Dependence modeling with copulas
Part II

Roberta Pappadà

June 6, 2022

Contact: rpappada@units.it

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Background

Numerical summaries of (aspects of) dependence are known as *measures of association* and are mostly studied in the bivariate case (for extensions to higher dimensions, see, e.g., Jaworski et al. (2010).

For a pair of rvs (X_1, X_2) we quantify the dependence by means of

- the usual linear Pearson's correlation coefficient

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- Copula-based measures $\begin{cases} \text{rank correlation coefficients} \\ \text{tail-dependence coefficients} \end{cases}$

Properties of Linear Correlation

Let (X_1, X_2) be a random vector whose components have finite variances. Then,

1. $\mathrm{Cor}(X_1, X_2) \in [-1, 1]$
2. $|\mathrm{Cor}(X_1, X_2)| = 1$ if and only if there exist $a, b \in \mathbb{R}$, $a \neq 0$, such that $X_2 = aX_1 + b$ almost surely (X_1 and X_2 are perfectly linearly dependent)
3. If X_1 and X_2 are independent, then $\mathrm{Cor}(X_1, X_2) = 0$
4. For any $a_1, a_2 > 0$, or any $a_1, a_2 < 0$, and for any $b_1, b_2 \in \mathbb{R}$,

$$\mathrm{Cor}(a_1X_1 + b_1, a_2X_2 + b_2) = \mathrm{Cor}(X_1, X_2)$$

   In particular, Pearson's correlation coefficient is invariant under strictly increasing linear transformations.

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Understanding the limitations of correlation

Fallacy 1 (Existence) $\text{Cor}(X_1, X_2)$ exists for every random vector (X_1, X_2)

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Fallacy 2 (Invariance) $\mathrm{Cor}(X_1, X_2)$ is invariant under strictly increasing transformations on $\mathrm{ran} X_1$ or $\mathrm{ran} X_2$.

Counterexample to Fallacies 1 and 2. If $X_1, X_2 \sim F(x) = 1 - x^{-3}, x \geq 1$ and X_1, X_2 ind., then $\mathrm{Cor}(X_1, X_2) = 0$ but $\mathrm{Cor}(X_1^3, X_2)$ does not exist since neither $E((X_1^3)^2)$ nor $E(X_1^3)$ are finite.

Understanding the limitations of correlation/ 2

Fallacy 3 (Uniqueness) The marginal distributions and the correlation coefficient uniquely determine the joint distribution

Fallacy 4 (Uncorrelatedness Implies Independence) $\mathrm{Cor}(X_1, X_2) = 0$ implies that X_1 and X_2 are independent.

Counterexamples to Fallacies 3 and 4. Fallacy 4 alone can be easily falsified by taking $X_1 \sim N(0, 1)$ and $X_2 = X_1^2$; then $\mathrm{Cor}(X_1, X_2) = 0$ but X_1 and X_2 are dependent.

A counterexample to both Fallacies 3 and 4 can be constructed by a mixture of the two Fréchet-Hoeffding bounds W and M.

Understanding the limitations of correlation/ 3

Model 1: $\mathbf{X} = (X_1, X_2) \sim N_2(\mathbf{0}, I_2)$, i.e., (X_1, X_2) has $N(0,1)$ margins and zero correlation.
Model 2: $(Y_1, Y_2) = (X_1, VX_1)$, with X_1 as in Model 1 and V and independent discrete rv such that $P(V = 1) = P(V = -1) = 1/2$. Hence

$$\mathrm{Cor}(Y_1, Y_2) = \mathrm{Cov}(Y_1, Y_2) = \mathrm{E}(Y_1 Y_2) = \mathrm{E}(VX_1^2) = \mathrm{E}(V)\mathrm{E}(X_1^2) = 0$$

Conditional on $V = -1$ (respectively, $V = 1$), the copula C of (Y_1, Y_2) is the countermonotonicity W (respectively, the comonotonicity copula M):

$$C(u_1, u_2) = 0.5 \max(u_1 + u_2 - 1, 0) + (1 - 0,5) \min(u_1, u_2)$$

which is a *mixture* of the two-dimensional Fréchet-Hoeffding bounds.

