Advanced Geometry 3

Notes of the course

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Contents

Chapter 1

Affine algebraic sets and Zariski topology.

1.1 Introduction

The aim of this course is to introduce the notion of algebraic variety in the classical sense, over a field K.

Roughly speaking, algebraic varieties are sets of solutions of a system of algebraic equations, i.e. equations given by polynomials. The natural space where to look at these solutions seems to be the affine space, but one realizes that the projective ambient is more convenient. On one hand the projective space extends the affine space and includes it naturally, on the other hand the projective ambient allows to prove more general and complete results.

After introducing the notions of affine and projective varieties, we will study the notion of dimension. Then we will introduce two kinds of transformations of algebraic varieties: regular and rational maps. They give rise to two types of equivalence or isomorphism: biregular isomorphism and birational equivalence, and therefore to two classification problems.

In this course we will see many examples of varieties, and of regular and rational maps. In particular we will see some classes of varieties related to the notion of tensor (without symmetries, symmetric, skew-symmetric); they are much studied because of many recent applications in fields as control theory, signal transmission, etc. We will see also examples of rational and unirational varieties, hopefully this will give a taste of the modern classification problems. We will then study the notions of tangent space, and of smoothness.

Classical algebraic geometry is the basis and gives the motivations for modern algebraic geometry: from schemes, introduced by Grothendieck in the sixties of last century, to the stacks, due to Mumford and Artin. All these notions are strongly based on commutative algebra, i.e. the theory of commutative rings, in particular polynomial rings and their quotients, local rings, and homological algebra.

The reference books I've chosen, all of which have become classics, have different flavours: the book [S] of Šafarevič is complete and precise, and contains almost all algebraic notions needed; Harris' book [jH] has a more geometric flavour, proofs are not complete but there are many many examples and ideas; Harthshorne's book [rH], the "Bible" of algebraic geometry since its appearance, treats classical varieties quickly in the first chapter, then moves to modern language, but always with an eye to classical problems. Further bibliographic references will be recommended later, for particular topics or as in-depth reading.

Notation. The ideal generated by a set S will be denoted by $\langle S\rangle$.

1.2 Affine and projective spaces.

In this first section, we begin by fixing the ambient in which we will work: the affine and the projective space over any field K. In particular we recall some basic facts about the projective space.

Let K be a field. For us the *affine space* of dimension n over K will simply be the set K^n : on it, the additive group of K^n acts naturally by translation. The affine space will be denoted by $\mathbb{A}^n_K$ or simply $\mathbb{A}^n$. So the points of $\mathbb{A}^n_K$ are $n-$tuples $(a_1,\dots,a_n)$, where $a_i\in K$ for $i=1,\dots,n$.

Let V be a $K-$vector space of dimension $n+1$. Let $V^*=V\setminus\{0\}$ be the subset of non–zero vectors. The following relation in V^* is an equivalence relation (relation of proportionality): $v\sim v'$ if and only if $\exists\lambda\neq 0,\lambda\in K$, such that $v'=\lambda v$.

The quotient set $V^*/\!\sim$ is called the *projective space* associated to V and is denoted by $\mathbb{P}(V)$. The points of $\mathbb{P}(V)$ are the lines in V (through the origin) deprived of the origin. In particular, $\mathbb{P}(K^{n+1})$ is denoted by $\mathbb{P}^n_K$ (or simply $\mathbb{P}^n$) and called the *numerical projective n-space*. By definition, the dimension of $\mathbb{P}(V)$ is equal to $\dim V-1$.

There is a canonical surjection $p:V^*\to\mathbb{P}(V)$ which maps a vector v to its equivalence class $[v]$. If $(x_0,\dots,x_n)\in(K^{n+1})^*$, we will denote the corresponding point of

$\mathbb{P}^n$ by $[x_0, \ldots, x_n]$. Another notation, used for instance in [S], is $(x_0 : \ldots : x_n)$. So $[x_0, \ldots, x_n]$=$[x'_0, \ldots, x'_n]$ if and only if $\exists \lambda \in K^*$ such that $x'_0 = \lambda x_0, \ldots, x'_n = \lambda x_n$.

If we fix a basis $e_0, \ldots, e_n$ of V, then there is an associated system of *homogeneous coordinates* in V, in the following way: if $v = x_0 e_0 + \ldots + x_n e_n$, then $x_0, \ldots, x_n$ are called homogeneous coordinates of the corresponding point $P =[v]= p(v)$ in $\mathbb{P}(V)$. We also write $P[x_0, \ldots, x_n]$. Note that homogeneous coordinates of a point P are not uniquely determined by P, but are defined only up to multiplication by a non–zero constant. If $\dim V = n+1$, a system of homogeneous coordinates allows to define a *bijection*

$$\mathbb{P}(V) \longrightarrow \mathbb{P}^n$$

$$P = [v] \longrightarrow [x_0, \ldots, x_n]$$

where $v = x_0 e_0 + \ldots + x_n e_n$.

The points $E_0[1, 0, \ldots, 0], \ldots, E_n[0, 0, \ldots, 1]$ are called fundamental points, and $U[1, \ldots, 1]$ unit point of the given system of coordinates.

A *projective* (or *linear*) *subspace* of $\mathbb{P}(V)$ is a subset of the form $\mathbb{P}(W)$, where $W \subset V$ is a vector subspace of V.

If W, U are vector subspaces of V, the following *Grassmann relation* holds:

$$\dim U + \dim W = \dim(U \cap W) + \dim(U + W).$$

From this relation, observing that $\mathbb{P}(U \cap W) = \mathbb{P}(U) \cap \mathbb{P}(W)$, we get in $\mathbb{P}(V)$:

$$\dim \mathbb{P}(U) + \dim \mathbb{P}(W) = \dim(\mathbb{P}(U) \cap \mathbb{P}(W)) + \dim \mathbb{P}(U + W).$$

Note that $\mathbb{P}(U+W)$ is the minimal linear subspace of $\mathbb{P}(V)$ containing both $\mathbb{P}(U)$ and $\mathbb{P}(W)$: it is denoted $\mathbb{P}(U) + \mathbb{P}(W)$.

Example 1.2.1. *Let $V = K^3$, $\mathbb{P}(V) = \mathbb{P}^2$, $U, W \subset K^3$ subspaces of dimension 2. Then $\mathbb{P}(U), \mathbb{P}(V)$ are lines in the projective plane. There are two cases:*

(i) $U = W = U + W = U \cap W$;

(ii) $U \neq W$, $\dim U \cap W = 1$, $U + W = K^3$.

