq \mathbf{x}) &= P(F_1^{\leftarrow}(U_1) \leq x_1, \dots, F_d^{\leftarrow}(U_d) \leq x_d) \\ &= P(U_1 \leq F_1(x_1), \dots, U_d \leq F_d(x_d)) \quad (QT) \\ &= C(F_1(x_1), \dots, F_d(x_d)) = H(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d \end{aligned}$$

- New multivariate dfs can be constructed with given univariate margins
- Copulas can be used to formulate dependence scenarios and to evaluate risk measures of interest by means of simulation.

Sklar's Theorem: remarks

Let $\mathbf{X} = (X_1, \dots, X_d)$; a **copula model**

$$H(\mathbf{x}) = C(F_1(x_1), \dots, F_d(x_d)), \quad \mathbf{x} \in \mathbb{R}^d$$

may belong to

(i) the class of all multivariate dfs with given margins $F_1, \dots, F_d$ known as a Fréchet class

(ii) the class of all dfs obtained from a given d-dimensional copula C known as meta-C models

Example 1 The meta-Π model consists of all multivariate df H such that $H(\mathbf{x}) = F_1(x_1) \cdots F_d(x_d)$. For fixed univariate dfs F_j ($j \in \{1, \dots, d\}$), H is a member of the Fréchet class

Example 2 The meta-Gaussian model originates from the Gaussian copula

Table of Contents

Invariance property

Let $\mathbf{X} \sim H$ with continuous margins F_j ($j \in \{1, \ldots, d\}$) and (unique) copula C. If $T_1, \ldots, T_d$ are strictly increasing functions, then

$$(T_1(X_1), \ldots, T_d(X_d)) \sim C$$

that is, copulas are invariant under strictly increasing transformations (on the ranges) of the underlying random variables.

We show that C is also the unique copula of $(T_1(X_1), \ldots, T_d(X_d))$:

$$\begin{aligned} C(u_1, \ldots, u_d) &= P(X_1 \le F_1^{\leftarrow}(u_1), \ldots, X_d \le F_d^{\leftarrow}(u_d)) \\ &= P(T_1(X_1) \le T_1(F_1^{\leftarrow}(u_1)), \ldots, T_d(X_d) \le T_d(F_d^{\leftarrow}(u_d)) \\ &= P\left(T_1(X_1) \le F_{T_1(X_1)}^{\leftarrow}(u_1), \ldots, T_d(X_d) \le F_{T_d(X_d)}^{\leftarrow}(u_d)\right) \end{aligned}$$

Invariance property/ 2

The invariance property allows us to transform $\mathbf{X} = (X_1, \ldots, X_d)$ to $\mathbf{U} = (F_1(X_1), \ldots, F_d(X_d))$ without changing the underlying copula

$$\mathbf{X} \text{ has copula } C \iff (F_1(X_1), \ldots, F_d(X_d)) \sim C.$$

that is, **X** and **U** have the same copula!
Hence, regardless of the marginals, we can study the dependence between $X_1, \ldots, X_d$ by studying the dependence between the components of **U**

Assume $d = 2$, and $(X_1, X_2) \sim H$ with continuous margins F_1, F_2. Then

$$(U, V) = (F_1(X_1), F_2(X_2))$$

gives the corresponding copula defined on $[0, 1]^2$.

Invariance property: Examples

From bivariate normal to normal copula

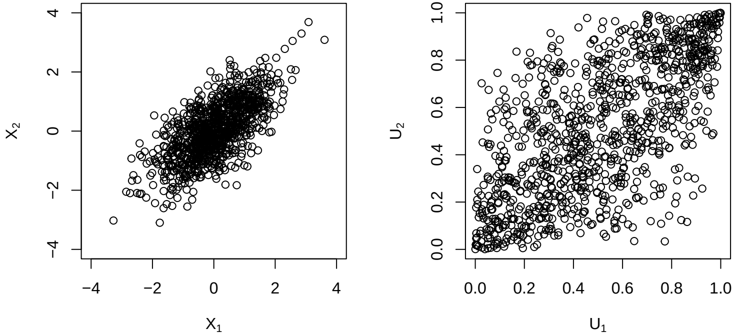


Figure: (Left) Scatter plot of $n = 1000$ independent observations from (X_1, X_2) having a joint bivariate Gaussian distribution $\mathcal{N}_2(\mathbf{0}, P)$, $P = \begin{pmatrix} 1 & 0.7 \\ 0.7 & 1 \end{pmatrix}$.
(Right) The corresponding (probability transformed) sample from the Gaussian copula is obtained by applying the df Φ (the F_j's here) to each pair of points.