Uncorrelatedness Versus Independence

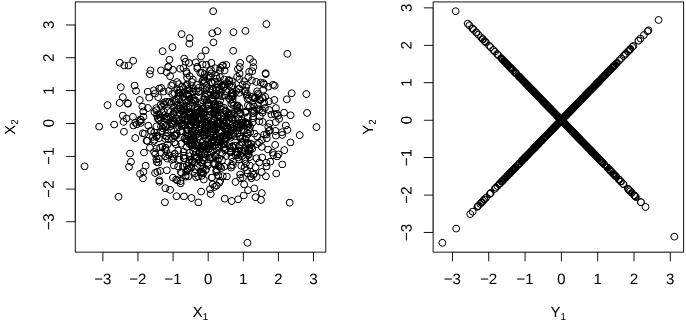

Figure: $n = 1000$ independent realizations from (X_1, X_2), whose copula is the independence copula (left) and (Y_1, Y_2), whose copula is a mixture between the Fréchet–Hoeffding bounds W and M (right); both have $N(0, 1)$ margins and zero correlation.

Uncorrelatedness Versus Independence/ 2

Assume X_1, X_2 from Model 1 and Y_1, Y_2 from Model 2 are losses, with $N(0,1)$ margins and zero correlation. it can be proved that for $\alpha > 0.75$, $VaR_\alpha(X_1 + X_2) = \sqrt{2}\Phi^{-1}(\alpha)$; $VaR_\alpha(Y_1 + Y_2) = 2\Phi^{-1}(2\alpha - 1)$

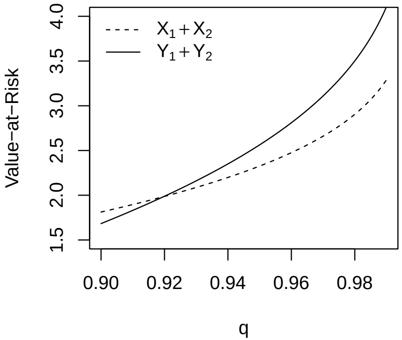

Figure: VaR for the risks $X_1 + X_2$ and $Y_1 + Y_2$. The VaR of a sum of risks is not determined by marginal distributions and pairwise correlations.

Attainable Correlations

Fallacy 5 (Attainable Correlations) Given margins F_1, F_2, all $\mathrm{Cor}(X_1, X_2) \in [-1, 1]$ can be attained by choosing a suitable copula for (X_1, X_2).

Counterexample to Fallacy 5. Let $\ln X_1 \sim N(0, 1)$ and $\ln X_2 \sim N(0, \sigma^2)$. Hence, (X_1, X_2) has lognormally distributed margins. Let $Z \sim N(0, 1)$. Then

- X_1,X_2 comonotonic: $(X_1, X_2) \overset{d}{=} (e^Z, e^{\sigma Z})$
  $$\mathrm{Cor}(X_1, X_2) = \rho_{\max} = \frac{e^{\sigma} - 1}{\sqrt{(e-1)(e^{\sigma^2} - 1)}}$$
- X_1,X_2 countermonotonic: $(X_1, X_2) \overset{d}{=} (e^Z, e^{-\sigma Z})$
  $$\mathrm{Cor}(X_1, X_2) = \rho_{\min} = \frac{e^{-\sigma} - 1}{\sqrt{(e-1)(e^{\sigma^2} - 1)}}$$

