UNIVERSITÀ
DEGLI STUDI DI TRIESTE

Master degree course in Mathematics
Advanced Geometry 3 - Program of the course
a.y. 2021/22
Prof.ssa Emilia Mezzetti

Affine and projective varieties
Basic facts on affine and projective space. Affine varieties. Zariski topology.
Graded rings, homogeneous ideals. Projective zeros of a polynomial. Projective varieties and Zariski topology on the projective space.
Hypersurfaces, product of affine spaces, Segre quadric, embedding of affine space in projective space.
Ideal of a subset of the affine or projective space. The Hilbert's Nullstellensatz. Maximal ideals in a polynomial ring. Projective version of the Nullstellensatz. Irrelevant homogeneous ideals.
Integral elements over a ring. Transitivity of integral dependence. Noether's Normalization Lemma. Projective closure, its ideal. The twisted cubic.

Topology of algebraic varieties
Irreducible topological spaces and their properties, characterization of irreducible algebraic sets in terms of prime ideals. Noetherian topological spaces, irreducible components.
Topological dimension. Dimension of K-algebras. Theorems of Cohen-Seidenberg (without proof). Chains of prime ideals of a finitely generated K-algebra. Coordinate ring of an affine variety, its Krull dimension, the relation with the transcendence degree of its quotient field. Dimension of hypersurfaces.
Locally closed sets, quasi-projective varieties.

Regular and rational functions and maps
Regular functions on a locally closed set. The K-algebra O(X), its characterization in the affine case. Rational functions. The field of rational functions K(X). Local ring of a point. Its algebraic characterization.
Regular maps (morphisms), comorphism, isomorphisms. Bijection between Hom(X,Y) and Hom(O(Y),O(X)) if Y is affine. Comorphism between K(Y) and K(X) if f is dominant. Characterization of regular maps as maps locally defined by homogeneous polynomials of the same degree. Stereographic projection, affinities, projectivities, projectively equivalent varieties. Special affine open subsets. Affine varieties as quasi-projective varieties isomorphic to a closed subset in an affine space. A^2-{O} is not affine. Every quasi-projective variety admits an open covering with affine varieties. The Veronese maps and the Veronese varieties. Geometrical properties of the Veronese surface.
Categories and functors. Natural transformations. Isomorphisms of categories. Isomorphism between the category of affine varieties and regular maps, and the category of finitely generated reduced K-algebras (K algebraically closed).
Rational maps. Birational equivalence. Projections. Rational and unirational varieties. Quadric surfaces are rational, explicit construction of a birational map by projection. Quadratic transformations. X-V(F), with X projective variety, is an affine variety.
Product of projective spaces, Segre varieties, open subsets, dimension and rationality, product of projective and quasi-projective varieties. Coordinate-free description in terms of tensor product of vector spaces.

Dimension of an intersection

Theorem on the dimension of an intersection. Expected intersection dimension. Intersection of projective varieties is non-empty if the expected dimension is non-negative. Complete intersection varieties. Set-theoretical complete intersections. The Krull principal ideal theorem (without proof).

Complete varieties

Complete varieties; the case of affine varieties. Completeness of projective varieties. Regular functions on a complete variety are constant. Quasi-projective varieties isomorphic to a projective variety are projective.

Tangent space and smoothness

Embedded tangent space to an affine variety at a point. Differential of regular functions.
Zariski tangent space. Jacobian matrix at a point and generic Jacobian matrix. Smooth and singular points. Dimension of the tangent space: case of a hypersurface. Every variety is birationally equivalent to a hypersurface. Equations of the embedded tangent space in the projective case. The search for singular points. Euler relation. Characterization of the local rings of smooth points. Tangent cone at a point of an affine variety.

Finite morphisms, blow-ups

Finite morphisms. Geometrical interpretation of Normalization lemma. Blow-up of the affine space $\mathbf{A}^m$ at the origin, exceptional divisor, total and strict transform of a subvariety; resolution of the singularity of some curves.

Reference books

The main references are:

I.R. Shafarevich: Basic Algebraic Geometry 1: Varieties in Projective Space, Second or Third edition. Springer-Verlag

R. Hartshorne: Algebraic geometry, Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977. (First chapter)

Other reference books quoted in the lecture notes:

M.F. Atiyah - I.G. MacDonald, Introduction to Commutative Algebra, Addison-Wesley, 1969

S.D. Cutkosky, Introduction to Algebraic Geometry, Graduate Studies in Mathematics 188, AMS 2018

W. Fulton, Algebraic curves, http://www.math.lsa.umich.edu/ wfulton/

J. Harris: Algebraic geometry. A first course, Graduate Texts in Mathematics, 133, Springer-Verlag, New York, 1995.

E. Kunz, Introduction to Commutative Algebra and Algebraic Geometry, Birkhauser, 1985

S. Lang, Algebra, 3rd ed. revised, Springer, 2005

C. Peskine, An algebraic introduction to Complex Projective Geometry, Cambridge University Press, 2009

M. Reid, Undergraduate Algebraic Geometry, Cambridge University Press, 1989

O. Zariski – P. Samuel, Commutative Algebra, I ed. Van Nostrand, 1958; Springer, 1975

The notes of the lectures of Prof.ssa Mezzetti can be downloaded from Moodle: https://moodle2.units.it/