Simulation of Implicit Copulas

Algorithm 1 can be used to sample implicit copulas (that is, copulas defined by (4) in Sklar's Theorem) such as the normal and t copulas (as in the previous example)

Algorithm 1

1. Sample $\mathbf{X} \sim H$, where H is a d-dimensional df with continuous margins $F_1, \dots, F_d$
2. Return $\mathbf{U} = (F_1(X_1), \dots, F_d(X_d))$

Invariance property: Examples

From normal copula to meta-Gaussian sample with exponential margins

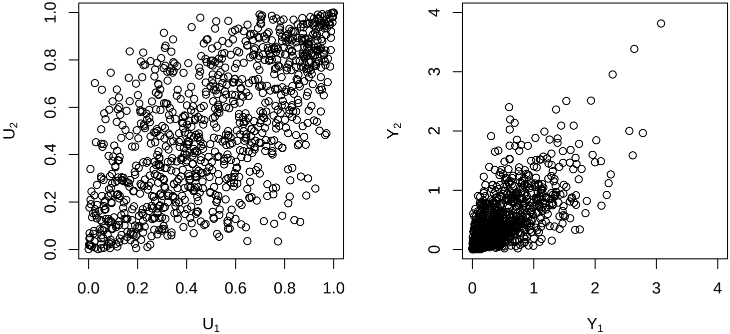


Figure: (Left) Same Gaussian copula scatter plot as before. (Right) The corresponding (quantile transformed) sample having a Gaussian copula and exponentially distributed marginals $F_j \sim \exp(2)$ (apply $F_j^{-1}(u) = -log(1-u)/2$ to each pair of points on the left plot.)

Simulation of Meta-C Models

Algorithm 2 can be used to sample meta-C models (as done in the previous example)

Algorithm 2

1. Sample $\mathbf{U} \sim C$
2. Return $\mathbf{X} = (F_1^{\leftarrow}(U_1), \ldots, F_d^{\leftarrow}(U_d))$

Simulation: R code

R code: *from a multivariate Gaussian distribution to a normal copula to a Meta-C Model*

```
> set.seed(332)
> d<-2
> rho<-0.7
> P<-matrix(rho, nrow=d, ncol=d) # correlation matrix
> diag(P)<-1
> X<-rmvnorm(1000, sigma = P) # bivariate normal obs.
> U<-pnorm(X) # copula sample
> # same as
> set.seed(332)
> U.<-rCopula(1000, normalCopula(rho, dim=2))
> Y<-qexp(U, 2) # transform U to exp(2) margins
```

Table of Contents

Some parametric copula families

Parametric copula families play a key role in the applications of copulas:

- **Implicit** copula families arise from well-known multivariate distributions via Sklar's Theorem

Some parametric copula families

Parametric copula families play a key role in the applications of copulas:

- **Implicit** copula families arise from well-known multivariate distributions via Sklar's Theorem

  $\rightarrow$ Elliptical copulas Gaussian copulas and t copulas from the Gaussian and t distributions, respectively

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  $\rightarrow$ Archimedean copulas: Clayton, Frank, Gumbel

Gaussian copulas

If $\mathbf{Y} \sim \mathcal{N}_d(\mu, \Sigma)$, then its copula is the same as the copula of $\mathbf{X} \sim \mathcal{N}_d(\mu, P)$, where P is the correlation matrix of $\mathbf{Y}$, and is the so-called Gaussian Copula (family)

$$\begin{aligned} C_P^{Ga}(\mathbf{u}) &= P(\Phi(X_1) \leq u_1, \ldots, \Phi(X_d) \leq u_d) \\ &= P(X_1 \leq \Phi^{-1}(u_1), \ldots, X_d \leq \Phi^{-1}(u_d)) \\ &= \boldsymbol{\Phi}_P(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d)) \end{aligned}$$

where $\boldsymbol{\Phi}_P$ is the joint df of $\mathbf{X}$, and Φ is the cdf of $\mathcal{N}(0,1)$.