In case (i) the two lines in $\mathbb{P}^2$ coincide; in case (ii) $\mathbb{P}(U) \cap \mathbb{P}(W) = \mathbb{P}(U \cap W) = [v]$, if $v \neq 0$ is a vector generating $U \cap W$. Observe that never $\mathbb{P}(U) \cap \mathbb{P}(W) = \emptyset$.

What are the possible reciprocal positions of two lines in $\mathbb{P}^3$? Of two planes? Of a line and a plane?

Let $T \subset \mathbb{P}(V)$ be a non–empty set. The linear span $\langle T \rangle$ of T is the intersection of the projective subspaces of $\mathbb{P}(V)$ containing T, i.e. the minimum subspace containing T.

For example, assume that $T = \{P_1, \ldots, P_t\}$ is a finite set, and that $v_1, \ldots, v_t$ are vectors such that $P_1 = [v_1], \ldots, P_t = [v_t]$. Then $\langle P_1, \ldots, P_t \rangle = \mathbb{P}(W)$, where W is the vector subspace of V generated by $v_1, \ldots, v_t$.

So $\dim\langle P_1, \ldots, P_t \rangle \leq t-1$ and equality holds if and only if $v_1, \ldots, v_t$ are linearly independent; in this case, also the points $P_1, \ldots, P_t$ are called *linearly independent*. In particular, if $t = 2$, two points are linearly independent if they generate a line; if $t = 3$, three points are linearly independent if they generate a plane, etc. It is clear that, if $P_1, \ldots, P_t$ are linearly independent, then $t \leq n+1$, and any subset of $\{P_1, \ldots, P_t\}$ is formed by linearly independent points.

Definition 1.2.2 (Points in general position in $\mathbb{P}^n$)**.** $P_1, \ldots, P_t$ are said to be *in general position* if either $t \leq n+1$ and they are linearly independent, or $t > n+1$ and they are $n+1$ by $n+1$ linearly independent.

Proposition 1.2.3. *The fundamental points $E_0, \ldots, E_n$ and the unit point U of a system of homogeneous coordinates on $\mathbb{P}^n$ are $n+2$ points in general position. Conversely, if $P_0, \ldots, P_n, P_{n+1}$ are $n+2$ points in general position, then there exists a system of homogeneous coordinates in which $P_0, \ldots, P_n$ are the fundamental points and P_{n+1} is the unit point.*

Proof. The proof is linear algebra. If $e_0, \ldots, e_n$ is a basis, then clearly the $n+1$ vectors $e_0, \ldots, \hat{e_i}, \ldots, e_n, e_0 + \cdots + e_n$ are linearly independent: this proves the first claim. To prove the second claim, we fix vectors $v_0, \ldots, v_{n+1}$ such that $P_i = [v_i]$ for all i. So $v_0, \ldots, v_n$ is a basis and there exist $\lambda_0, \ldots, \lambda_n$ in K such that $v_{n+1} = \lambda_0 v_0 + \cdots + \lambda_n v_n$. The assumption of general position easily implies that $\lambda_0, \ldots, \lambda_n$ are all different from 0, hence $\lambda_0 v_0, \ldots, \lambda_n v_n$ is a new basis such $[\lambda_i v_i] = P_i$ and P_{n+1} is the corresponding unit point. □

1.3 Embedding of the affine space in the projective space

Let a system of homogeneous coordinates be fixed in $\mathbb{P}^n$. We introduce the subspaces $H_0 = \langle E_1, \ldots, E_n \rangle, H_1 = \langle E_0, E_2, \ldots, E_n \rangle, \ldots, H_n = \langle E_0, \ldots, E_{n-1} \rangle$: they are $n+1$ hyperplanes in $\mathbb{P}^n$ (subspaces of codimension 1). Note that H_i is defined by the equation $x_i = 0$. These hyperplanes are called the *fundamental hyperplanes*.

Let $U_i = \mathbb{P}^n \setminus H_i = \{P[x_0, \ldots, x_n] \mid x_i \neq 0\}$. Note that $\mathbb{P}^n = U_0 \cup U_1 \cup \ldots \cup U_n$, because no point in $\mathbb{P}^n$ has all coordinates equal to zero.

There is a map $\varphi_0 : U_0 \longrightarrow \mathbb{A}^n(= K^n)$ defined by

$$\varphi_0([x_0, \ldots, x_n]) = (\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}).$$

φ_0 is bijective and the inverse map is $j_0 : \mathbb{A}^n \longrightarrow U_0$ such that $j_0(y_1, \ldots, y_n) = [1, y_1, \ldots, y_n]$. So φ_0 and j_0 establish a bijection between the affine space $\mathbb{A}^n$ and the subset U_0 of the projective space $\mathbb{P}^n$. Similarly, there are maps φ_i and j_i for any $i = 1, \ldots, n$, that establish bijections between $\mathbb{A}^n$ and U_i. So $\mathbb{P}^n$ is covered by $n+1$ subsets, each one in natural bijection with $\mathbb{A}^n$.

There is a natural way of thinking of $\mathbb{P}^n$ as a completion of $\mathbb{A}^n$; this is done by identifying $\mathbb{A}^n$ with U_i via φ_i, and by interpreting the points of $H_i(= \mathbb{P}^n \setminus U_i)$ as *points at infinity* of $\mathbb{A}^n$, or directions in $\mathbb{A}^n$. We do this explicity for $i = 0$. First of all we identify $\mathbb{A}^n$ with U_0 via φ_0 and j_0. So if $P[a_0, \ldots, a_n] \in \mathbb{P}^n$, either $a_0 \neq 0$ and $P \in \mathbb{A}^n$, or $a_0 = 0$ and $P[0, a_1, \ldots, a_n] \notin \mathbb{A}^n$. Then we consider in $\mathbb{A}^n$ the line L, passing through $O(0, \ldots, 0)$ and of direction given by the vector $(a_1, \ldots, a_n)$. The following are parametric equations of L:

$$\begin{cases} x_1 = & a_1 t \\ x_2 = & a_2 t \\ \ldots & \ldots \\ x_n = & a_n t \end{cases}$$

with $t \in K$. The points of L are identified (via j_0) with the points of U_0 with homogeneous coordinates $x_0, \ldots, x_n$ given by:

$$\begin{cases} x_0 = & 1 \\ x_1 = & a_1 t \\ x_2 = & a_2 t \\ \ldots & \ldots \end{cases}$$

or equivalently, if $t \neq 0$, by:

$$\begin{cases} x_0 = & \frac{1}{t} \\ x_1 = & a_1 \\ x_2 = & a_2 \\ \ldots & \ldots \end{cases}$$

Now, roughly speaking, if t *tends to infinity*, this point “tends” to $P[0, a_1, \ldots, a_n]$. Clearly this is not a rigorous argument, but just a hint to the intuition.