Attainable Correlations/ 2

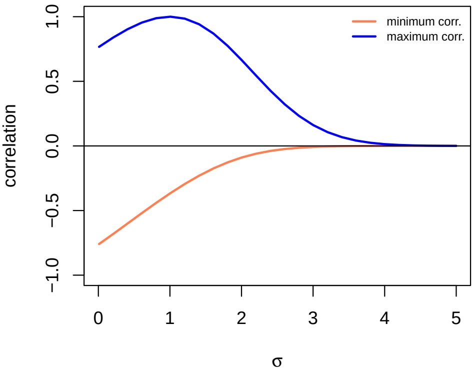

Figure: Correlation bounds for lognormal rvs X_1 and X_2 where $\ln X_1 \sim N(0,1)$ and $\ln X_2 \sim N(0,\sigma^2)$

Attainable Correlations: Remark

Let $\alpha \in (0,1)$. For comonotonic rvs $X_1, \ldots, X_d$ the *comonotone additivity of quantiles* holds:

$$F^{\leftarrow}_{X_1+X_2+\ldots X_d}(\alpha) = F^{\leftarrow}_{X_1}(\alpha) + \cdots + F^{\leftarrow}_{X_d}(\alpha)$$

Assume $d = 2$. In a **superadditive** VaR case we have

$$F^{\leftarrow}_{X_1+X_2}(\alpha) > F^{\leftarrow}_{X_1}(\alpha) + F^{\leftarrow}_{X_2}(\alpha)$$

for some level $\alpha \in (0,1)$. Thus

$$F^{\leftarrow}_{X_1}(\alpha) + F^{\leftarrow}_{X_2}(\alpha) = F^{\leftarrow}_{Y_1+Y_2}(\alpha)$$

for comonotonic rvs (Y_1, Y_2), $Y_1 \stackrel{d}{=} X_1$ and $Y_2 \stackrel{d}{=} X_2$. Since the maximal correlation ρ_{max} is attained if and only if two rvs are comonotonic, (Y_1, Y_2) will attain $\rho_{\text{max}} > \text{Cor}(X_1, X_2)$, but $VaR_{X_1+X_2} > VaR_{Y_1+Y_2}$.

Linear correlation: Summary

The main limitations of the linear correlation coefficient are:

1. $\text{Cor}(X_1, X_2)$ does not exist for all random vectors (X_1, X_2) (only for those with finite second moments);
2. $\text{Cor}(X_1, X_2)$ depends on the marginal dfs of (X_1, X_2) even when the latter are continuous
3. $\text{Cor}(X_1, X_2)$ is invariant only under strictly increasing linear transformations (not under strictly increasing transformations in general)

By only depending on the underlying copula C in the case of continuous random vectors, *rank correlation coefficients* overcome several of the aforementioned issues.

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Rank correlation measures

Definition (Spearman's Rho, Kendall's Tau)

Let (X_1, X_2) be a bivariate random vector with continuous marginal dfs F_1 and F_2.

1. The *(population version of) Spearman's rho* is defined by

$$\rho_S = \rho_S(X_1, X_2) = \mathrm{Cor}(F_1(X_1), F_2(X_2))$$

2. Let (X_1', X_2') be an independent copy (same distribution) of (X_1, X_2) The *(population version of) Kendall's tau* is defined by

$$\tau = \tau(X_1, X_2) = \mathrm{E}(\mathrm{sign}((X_1 - X_1')(X_2 - X_2')))$$

where $\mathrm{sign}(x) = I_{\{x>0\}} - I_{\{x<0\}}$

Remark: ρ_S measures dependence independently of the margins and it is invariant under strictly increasing transformations of X_1 and X_2

ρ_S and τ as measures of concordance

Both Spearman's Rho and Kendall's Tau can be defined in terms of the *concordance* and *discordance* of (X_1, X_2) and an independent copy (X_1', X_2').