- if $d = 2$, then $C_P^{Ga} \equiv C_\rho^{Ga}$, where $\rho = \mathrm{corr}(\mathrm{X}_1, \mathrm{X}_2)$
- $P = I_d$ gives independence
- If $P = J_d$, a $d \times d$ matrix of ones, then C is the comonotonicity copula (M)
- For $d = 2$ and $\rho = -1$, C_ρ^{Ga} is the countermonotonicity copula (W)

Gaussian copulas /2

The Gaussian copula does not have a simple closed form, but can be expressed as an integral over the density of **X**; in two dimensions, we have the distribution function

$$C_\rho^{Ga}(u_1, u_2) = \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{a} \exp\left(-\frac{s_1^2 + s_2^2 - 2\rho s_1 s_2}{2(1-\rho^2)}\right) \mathrm{d}s_1 \mathrm{d}s_2$$

where $a = 2\pi(1-\rho^2)^{1/2}$; the copula density is

$$c(u_1, u_2) = \frac{h_P(\Phi_1^{-1}(u_1), \Phi_2^{-1}(u_2))}{\phi(\Phi_1^{-1}(u_1))\,\phi(\Phi_2^{-1}(u_2))}$$

where h_P is the density of the bivariate standard normal distribution whose correlation matrix P has off-diagonal elements ρ; ϕ and Φ are the density and cdf of a standard normal $\mathcal{N}(0,1)$.

Gaussian copulas /3

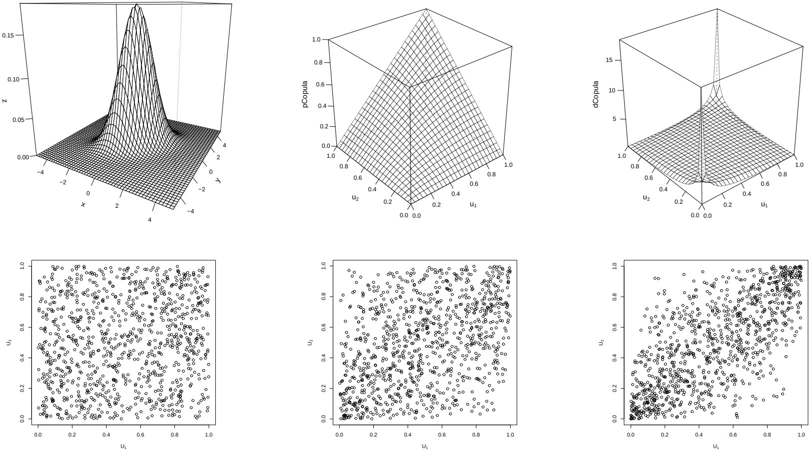


Figure: (Top) Density of the bivariate normal df with $\rho = 0.5$ (left), perspective plot of C_ρ^{Ga} (middle), and corresponding copula density c_ρ^{Ga} (right). (Bottom) Sample of size 1000 from C_ρ^{Ga} with $\rho = 0.1, 0.5, 0.7$ (from left to right).

t copulas

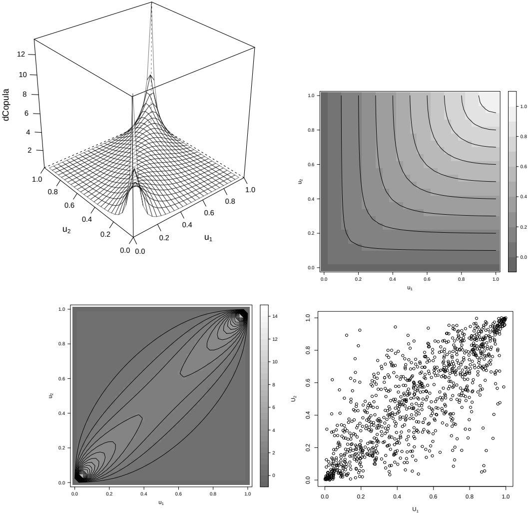
The d-dimensional t copula (family) $C^t_{P,\rho}$ arises from Sklar's Theorem applied to the multivariate t distribution, $t_{P,\nu}$, with location vector $\mathbf{0}$, scale matrix P, and $\nu > 0$ degrees of freedom:

$$C^t_{P,\nu}(\mathbf{u}) = t_{P,\nu}(t_\nu^{-1}(u_1), \dots, t_\nu^{-1}(u_d))$$
$$= \int_{-\infty}^{t_\nu^{-1}(u_d)} \cdots \int_{-\infty}^{t_\nu^{-1}(u_1)} \frac{\Gamma((\nu+d)/2)}{\Gamma\left(\frac{\nu}{2}\right)(\pi\nu)^{\frac{d}{2}}\sqrt{\det P}} \left(1 + \frac{\mathbf{x}' P^{-1} \mathbf{x}}{\nu}\right)^{-\frac{\nu+d}{2}} \mathrm{d}x_1 \dots \mathrm{d}x_d$$

where t_ν^{-1} denotes the quantile function of the df t_ν of the univariate Student t distribution with ν degrees of freedom.

- For $d = 2$, $C^t_{-1,\nu}$ is the lower Fréchet-Hoeffding bound W,
- For $d \geq 2$, if P only consists of entries equal to 1, $C^t_{P,\nu}$ is the upper Fréchet–Hoeffding bound M
- $P = I_d$ does not lead to the independence copula

t copulas /2



Wireframe plot of density $c^t_{\rho,\nu}$ for $\rho \approx 0.81$ (Kendall'tau $\tau = 0.6$) and $\nu = 4$ degrees of freedom, contour plots of $C^t_{\rho,\nu}$ (top right) and $c^t_{\rho,\nu}$ (bottom left); scatter plot of a sample of size $n = 1000$ from $C^t_{\rho,\nu}$.
Notice that bivariate t copulas are both radially symmetric and exchangeable (will be discussed later)

Explicit copulas

A number of copula families have simple closed forms.

Some examples:

Gumbel-Hougaard Copula

(d=2) $C_\theta^{Gu}(u_1, u_2) = \exp(-((-log(u_1))^\theta + (-log(u_2))^\theta)^{1/\theta})$

$\theta \geq 1$: $\theta = 1$ gives independence; $\theta \to \infty$ gives comonotonicity

Clayton copula (d=2) $C_\theta^{C}(u_1, u_2) = (u_1^{-\theta} + u_2^{-\theta} - 1)^{-1/\theta}, \theta > 0$

$\theta \to 0$ gives independence; $\theta \to \infty$ gives comonotonicity

Frank copula $C_\theta^{F}(u_1, u_2) = -\frac{1}{\theta} \log\left(1 + \frac{(e^{-\theta u_1} - 1)(e^{-\theta u_2} - 1)}{e^{-\theta} - 1}\right)$

$\theta \to 0$ gives independence; $\theta \to \infty$ gives comonotonicity

Comparison of some copulas

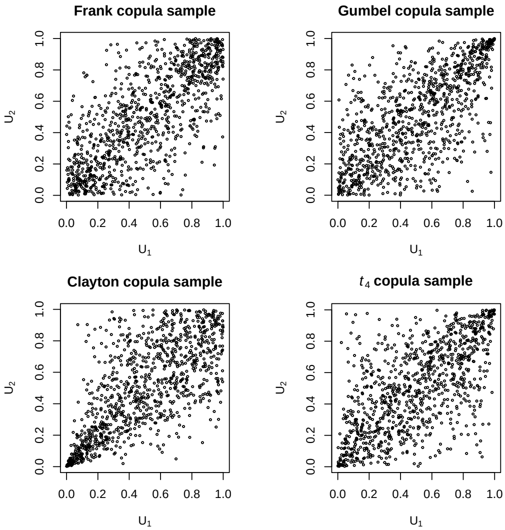
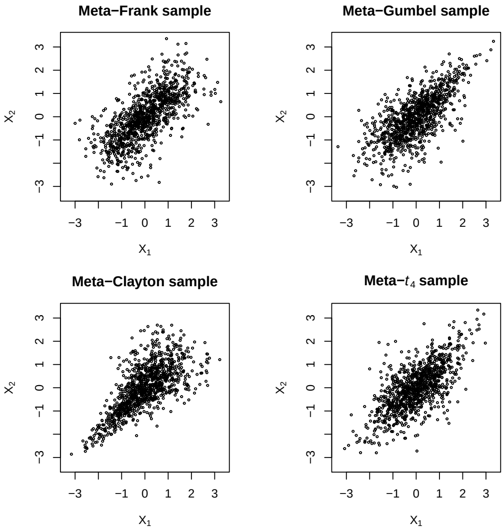