In this way $\mathbb{P}^n$ can be interpreted as $\mathbb{A}^n$ with the points at infinity added, each point at infinity corresponding to one direction in $\mathbb{A}^n$.

Exercises 1.3.1. Let V be a vector space of finite dimension over a field K. Let $\check{V}$ denote the dual of V, i.e. the space of linear forms (or functionals) on V. Prove that $\mathbb{P}(\check{V})$ can be put in bijection with the set of the hyperplanes of $\mathbb{P}(V)$ (hint: the kernel of a non-zero linear form on V is a subvector space of V of codimension one). $\mathbb{P}(\check{V})$ is the *dual projective space.*

1.4 Algebraic sets.

Roughly speaking, algebraic subsets of the affine or of the projective space are sets of solutions of systems of algebraic equations, i.e. common roots of sets of polynomials.

Examples of algebraic sets are: linear subspaces of both the affine and the projective space, plane algebraic curves, quadrics, graphs of polynomials functions, ...

Algebraic geometry is the branch of mathematics which studies algebraic sets (and their generalizations). Our first aim is *to give a formal definition of algebraic sets in the affine space.*

1.4.1 Affine algebraic sets

Let $K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over the field K. If $P(a_1, \ldots, a_n) \in \mathbb{A}^n$, and $F = F(x_1, \ldots, x_n) \in K[x_1, \ldots, x_n]$, we can consider the value of F at P, i.e. $F(P) = F(a_1, \ldots, a_n) \in K$. We say that P is a *zero of F* if $F(P) = 0$.

For example the points $P_1(1,0)$, $P_2(-1,0)$, $P_3(0,1)$ are zeros of $F = x_1^2 + x_2^2 - 1$ over any field. If $G = x_1^2 + x_2^2 + 1$ then G has no zeros in $\mathbb{A}^2_{\mathbb{R}}$, but does have zeros in $\mathbb{A}^2_{\mathbb{C}}$.

Definition 1.4.1. A subset X of $\mathbb{A}^n_K$ is an *affine algebraic set*, or an *affine variety*, if X is the set of common zeros of a family of polynomials of $K[x_1, \ldots, x_n]$.

Remark 1. In some texts the term "variety" is reserved to the affine algebraic sets which are *irreducible*. The notion of irreducible algebraic set will be introduced in Chapter 6.

X is an affine algebraic set means that there exists a subset $S \subset K[x_1, \ldots, x_n]$ such that

$$X = \{P \in \mathbb{A}^n \mid F(P) = 0 \ \forall \ F \in S\}.$$

In this case X is called the zero set of S and is denoted by $V(S)$ (or in some books $Z(S)$, e.g. this is the notation of Hartshorne's book [rH]). In particular, if $S = \{F\}$, then $V(S)$ will be denoted simply by $V(F)$.

Example 1.4.2. *1. $S = K[x_1, \ldots, x_n]$: then $V(S) = \emptyset$, because S contains non–zero constants.*

2. $S=\{0\}$: *then* $V(S)=\mathbb{A}^n$.

3. $S=\{xy-1\}$: *then* $V(xy-1)$ *is a hyperbola in the affine plane.*

4. *If* $S\subset T$, *then* $V(S)\supset V(T)$.

5. $V(F_1,\ldots,F_r)=V(F_1)\cap\ldots\cap V(F_r)$.

Let $S\subset K[x_1,\ldots,x_n]$ be a set of polynomials, let $\alpha:=\langle S\rangle$ be the ideal generated by S. Recall that $\alpha=\{$finite sums of products of the form HF where $F\in S,\ H\in K[x_1,\ldots,x_n]\}$.

Proposition 1.4.3. *Let* $\alpha=\langle S\rangle$. *Then* $V(S)=V(\alpha)$.

Proof. From $S\subset\alpha$ it follows that $V(S)\supset V(\alpha)$.

Conversely, if $P\in V(S)$, let $G=\sum_i H_iF_i$ be a polynomial of α ($F_i\in S\ \forall\ i$). Then $G(P)=(\sum H_iF_i)(P)=\sum H_i(P)F_i(P)=0$. □

Proposition 1.4.3 is important in view of the following:

Theorem 1.4.4 (Hilbert's Basis Theorem)**.** *If R is a Noetherian ring, then the polynomial ring $R[x]$ is Noetherian.*

Proof. Assume by contradiction that $R[x]$ is not Noetherian. Let $I\subset R[x]$ be a non-finitely generated ideal. Let $f_1\in I$ be a non-zero polynomial of minimum degree. We define by induction a sequence $\{f_k\}_{k\in\mathbb{N}}$ of polynomials as follows: if f_k ($k\geq 1$) has already been chosen, let f_{k+1} be a polynomial of minimum degree in $I\setminus\langle f_1,\ldots,f_k\rangle$. Let n_k be the degree of f_k, and let a_k be its leading coefficient. Note that, due to the choice of f_k, the chain of the degrees is increasing: $n_1\leq n_2\leq\ldots$.

We will prove now that $\langle a_1\rangle\subset\langle a_1,a_2\rangle\subset\ldots$ is a chain of ideals, that does not become stationary: this will give the required contradiction. Indeed, if $\langle a_1,\ldots,a_r\rangle=\langle a_1,\ldots,a_r,a_{r+1}\rangle$ for some r, then $a_{r+1}=\sum_{i=1}^r b_ia_i$, for suitable $b_i\in R$. In this case, we consider the polynomial $g:=f_{r+1}-\sum_{i=1}^r b_ix^{n_{r+1}-n_i}f_i$: g belongs to I, but $g\notin\langle f_1,\ldots,f_r\rangle$, and its degree is strictly lower than the degree of f_{r+1}: contradiction. □

Corollary 1.4.5. *Any affine algebraic set $X\subset\mathbb{A}^n$ is the zero set of a finite number of polynomials, i.e. there exist $F_1,\ldots,F_r\in K[x_1,\ldots,x_n]$ such that $X=V(F_1,\ldots,F_r)$.*

Note that $V(F_1,\ldots,F_r)=V(F_1)\cap\ldots\cap V(F_r)$, so every algebraic set is a finite intersection of algebraic sets of the form $V(F)$, i.e. zeros of a unique polynomial F. If $F=0$, then $V(0)=\mathbb{A}^n$; if $F=c\in K\setminus\{0\}$, then $V(c)=\emptyset$; if *deg* $F>0$, then $V(F)$ is called a *hypersurface*.

