5.1 INTERPRETING PARAMETERS IN LOGISTIC REGRESSION

For a binary response variable Y and an explanatory variable X, let $\pi(x) = P(Y = 1 | X = x) = 1 - P(Y = 0 | X = x)$. The logistic regression model is

$$\pi(x) = \frac{\exp(\alpha + \beta x)}{1 + \exp(\alpha + \beta x)}. \tag{5.1}$$

Equivalently, the log odds, called the *logit*, has the linear relationship

$$\text{logit}[\pi(x)] = \log \frac{\pi(x)}{1 - \pi(x)} = \alpha + \beta x. \tag{5.2}$$

This equates the logit link function to the linear predictor.

5.1.1 Interpreting β: Odds, Probabilities, and Linear Approximations

How can we interpret β in (5.2)? Its sign determines whether $\pi(x)$ is increasing or decreasing as x increases. The rate of climb or descent increases as $|\beta|$ increases; as $\beta \to 0$ the curve flattens to a horizontal straight line. When $\beta = 0$, Y is independent of X. For quantitative x with $\beta > 0$, the curve for $\pi(x)$ has the shape of the cdf of the logistic distribution (recall Section 4.2.5). Since the logistic density is symmetric, $\pi(x)$ approaches 1 at the same rate that it approaches 0.

Exponentiating both sides of (5.2) shows that the odds are an exponential function of x. This provides a basic interpretation for the magnitude of β: The odds increase multiplicatively by e^{β} for every 1-unit increase in x. In other words, e^{β} is an odds ratio, the odds at $X = x + 1$ divided by the odds at $X = x$.

Most scientists are not familiar with odds or logits, so the interpretation of a multiplicative effect of e^{β} on the odds scale or an additive effect of β on the logit scale is not helpful to them. A simpler, although approximate slope interpretation uses a linearization argument (Berkson 1951). Since it has a curved rather than a linear appearance, the logistic regression function (5.1) implies that the rate of change in $\pi(x)$ per unit change in x varies. A straight line drawn tangent to the curve at a particular x value, shown in Figure 5.1, describes the rate of change at that point. Calculating $\partial\pi(x)/\partial x$ using (5.1) yields a fairly complex function of the parameters and x, but it simplifies to the form $\beta\pi(x)[1 - \pi(x)]$.

For instance, the line tangent to the curve at x for which $\pi(x) = \frac{1}{2}$ has slope $\beta(\frac{1}{2})(\frac{1}{2}) = \beta/4$; when $\pi(x) = 0.9$ or 0.1, it has slope 0.09β. The slope approaches 0 as $\pi(x)$ approaches 1.0 or 0. The steepest slope occurs at x for which $\pi(x) = \frac{1}{2}$; that x value is $x = -\alpha/\beta$. [To check that $\pi(x) = \frac{1}{2}$ at this

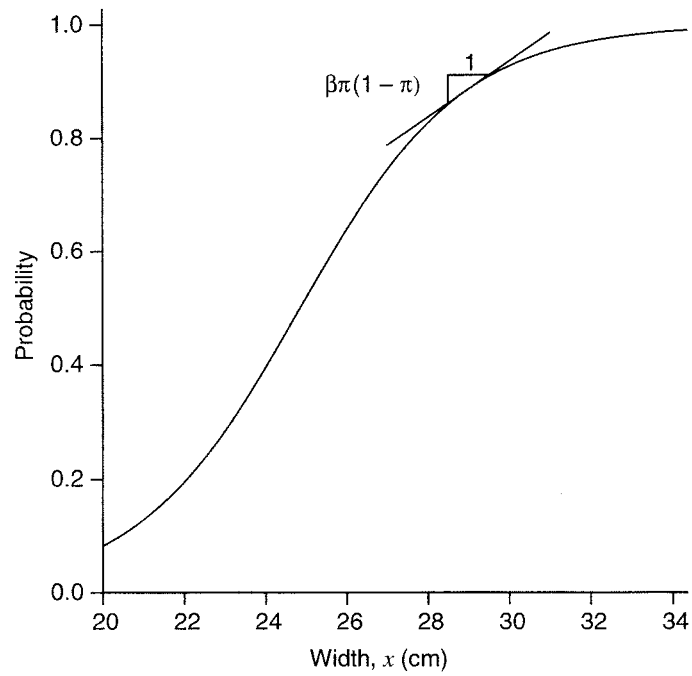

FIGURE 5.1 Linear approximation to logistic regression curve.

point, substitute $-\alpha/\beta$ for x in (5.1), or substitute $\pi(x) = \frac{1}{2}$ in (5.2) and solve for x.] This x value is sometimes called the *median effective level* and denoted EL_{50}. In toxicology studies it is called LD_{50} (LD = lethal dose), the dose with a 50% chance of a lethal result.

From this linear approximation, near x where $\pi(x) = \frac{1}{2}$, a change in x of $1/\beta$ corresponds to a change in $\pi(x)$ of roughly $(1/\beta)(\beta/4) = \frac{1}{4}$; that is, $1/\beta$ approximates the distance between x values where $\pi(x) = 0.25$ or 0.75 (in reality, 0.27 and 0.73) and where $\pi(x) = 0.50$. The linear approximation works better for smaller changes in x, however.

An alternative way to interpret the effect reports the values of $\pi(x)$ at certain x values, such as their quartiles. This entails substituting those quartiles for x into formula (5.1) for $\pi(x)$. The change in $\pi(x)$ over the middle half of x values, from the lower quartile to the upper quartile of x, then describes the effect. It can be compared to the corresponding change over the middle half of values of other predictors.

The intercept parameter α is not usually of particular interest. However, by centering the predictor about 0 [i.e., replacing x by $(x - \bar{x})$], α becomes the logit at that mean, and thus $e^{\alpha}/(1 + e^{\alpha}) = \pi(\bar{x})$. (As in ordinary regression, centering is also helpful in complex models containing quadratic or interaction terms to reduce correlations among model parameter estimates.)