Figure: Copula parameters are chosen such that linear correlation between the (quantile transformed) $N(0,1)$ margins is roughly 0.7

Comparison of some copulas/ 2



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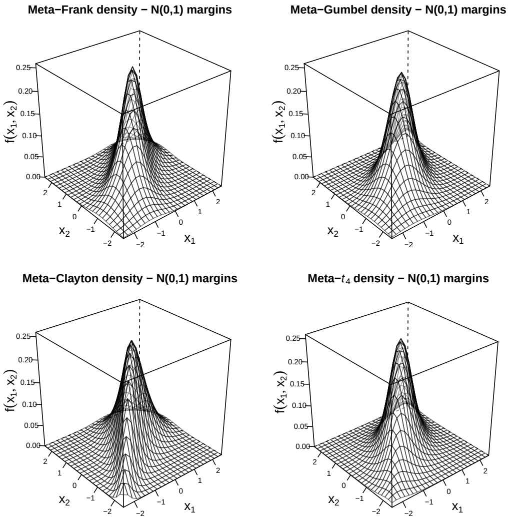
Comparison of some copulas/ 3



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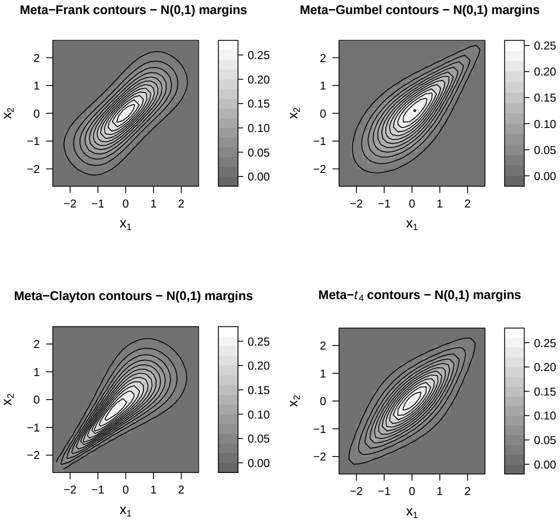
Comparison of some copulas/ 4



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A trivariate example

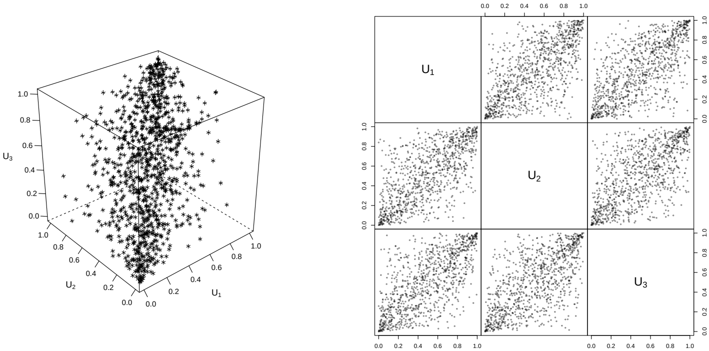

Figure: (Left) 3d cloud plot and (right) scatter-plot matrix of $n = 1000$ independent observations from a trivariate Student t copula with $\nu = 5$ and correlation $\rho \approx 0.71$.