Kendall's tau can be written as the probability of concordance minus the probability of discordance

$$\tau = P((X_1 - X_1')(X_2 - X_2') > 0) - P((X_1 - X_1')(X_2 - X_2') < 0)$$

For Spearman's Rho, if $X_1' \stackrel{d}{=} X_1$ and $X_2' \stackrel{d}{=} X_2$, and (X_1, X_2), X_1', X_2' are all independent, then

$$\rho_S = 3[P((X_1 - X_1')(X_2 - X_2') > 0) - P((X_1 - X_1')(X_2 - X_2') < 0)]$$

Representation of ρ_S and τ

Let (X_1, X_2) be a bivariate random vector with continuous marginal dfs and copula C. Then,

$$\begin{aligned} \rho_S = \rho_S(C) =& 12 \int_0^1 \int_0^1 (C(u_1, u_2) - u_1 u_2) \mathrm{d}u_1 \mathrm{d}u_2 \\ =& 12 \int_{[0,1]^2} C(\mathbf{u}) \mathrm{d}\mathbf{u} - 3 \end{aligned} \tag{2}$$

and

$$\tau = \tau(C) = 4 \int_{[0,1]^2} C(\mathbf{u}) \mathrm{d}C(\mathbf{u}) - 1 \tag{3}$$

Rank correlation VS linear correlation

Spearman's rho and Kendall's tau only depend on the underlying copula; they can be viewed as moments of the copula. Moreover,

- they always exist, and are not limited to continuous random vectors with finite second moments (compare with Fallacy 1)
- are invariant under strictly increasing transformations (compare with Fallacy 2)
- can reach any value in $[-1, 1]$ (compare with Fallacy 5); the minimal and maximal values of ρ_S and τ are attained for the lower and upper Fréchet–Hoeffding bounds

Analogs of Fallacies 3 and 4 still apply to rank correlation coefficients.

Method-of-moments estimators

Consider a one-parameter family of copulas $\{C_\theta : \theta \in \Theta\}$, where $\Theta \subseteq \mathbb{R}$. For many such copula families, the functions

$$g_{\rho_S}(\theta) = \rho_S(C_\theta) \quad \text{and} \quad g_\tau(\theta) = \tau(C_\theta), \quad \theta \in \Theta$$

are one-to-one. For example:

- Clayton family: $g_\tau(\theta) = \theta/(\theta+2), \quad \theta \in (0,\infty)$;
- Gumbel-Hougaard family: $g_\tau(\theta) = 1 - 1/\theta, \quad \theta \in [1,\infty)$;
- Normal family: for $\theta \in [-1,1]$
  $g_{\rho_S}(\theta) = (6/\pi)\arcsin(\theta/2); \quad g_\tau(\theta) = (2/\pi)\arcsin\theta$

From $g_{\rho_S}^{-1}$ and g_τ^{-1} one can obtain the unique value of θ corresponding to an admissible value of ρ_S or τ.

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From $g_{\rho_S}^{-1}$ and g_τ^{-1} one can obtain the unique value of θ corresponding to an admissible value of ρ_S or τ.

In the `R` package **copula** the functions ρ_S, $g_{\rho_S}^{-1}$, τ, g_τ^{-1} are `rho()`, `iRho()`, `tau()` and `iTau()`

Rank correlations for Normal Copulas

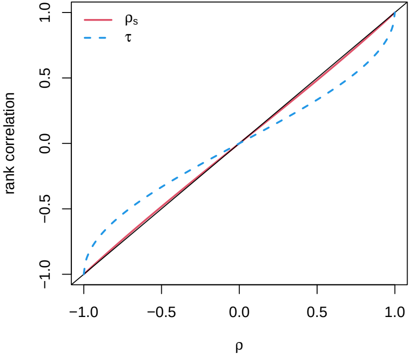

Figure: ρ_S and Kendall's τ as functions of the correlation parameter ρ of a normal copula C^n_ρ. ρ is very well approximated by Spearman's rho in this case. The relationship between τ and ρ holds more generally for other elliptical copulas, such as the t Copula $C^t_{\nu,\rho}$.