1.4.2 The Zariski topology on the affine space

Proposition 1.4.6. *The affine algebraic sets of $\mathbb{A}^n$ satisfy the axioms of the closed sets of a topology, called the Zariski topology.*

Proof. It is enough to check that finite unions and arbitrary intersections of algebraic sets are again algebraic sets.

Let $V(\alpha), V(\beta)$ be two algebraic sets, with α, β ideals of $K[x_1, \dots, x_n]$. We recall that the product ideal of α and β is

$$\alpha\beta = \{ \sum_{\text{finite}} a_i b_i \mid a_i \in \alpha, b_i \in \beta \}.$$

Then $V(\alpha) \cup V(\beta) = V(\alpha \cap \beta) = V(\alpha\beta)$. Indeed: $\alpha\beta \subset \alpha \cap \beta$ so $V(\alpha \cap \beta) \subset V(\alpha\beta)$, and both $\alpha \cap \beta \subset \alpha$ and $\alpha \cap \beta \subset \beta$ so $V(\alpha) \cup V(\beta) \subset V(\alpha \cap \beta)$. Assume now that $P \in V(\alpha\beta)$ and $P \notin V(\alpha)$: hence $\exists F \in \alpha$ such that $F(P) \neq 0$; on the other hand, if $G \in \beta$ then $FG \in \alpha\beta$ so $(FG)(P) = 0 = F(P)G(P)$, which implies $G(P) = 0$.

Let $V(\alpha_i), i \in I$, be a family of algebraic sets, $\alpha_i \subset K[x_1, \dots, x_n]$. Then $\cap_{i \in I} V(\alpha_i) = V(\sum_{i \in I} \alpha_i)$, where $\sum_{i \in I} \alpha_i$ is the sum ideal of α_i's. Indeed $\alpha_i \subset \sum_{i \in I} \alpha_i \ \forall i$, hence $V(\sum_i \alpha_i) \subset V(\alpha_i) \ \forall i$ and $V(\sum_i \alpha_i) \subset \cap_i V(\alpha_i)$. Conversely, if $P \in V(\alpha_i) \ \forall i$, and $F \in \sum_i \alpha_i$, then $F = \sum_i F_i$; therefore $F(P) = \sum F_i(P) = 0$. $\square$

Example 1.4.7. 0. Every point of $\mathbb{A}^n$ is closed in the Zariski topology, indeed $A = (a_1, \dots, a_n) = V(x_1 - a_1, \dots, x_n - a_n)$.

1. The Zariski topology on the affine line $\mathbb{A}^1$.

Let us recall that the polynomial ring $K[x]$ in one variable is a PID (principal ideal domain), so every ideal $I \subset K[x]$ is of the form $I = \langle F \rangle$. Hence every closed subset of $\mathbb{A}^1$ is of the form $X = V(F)$, the set of zeros of a unique polynomial $F(x)$. If $F = 0$, then $V(F) = \mathbb{A}^1$, if $F = c \in K^*$, then $V(F) = \emptyset$, if $\deg F = d > 0$, then F can be decomposed in linear factors in the polynomial ring over the algebraic closure of K; it follows that $V(F)$ has at most d points.

We conclude that the closed sets in the Zariski topology of $\mathbb{A}^1$ are: $\mathbb{A}^1$, $\emptyset$ and the finite sets.

2. If $K = \mathbb{R}$ or $\mathbb{C}$, then the Zariski topology and the Euclidean topology on $\mathbb{A}^n_K$ can be compared, and it results that the Zariski topology is coarser. Indeed every open set in the Zariski topology is open also in the usual topology. Let $X = V(F_1, \dots, F_r)$ be a closed set in the Zariski topology, and $U := \mathbb{A}^n \setminus X$; if $P \in U$, then $\exists \ F_i$ such that $F_i(P) \neq 0$, so there exists an open neighbourhood of P in the usual topology in which F_i does not vanish.

Conversely, there exist closed sets in the usual topology which are not Zariski closed, for example the balls. The first case, of an interval in the real affine line, follows from part 1.

1.4.3 Projective algebraic sets

We want to define now the projective algebraic sets, or projective varieties, in $\mathbb{P}^n$.

The idea is the same as in the affine space: a projective variety is the set of solutions of a system of polynomial equations. The difference is that a point in the projective space does not have a well defined set of coordinates: homogeneous coordinates are defined only up to proportionality. So it may happen that, given a polynomial F and a point $P \in \mathbb{P}^n$ with homogeneous coordinates $[x_0, \dots, x_n]$, the n-tuple $x_0, \dots, x_n$ is a zero of F, but other proportional n-tuples of the form $[\lambda x_0, \dots, \lambda x_n]$ are not.

To give a good definition, we have to consider only homogeneous polynomials, because for them the problem does not occur. Otherwise, to say that a point $p \in \mathbb{P}^n$ is a zero of a polynomial F, we must ask that it annihilates F for each choice of its homogeneous coordinates.

Let's now formalize what I have anticipated.

Let $K[x_0, x_1, \dots, x_n]$ be the polynomial ring in $n+1$ variables. If we fix a polynomial $G(x_0, x_1, \dots, x_n) \in K[x_0, x_1, \dots, x_n]$ and a point $P[a_0, a_1, \dots, a_n] \in \mathbb{P}^n$, then, in general,

$$G(a_0, \dots, a_n) \neq G(\lambda a_0, \dots, \lambda a_n),$$

so the value of G at P cannot be defined.

Example 1.4.8. *Let $G = x_1 + x_0x_1 + x_2^2$, $P[0,1,2] = [0,2,4] \in \mathbb{P}^2_{\mathbb{R}}$. Note that $G(0,1,2) = 1 + 4 \neq G(0,2,4) = 2 + 16$. But if $Q = [1,0,0] = [\lambda, 0, 0]$, then $G(1,0,0) = G(\lambda,0,0) = 0$ for each λ.*

Definition 1.4.9 (Homogeneous polynomials)**.** Let $G \in K[x_0, x_1, \dots, x_n]$: G is *homogeneous of degree* d, or G is *a form* of degree d, if G is a linear combination of monomials of degree d.