Table of Contents

Survival copula

Let $\mathbf{X}$ be a random vector with multivariate survival function $\bar{F}$, marginal dfs F_i, and marginal survival functions $\bar{F}_i = 1 - F_i$, $i \in \{1, \ldots, d\}$. Then

$$\bar{F}(x_1, \ldots, x_d) = \hat{C}(\bar{F}_1(x_1), \ldots, \bar{F}_d(x_d))$$

where $\hat{C}$ is the *survival copula*.

Remark: Let $\mathbf{x} \in \mathcal{R}$, if $d > 1$, $\bar{F}(\mathbf{x}) \neq 1 - H(\mathbf{x})$ in general.

Survival copula/ 2

Suppose the marginal dfs F_i are continuous and strictly increasing. Then

$$\begin{aligned}\bar{F}(x_1,\dots,x_d) &= P(X_1 > x_1,\dots,X_d > x_d)\\ &= P(1-F_1(X_1) \le \bar{F}_1(x_1),\dots,1-F_d(X_d) \le \bar{F}_d(x_d))\\ &= \hat{C}(\bar{F}_1(x_1),\dots,\bar{F}_d(x_d))\end{aligned}$$

where $\hat{C}$ is the df of $\mathbf{1}-\mathbf{U}$, $\mathbf{U} = (F_1(x_1),\dots,F_d(x_d))$ and $\mathbf{U} \sim C$. A representation of $\hat{C}$ is

$$\hat{C}(u_1,\dots,u_d) = \bar{F}(\bar{F}_1^{-1}(u_1),\dots,\bar{F}_d^{-1}(u_d))$$

Note that $\hat{C}$ is a copula (and thus a df). However, neither $\bar{F}$ nor $\bar{F}_1,\dots,\bar{F}_d$ are dfs.

Survival copula/ 3

For $d = 2$ the following relationships holds:

$$\hat{C}(1 - u_1, 1 - u_2) = 1 - u_1 - u_2 + C(u_1, u_2)$$

Survival copulas should **not** be confused with *survival functions* of copulas. We denote the survival function of a copula C by $\bar{C}$. Then, if $\mathbf{U} \sim C$ and the survival copula of $\mathbf{U}$ is $\hat{C}$ (the df of $1 - \mathbf{U}$), we have

$$\begin{aligned}\bar{C}(u_1, \ldots, u_d) &= P(U_1 > u_1, \ldots, U_d > u_d) \\ &= P(1 - U_1 \leq 1 - u_1, \ldots, 1 - U_d \leq u_d) \\ &= \hat{C}(1 - u_1, \ldots, 1 - u_d)\end{aligned}$$

Survival copula: example

Bivariate Marshall-Olkin copulas as in Eq.(2) were originally constructed as survival copulas of lifetimes of the form

$$X_1 = \min\{Z_1, Z_{12}\} \quad X_2 = \min\{Z_2, Z_{12}\}$$

where Z_1, Z_2, Z_{12} are independent exponential rvs with parameters $\lambda_1, \lambda_2, \lambda_{12}$, respectively, representing arrival times of two individual and one joint fatal shock to a system.
By using the parametrization $\alpha_j = \lambda_j/(\lambda_j + \lambda_{12})$, $j \in \{1, 2\}$, Marshall-Olkin copulas arise as survival copulas of (X_1, X_2).

Sampling from survival copulas

R code: *Sampling from a Clayton and a survival Clayton copula with parameter* $\theta = 2$

```
> set.seed(332)
> cop<-claytonCopula(2)
> U<-rCopula(1000, copula=cop)
> V<-1-U #sample from the survival Clayton copula
> plot(U, xlab = quote(U[1]), ylab = quote(U[2]))
> plot(V, xlab = quote(V[1]), ylab = quote(V[2]))
> wireframe2(rotCopula(cop), FUN=dCopula, delta=0.025)
```