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Tail dependence

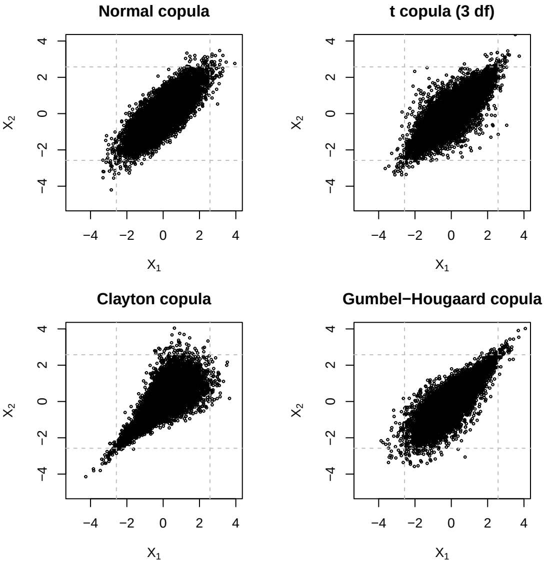
Coefficients of tail dependence aim at summarizing the *extremal dependence*, i.e., the dependence in the (joint) tails of bivariate distributions.

Formally, the coefficients of tail dependence (TD) are limits of conditional probabilities of quantile exceedances (see Nelsen (2006)).

Scatter plots from bivariate distributions with $N(0,1)$ margins and the same Kendall's tau but different copulas can exhibit very different *tail behaviors*.

Tail dependence: example

The plots show generated random samples of size $n = 10000$ from four bivariate distributions in the Fréchet class, with standard normal margins, to investigate how the copula affects the dependence in the tails (Kendall's tau is 0.6 for all of them)



Tail dependence coefficients

Definition (Upper and Lower TDC)

Let (X_1, X_2) be a random vector with marginal dfs F_1 and F_2. Provided that the limits exist, the coefficient of *lower* and *upper tail dependence* of X_1 and X_2 are defined by

1. $\lambda_l = \lambda_l(X_1, X_2) = \lim_{q \to 0^+} P(X_2 \leq F_2^{\leftarrow}(q) | X_1 \leq F_1^{\leftarrow}(q))$
2. $\lambda_u = \lambda_u(X_1, X_2) = \lim_{q \to 1^-} P(X_2 > F_2^{\leftarrow}(q) | X_1 > F_1^{\leftarrow}(q))$

respectively.
If $\lambda_l \in (0, 1]$ (respectively, $\lambda_u \in (0, 1]$), then X_1 and X_2 are said to be lower (respectively, upper) tail dependent; if $\lambda_l = 0$ (respectively, $\lambda_u = 0$) they are *asymptotically independent* in the lower (respectively, upper) tail.

Representation of TDC

If F_1 and F_2 are continuous dfs, then we get simple expressions for λ_l and λ_u in terms of the unique copula C of (X_1, X_2):

$$\lambda_l = \lambda_l(C) = \lim_{q \to 0^+} \frac{P(X_2 \leq F_2^{\leftarrow}(q), X_1 \leq F_1^{\leftarrow}(q))}{P(X_1 \leq F_1^{\leftarrow}(q))} = \lim_{q \to 0^+} \frac{C(q,q)}{q}$$

$$\lambda_u = \lambda_u(C) = \lim_{q \to 1^-} \frac{\hat{C}(1-q, 1-q)}{1-q} = \lim_{q \to 1^-} \frac{1 - 2q + C(q,q)}{1-q}$$

where $\hat{C}$ is the survival copula of C. Hence, for radially symmetric copulas we must have $\lambda_u = \lambda_l$, since $C = \hat{C}$ and $\lambda_u = \lim_{q \to 0^+} \dfrac{\hat{C}(q,q)}{q}$.