Lemma 1.4.10. *If $G \in K[x_0, x_1, \dots, x_n]$ and t is a new variable, then G is homogeneous of degree d if and only if $G(tx_0, \dots, tx_n) = t^d G(x_0, \dots, x_n)$.*

Proof. To prove the "only of" implication, it is enough to prove the equality for monomials, i.e. for

$$G = ax_0^{i_0}x_1^{i_1}\dots x_n^{i_n} \text{ with } i_0 + i_1 + \dots + i_n = d:$$

$G(tx_0,\dots,tx_n) = a(tx_0)^{i_0}(tx_1)^{i_1}\dots(tx_n)^{i_n} = at^{i_0+i_1+\dots+i_n}x_0^{i_0}x_1^{i_1}\dots x_n^{i_n} = t^dG(x_0,\dots,x_n)$.

Conversely, if $G(tx_0,\dots,tx_n) = t^dG(x_0,\dots,x_n)$, we write G as sum of its homogeneous components $G = G_0 + G_1 + \dots + G_e$ and use the first implication. We get $t^dG(x_0,\dots,x_n) = G_0 + tG_1(x_0,\dots,x_n) + \dots + t^eG_e(x_0,\dots,x_n)$, which is an equality of polynomials in the variable t. So we get $d = e$ and $G = G_d$. □

Definition 1.4.11. Let G be a homogeneous polynomial of $K[x_0, x_1,\dots,x_n]$. A point $P[a_0,\dots,a_n] \in \mathbb{P}^n$ is a *zero of* G if $G(a_0,\dots,a_n) = 0$. In this case we write $G(P) = 0$.

Note that by Lemma 1.4.10 if $G(a_0,\dots,a_n) = 0$, then

$$G(\lambda a_0,\dots,\lambda a_n) = \lambda^{\deg G}G(a_0,\dots,a_n) = 0$$

for every choice of $\lambda \in K^*$. (Remind: K^* denotes $K \setminus \{0\}$.)

Definition 1.4.12. A subset Z of $\mathbb{P}^n$ is a *projective algebraic set*, or a *projective variety*, if Z is the set of common zeros of a set of homogeneous polynomials of $K[x_0, x_1,\dots,x_n]$.

If $T \subset K[x_0, x_1,\dots,x_n]$ is any subset formed by homogeneous polynomials, then the corresponding algebraic set will be denoted by $V_P(T)$.

1.5 Graded rings and homogeneous ideals

We want now to give an interpretation of projective varieties as sets of zeros of ideals, as we did in the affine case, see Proposition 1.4.3. But of course the ideal generated by a family of homogeneous polynomials contains also polynomials that are not homogeneous.

Let $\alpha = \langle T \rangle$ be the ideal generated by the polynomials of T, all assumed to be homogeneous. For any $F \in \alpha$, there is en expression $F = \sum_i H_iF_i$, $F_i \in T$.

So if $P[a_0,\dots,a_n] \in V_P(T)$, then

$$F(a_0,\dots,a_n) = \sum H_i(a_0,\dots,a_n)F_i(a_0,\dots,a_n) = 0$$

for any choice of coordinates of P, regardless if F is homogeneous or not. We say that P is a *projective zero* of F.

We want to formalize this situation in the context of the *graded rings*, of which the polynomial rings are a prototype. In particular in a graded ring there will be a situation similar to the following one: if F is a polynomial, then F can be written *in a unique way* as a sum of homogeneous polynomials, called the homogeneous components of F: $F = F_0 + F_1 + \cdots + F_d$, where, for any index i, either the degree of F_i is equal to i, or $F_i = 0$.

We give the following definition:

Definition 1.5.1. Let A be a ring (as usual assumed to be commutative with unit). A is called *a graded ring over* $\mathbb{Z}$ if there exists a family of additive subgroups of A, $\{A_i\}_{i\in\mathbb{Z}}$, such that:

(i) $A = \oplus_{i\in\mathbb{Z}} A_i$; and

(ii) $A_i A_j \subset A_{i+j}$ for any pair of indices.

The elements of A_i are called *homogeneous of degree* i, and A_i is the homogeneous component of A of degree i. Condition (i) regards the additive structure of A; it means that any element a of A has a unique finite expression $a = \Sigma_{i\in\mathbb{Z}} a_i$, finite sum of homogeneous elements. Condition (ii) regards the multiplicative structure: a product of homogeneous elements is homogeneous of degree the sum of the degrees. Notice that 0 belongs to all homogeneous components of A.

The standard example of graded ring is the polynomial ring with coefficients in a ring R. R is the homogeneous component of degree 0, the variables have all degree 1. In this case the homogeneous components of negative degrees are all zero.

Proposition 1.5.2 (Proposition - Definition of homogeneous ideal)**.** *Let $I \subset A$ be an ideal of a graded ring. I is called* **homogeneous** *if the following equivalent conditions are fulfilled:*

(i) I is generated by homogeneous elements (this means: there is a system of generators formed by homogeneous elements);

(ii) $I = \oplus_{k\in\mathbb{Z}}(I \cap A_k)$, i.e. if $F = \Sigma_{k\in\mathbb{Z}} F_k \in I$, then all homogeneous components F_k of F belong to I.

Proof of the equivalence. "(ii)⇒(i)": given a system of generators of I, write each of them as sum of its homogeneous components: $F_i = \Sigma_{k\in\mathbb{Z}} F_{ik}$. Then a set of homogeneous generators of I is formed by all the elements F_{ik}.

"(i)⇒(ii)": let I be generated by a family of homogeneous elements $\{G_\alpha\}$, with $\deg G_\alpha = d_\alpha$. If $F \in I$, then F is a combination of the elements G_α with suitable coefficients H_α; write each H_α as sum of its homogeneous components: $H_\alpha = \Sigma H_{\alpha k}$. Note that the product $H_{\alpha k} G_\alpha$ is homogeneous of degree $k + d_\alpha$. By the unicity of the expression of F as sum of homogeneous elements, it follows that all of them are combinations of the generators $\{G_\alpha\}$ and therefore they belong to I. □

Let $I \subset K[x_0, x_1, \dots, x_n]$ be a homogeneous ideal. Note that, by the noetherianity, I admits a finite set of homogeneous generators.