Survival Clayton copula

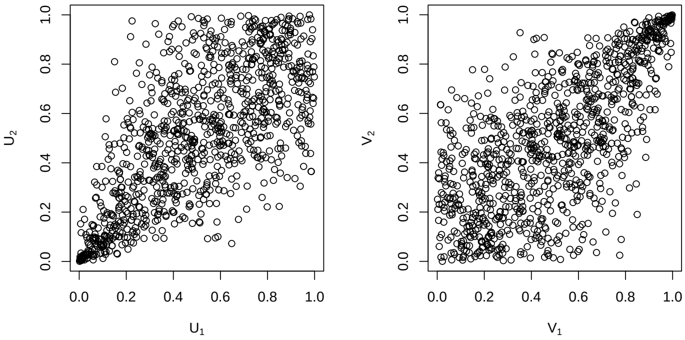

Figure: Scatter plots of a Clayton copula C_θ^C (left) and survival Clayton copula $\hat{C}_\theta^C$ (right) with parameter $\theta = 2$.

Radial Symmetry and Exchangeability

1. A random vector $\mathbf{X}$ is called radially symmetric about $\mathbf{a} \in \mathbb{R}^d$ if $\mathbf{X} - \mathbf{a} \stackrel{d}{=} \mathbf{a} - \mathbf{X}$, that is, if $\mathbf{X} - \mathbf{a}$ and $\mathbf{a} - \mathbf{X}$ are equal in distribution

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   If $C(u_{j1}, \ldots, u_{jd}) = C(u_1, \ldots, u_d)$ for all $u_1, \ldots, u_d \in [0, 1]$ and all permutations $(j_1, \ldots, j_d)$ of $\{1, \ldots, , d\}$, we call C *exchangeable*

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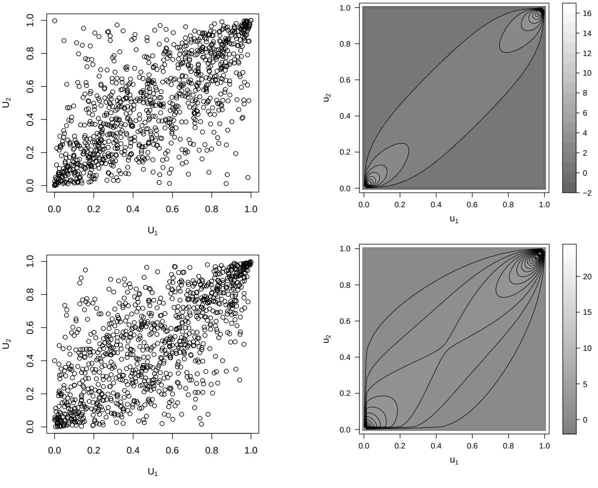
- elliptical distributions are radially symmetric
- $W_{d=2}$, Π, and M are both radially symmetric and exchangeable

Visually Assessing Symmetry and Exchangeability



Bivariate t copulas ($\rho = 0.7, \nu = 3.5$) are both radially symmetric (symmetry wrt the point $(1/2, 1/2)$) and exchangeable;

The copulas in the Gumbel-Hougaard family (here $\theta = 2$) are exchangeable (symmetry of the density with respect to the main diagonal) but not radially symmetric

Conditional distributions of copulas

Suppose $(U_1, U_2) \sim C$. Recall that a copula is an increasing continuous function in each argument. Hence

$$\begin{aligned} C_{U_2|U_1}(u_2|u_1) &= P(U_2 \leq u_2 | U_1 = u_1) \\ &= \lim_{\delta \to 0} \frac{C(u_1 + \delta, u_2) - C(u_1, u_2)}{\delta} = \frac{\partial}{\partial u_1} C(u_1, u_2) \end{aligned}$$

(see Nelsen (2006)). The conditional distribution $C_{U_2|U_1}(u_2|u_1)$ is a df on $[0, 1]$ which is uniform only in the case $C = \Pi$.

Interpretation in Risk management. (X_1, X_2) is a pair of two continuous risks having (unique) copula C. Then

$$\begin{aligned} 1 - C_{U_2|U_1}(q|p) &= 1 - P(U_2 \leq q | U_1 = p) \\ &= P(U_2 > q | U_1 = p) \\ &= P(X_2 > F_2^{-1}(q) | X_1 = F_1^{-1}(p)) \end{aligned}$$

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