TDCs: Examples

If the copula has a simple closed form, calculation of the coefficients of tail dependence λ_l and λ_u is simple:

- A **Clayton copula** with parameter $\theta \in (0, \infty)$ has lower TD:
  $\lambda_l = 2^{-1/\theta}; \quad \lambda_u = 0$
  (Clayton copulas converge to the M as $\theta \to \infty$)
- the **Gumbel copula** with $\theta > 1$ has upper TD
  $\lambda_l = 0; \quad \lambda_u = 2 - 2^{1/\theta}$
  (Gumbel-Hougaard copulas converge to the M as $\theta \to \infty$)
- **Normal copulas** are asymptotically independent in both tails
- the **t-copula** has both upper and lower tail dependence of the same magnitude.

In the `R` package **copula** the function for computing the coefficients of tail dependence is `lambda()`

Example: Tail Dependence of t Copulas

For the t copula $C^t_{\rho,\nu}$ with correlation ρ and degrees of freedom ν, the plots display the graphs of the TDC $\lambda = \lambda_u = \lambda_l$ as a function of ρ and ν

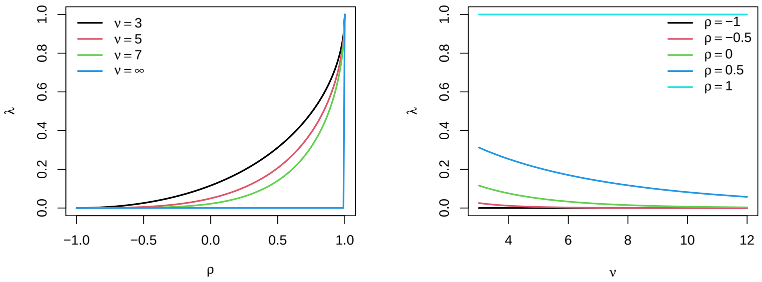


Figure: For fixed, finite ν, tail dependence increases as ρ increases (left). For fixed $|\rho| < 1$, tail dependence increases as ν decreases (right).

Joint quantile exceedances

Looking at joint exceedances of *finite high quantiles* can help to understand the practical consequences of the differences between the extremal behaviours of different models.

Example: Daily returns. Suppose $\mathbf{X} = (X_1, \ldots, X_5)$ represent a vector of five daily negative log-returns with fixed continuous marginal dfs and fixed common pairwise Kendall's tau equal to $1/3$. In addition, suppose that we are unsure whether a normal or a t copula should be used as underlying dependence structure C.

Under the normal copula (with parameter ρ) the probability that, on any day, all five negative log-returns lie above their $u = 0.99$ quantiles is

$$\begin{aligned} P(X_1 > F_1^{\leftarrow}(u), \ldots, X_5 > F_5^{\leftarrow}(u)) &= P(F_1(X_1) > u, \ldots, F_5(X_5) > u) \\ &= C_\rho^n(1-u, \ldots, 1-u), \end{aligned}$$

where the last equality follows by radial symmetry.

Joint quantile exceedances/ 2

Assuming 260 trading days in a year, his calculation can be carried out using the following code

R code:

```
>set.seed(231)
>d<-5
>rho<-iTau(normalCopula(), tau=1/3) #0.5
>u<-0.99
>prob<-pCopula(rep(1-u, d),copula=normalCopula(rho,
dim=d))
>1/(260*prob) # 51.42 years
```

Hence, the event of joint exceedances above the 99% quantile for the five daily negative log-returns happens about once every 51.42 years.

Joint quantile exceedances/ 3

If the copula of $\mathbf{X}$ is assume to be a 5-dimensional t copula $C^t_{0.5,3}$, such an event will happen approximately 9.31 times more often (roughly once every 5.63 years)

R code:

```
> prob.t<-pCopula(rep(1-u, d), copula=tCopula(rho, dim=d,
df=3))
> 1/(260*prob.t)
[1] 5.625567
```

Jaworski, P., Durante, F., Hardle, W. K., and Rychlik, T. (2010). *Copula theory and its applications*, volume 198. Springer.

Nelsen, R. B. (2006). *An Introduction to Copulas*. Springer Series in Statistics. Springer, New York, second edition.