Let $P[a_0, \dots, a_n] \in \mathbb{P}^n$. If $F \in I$, $F = F_0 + \dots + F_d$, then $F_0 \in I, \dots, F_d \in I$. We say that P *is a zero of* I if P is a projective zero of any polynomial of I or, equivalently, of any homogeneous polynomial of I. This also means that P is a zero of any homogeneous polynomial of a set generating I. The set of zeros of I will be denoted $V_P(I)$: all projective algebraic subsets of $\mathbb{P}^n$ are of this form.

As in the affine case, the projective algebraic subsets of $\mathbb{P}^n$ satisfy the axioms of the closed sets of a topology, called the Zariski topology of $\mathbb{P}^n$. This time the empty set can be expressed as $V_P(1)$, as well as $V_P(x_0, \dots, x_n)$: indeed the n-tuple $[0, ..., 0]$ is not a point of $\mathbb{P}^n$. As for the other axioms of closed sets, the idea is always the same: the equations of the intersection of a family of algebraic sets are the union of all the equations, while the union of two algebraic sets X and Y is defined by all the possible products of two equations, one of X and the other of Y.

From the point of view of ideals, it is useful to make the following remark, whose proof follows from Proposition 1.5.2. Let I, J be homogeneous ideals of $K[x_0, x_1, \dots, x_n]$. Then $I + J$, IJ and $I \cap J$ are homogeneous ideals. Indeed both I and J are generated by homogeneous polynomials, $I + J$ is generated by the union of all of them, IJ is generated by products of two of them, one in I and the other in J, so in both cases by homogeneous polynomials. For $I \cap J$ it is enough to use Proposition 1.5.2 (ii).

Note that also all subsets of $\mathbb{A}^n$ and $\mathbb{P}^n$ have a structure of topological space, with the induced topology, which is still called the Zariski topology.

Exercises 1.5.3.

1. Let $F \in K[x_1, \dots, x_n]$ be a non–constant polynomial. The set $\mathbb{A}^n \setminus V(F)$ will be denoted $\mathbb{A}^n_F$. Prove that $\{\mathbb{A}^n_F \mid F \in K[x_1, \dots, x_n] \setminus K\}$ is a topology basis for the Zariski topology.

2. Let $B \subset \mathbb{R}^n$ be a ball. Prove that B is not Zariski closed.

3. Prove that the map $\varphi : \mathbb{A}^1 \to \mathbb{A}^3$ defined by $t \to (t, t^2, t^3)$ is a homeomorphism between $\mathbb{A}^1$ and its image, for the Zariski topology.

4. Let $X \subset \mathbb{A}^2_{\mathbb{R}}$ be the graph of the map $\mathbb{R} \to \mathbb{R}$ such that $x \to \sin x$. Is X closed in the Zariski topology? (hint: intersect X with a line....)

Chapter 2

Examples of algebraic varieties.

2.1 Points

In the Zariski topology both in $\mathbb{A}^n$ and in $\mathbb{P}^n$ all points are closed. If $P(a_1, \ldots, a_n) \in \mathbb{A}^n$, then $P = V(x_1 - a_1, \ldots, x_n - a_n)$. But in the projective space, if $P[a_0, \ldots, a_n] \in \mathbb{P}^n$, the equations are different: $P = V_P(a_i x_j - a_j x_i)_{i,j=0,\ldots,n}$. In this way the polynomials defining P as closed set are homogeneous. They can be seen as minors of order 2 of the matrix

$$\begin{pmatrix} a_0 & a_1 & \ldots & a_n \\ x_0 & x_1 & \ldots & x_n \end{pmatrix}$$

with entries in $K[x_0, x_1, \ldots, x_n]$. This expresses the fact that $x_0, \ldots, x_n$ are proportional to $a_0, \ldots, a_n$. Equations of the form $V_P(x_0 - a_0, \ldots, x_n - a_n)$ don't make sense.

2.2 Affine and projective linear subspaces.

Generalizing the previous example, the linear subspaces, both in the affine and in the projective case, are examples of algebraic sets. As it is well known, they are defined by equations of degree 1.

2.3 Hypersurfaces

An affine hypersurface is an affine variety of the form $V(F)$, the set of zeros of a unique polynomial F of **positive** degree. Similarly, in the projective space, a projective hypersurface is of the form $V_P(G)$, where $G \in K[x_0, x_1, \ldots, x_n]$ is a homogeneous non-constant polynomial.

Examples of hypersurfaces are the curves in the affine or projective plane, and the surfaces in a space of dimension 3, as for instance the quadrics.

Let us recall that the polynomial ring $K[x_1, \dots, x_n]$ is a UFD (unique factorization domain), i.e., every non-constant polynomial F can be expressed in a unique way (up to the order and up to units) as $F = F_1^{r_1} F_2^{r_2} \dots F_s^{r_s}$, where $F_1, \dots, F_s$ are irreducible and two by two distinct polynomials, and $r_i \geq 1$ for any $i = 1, \dots, s$. Hence the hypersurface of $\mathbb{A}^n$ defined by F is

$$X := V(F) = V(F_1^{r_1} F_2^{r_2} \dots F_s^{r_s}) = V(F_1 F_2 \dots F_s) = V(F_1) \cup V(F_2) \cup \dots \cup V(F_s).$$

The equation $F_1 F_2 \dots F_s = 0$ is called the reduced equation of X. Note that $F_1 F_2 \dots F_s$ generates the radical $\sqrt{F}$. If $s = 1$, X is called an *irreducible* hypersurface; by definition its degree is the degree of its reduced equation. Therefore, any hypersurface is a finite union of irreducible hypersurfaces.

Assume now that $Z = V_P(G)$, with $G \in K[x_0, x_1, \dots, x_n]$, G homogeneous, is a projective hypersurface. Exercise 3 asks to prove that the irreducible factors of G are homogeneous. Therefore, as in the affine case, each projective hypersurface Z has a reduced equation (whose degree is, by definition, the degree of Z) and Z is a finite union of irreducible hypersurfaces.

If the field K is algebraically closed, the degree of a projective hypersurface has the following important geometrical meaning.

Proposition 2.3.1. *Let K be an algebraically closed field. Let $Z \subset \mathbb{P}^n$ be a projective hypersurface of degree d. Then any line in $\mathbb{P}^n$, not contained in Z, meets Z at exactly d points, counting multiplicities.*

In the proof we will see what we mean by saying "counting multiplicity".

Proof. Let G be the reduced equation of Z and $L \subset \mathbb{P}^n$ be any line.

We fix two points on L: $A = [a_0, \dots, a_n], B = [b_0, \dots, b_n]$. So L admits parametric equations of the form

$$\begin{cases} x_0 = \lambda a_0 + \mu b_0 \\ x_1 = \lambda a_1 + \mu b_1 \\ \dots \\ x_n = \lambda a_n + \mu b_n \end{cases}$$

The points of $Z \cap L$ are obtained from the homogeneous pairs $[\lambda, \mu]$ which are solutions of the equation $G(\lambda a_0 + \mu b_0, \dots, \lambda a_n + \mu b_n) = 0$. If $L \subset Z$, then this equation is an identity.

Otherwise, $G(\lambda a_0 + \mu b_0, \dots, \lambda a_n + \mu b_n)$ is a non-zero homogeneous polynomial of degree d in the two variables λ, μ. Since K is algebraically closed, it can be factorized in linear factors:

$$G(\lambda a_0 + \mu b_0, \dots, \lambda a_n + \mu b_n) = (\mu_1\lambda - \lambda_1\mu)^{d_1}(\mu_2\lambda - \lambda_2\mu)^{d_2} \dots (\mu_r\lambda - \lambda_r\mu)^{d_r}$$

with $d_1 + d_2 + \dots + d_r = d$. Every factor corresponds to a point in $Z \cap L$, to be counted with the same multiplicity as the corresponding factor.

□

If K is not algebraically closed, considering the algebraic closure of K and using Proposition 2.3.1, we get that d is an upper bound on the number of points of $Z \cap L$.

2.4 Product of affine spaces

Let $\mathbb{A}^n$, $\mathbb{A}^m$ be two affine spaces over the field K. The cartesian product $\mathbb{A}^n \times \mathbb{A}^m$ is the set of pairs (P, Q), $P \in \mathbb{A}^n$, $Q \in \mathbb{A}^m$: it is in natural bijection with $\mathbb{A}^{n+m}$ via the map

$$\varphi : \mathbb{A}^n \times \mathbb{A}^m \longrightarrow \mathbb{A}^{n+m}$$

such that $\varphi((a_1, \dots, a_n), (b_1, \dots, b_m)) = (a_1, \dots, a_n, b_1, \dots, b_m)$.

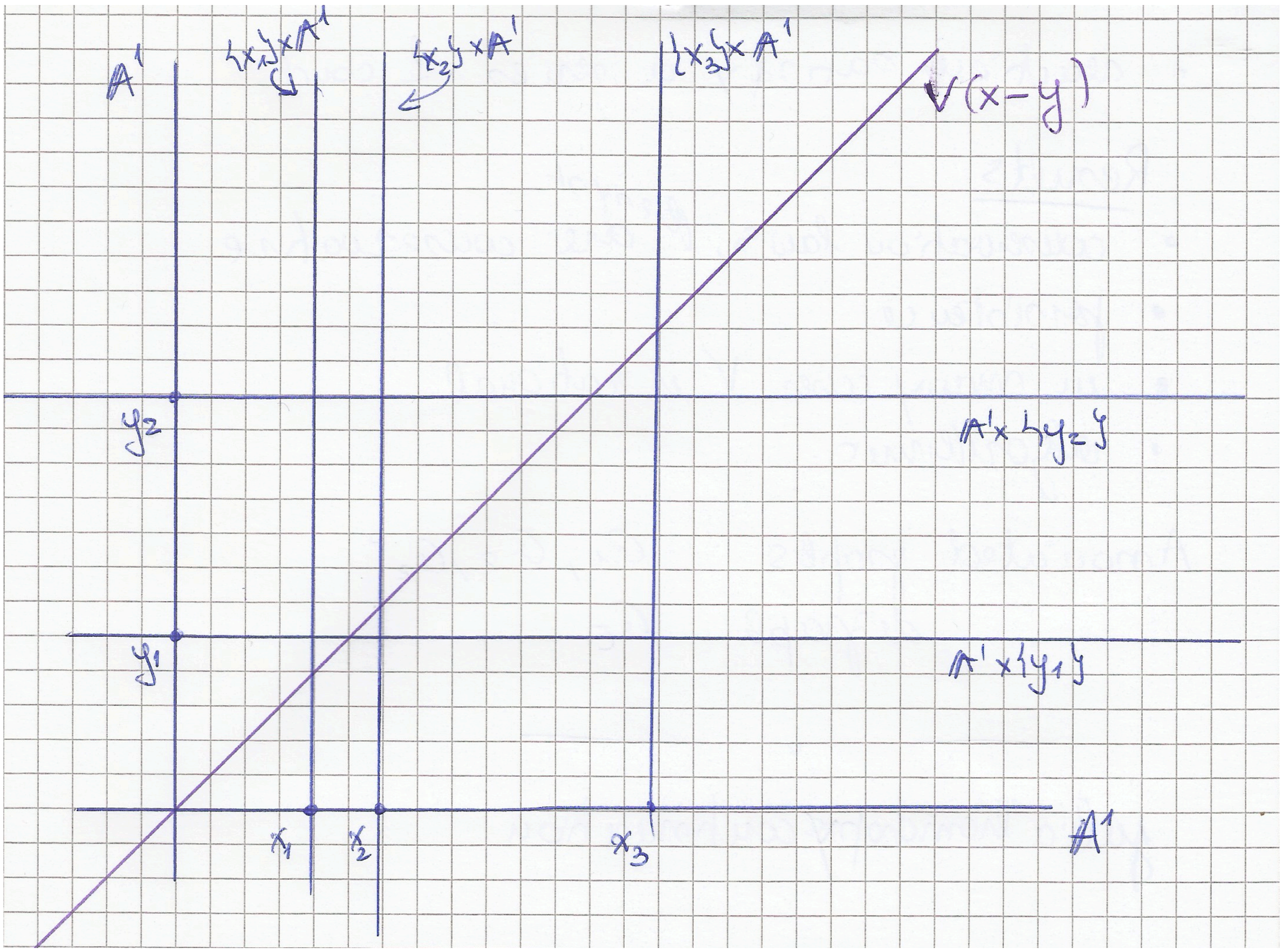
From now on we will always identify $\mathbb{A}^n \times \mathbb{A}^m$ with $\mathbb{A}^{n+m}$. Therefore we have two topologies on $\mathbb{A}^n \times \mathbb{A}^m$: the Zariski topology of $\mathbb{A}^{n+m}$ and the product topology of the Zariski topologies of $\mathbb{A}^n$ and $\mathbb{A}^m$.

Proposition 2.4.1. *The Zariski topology is strictly finer than the product topology.*

Proof. Let us first observe that, if $X = V(\alpha) \subset \mathbb{A}^n$, $\alpha \subset K[x_1, \dots, x_n]$ and $Y = V(\beta) \subset \mathbb{A}^m$, $\beta \subset K[y_1, \dots, y_m]$, then $X \times Y \subset \mathbb{A}^n \times \mathbb{A}^m$ is Zariski closed, precisely $X \times Y = V(\alpha \cup \beta)$ where the union is made in the polynomial ring in $n + m$ variables $K[x_1, \dots, x_n, y_1, \dots, y_m]$. Now we consider $U = \mathbb{A}^n \setminus X$ and $V = \mathbb{A}^m \setminus Y$, open subsets of $\mathbb{A}^n$ and $\mathbb{A}^m$ in the Zariski topology. Then $U \times V = \mathbb{A}^n \times \mathbb{A}^m \setminus ((\mathbb{A}^n \times Y) \cup (X \times \mathbb{A}^m))$: this is a set-theoretical fact that holds true in general. So it is open in $\mathbb{A}^n \times \mathbb{A}^m$ in the Zariski topology.

Conversely, we give an example to prove that the two topologies are different. Precisely we show that $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$ contains some subsets which are Zariski open, but are not open in the product topology.

The proper open subsets in the product topology are of the form $\mathbb{A}^1 \times \mathbb{A}^1 \setminus \{$ finite unions of "vertical" and "horizontal" lines$\}$. See the figure.


$\mathbb{A}^1$
$\{x_1\}\times\mathbb{A}^1$
$\{x_2\}\times\mathbb{A}^1$
$\{x_3\}\times\mathbb{A}^1$
$V(x-y)$
y_2
$\mathbb{A}^1\times\{y_2\}$
y_1
$\mathbb{A}^1\times\{y_1\}$
x_1
x_2
x_3
$\mathbb{A}^1$

Let $X = \mathbb{A}^2 \setminus V(x-y)$: it is Zariski open but does not contain any non-empty subset of the above form, so it is not open in the product topology. There are similar examples in $\mathbb{A}^n \times \mathbb{A}^m$ for any n, m. □

Note that there is no similar construction for $\mathbb{P}^n \times \mathbb{P}^m$. We will see in Chapter 13.1 that there is an injective map, the Segre map, of $\mathbb{P}^n \times \mathbb{P}^m$ to the projective space of dimension $(n+1)(m+1)-1$, whose image is a projective variety. This allows to give a geometric structure to the product of projective spaces . We see here only the first case.

2.5 $\mathbb{P}^1 \times \mathbb{P}^1$

The cartesian product $\mathbb{P}^1 \times \mathbb{P}^1$ is simply a set, but we are going to define an injective map σ from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^3$, whose image will be a projective variety: it will be identified with our product, and this will allow to interpret $\mathbb{P}^1 \times \mathbb{P}^1$ as a projective variety.

The map σ is defined in the following way: $\sigma([x_0, x_1], [y_0, y_1]) = [x_0y_0, x_0y_1, x_1y_0, x_1y_1]$. Using coordinates $z_0, ..., z_3$ in $\mathbb{P}^3$, σ is defined parametrically by

$$\begin{cases} z_0 = x_0y_0 \\ z_1 = x_0y_1 \\ z_2 = x_1y_0 \\ z_3 = x_1y_1 \end{cases}$$

It is easy to observe that σ is a well–defined map: the image is never $[0, 0, 0, 0]$, and depends uniquely on the pair of points and not on the choice of their coordinates. Moreover σ is injective. Assume that $\sigma([x_0, x_1], [y_0, y_1]) = \sigma([x'_0, x'_1], [y'_0, y'_1])$. Then there exists a non-zero constant λ such that

$$\begin{cases} x_0y_0 = \lambda x'_0y'_0 \\ x_0y_1 = \lambda x'_0y'_1 \\ x_1y_0 = \lambda x'_1y'_0 \\ x_1y_1 = \lambda x'_1y'_1 \end{cases}$$

Now, if $y_0 \neq 0$, then $x_0 = (\lambda y'_0/y_0)x'_0$ and $x_1 = (\lambda y'_0/y_0)x'_1$; if $y_1 \neq 0$, then $x_0 = (\lambda y'_1/y_1)x'_0$ and $x_1 = (\lambda y'_1/y_1)x'_1$; in both cases $[x_0, x_1] = [x'_0, x'_1]$. Similarly one proves that $[y_0, y_1] = [y'_0, y'_1]$.

Let Σ denote the image $\sigma(\mathbb{P}^1 \times \mathbb{P}^1)$. It is the quadric of equation $z_0z_3 - z_1z_2 = 0$; indeed, on one hand it is clear that $\sigma(\mathbb{P}^1 \times \mathbb{P}^1) \subset V_P(z_0z_3 - z_1z_2)$; conversely, assume that

$z_0z_3 = z_1z_2$ and $z_0 \neq 0$. Then, multiplying all coordinates by z_0, we get: $[z_0, z_1, z_2, z_3] = [z_0^2, z_0z_1, z_0z_2, z_0z_3]$; by assumption this coincides with $[z_0^2, z_0z_1, z_0z_2, z_1z_2]$, and is therefore equal to $\sigma([z_0, z_2], [z_0, z_1])$. If $z_0 = 0$, the argument is similar, using another non-zero coordinate.

The map σ is called the Segre map and Σ the **Segre variety**. The name comes from the Italian mathematician Corrado Segre (Torino, 1863–1924), the "father" of the Italian school of algebraic geometry.

2.6 Embedding of $\mathbb{A}^n$ in $\mathbb{P}^n$.

We will see now how to unify the two notions introduced so far of affine and projective variety. Precisely, after identifying $\mathbb{A}^n$ with the open subset $U_0 \subset \mathbb{P}^n$ (or with any U_i) (as in Section 1.3), we will prove that the Zariski topology on $\mathbb{A}^n$ coincides with the topology induced by the Zariski topology on $\mathbb{P}^n$.

Let H_i be the hyperplane of $\mathbb{P}^n$ of equation $x_i = 0$, $i = 0, \dots, n$; it is closed in the Zariski topology, and its complementar set U_i is open. So we have an open covering of $\mathbb{P}^n$: $\mathbb{P}^n = U_0 \cup U_1 \cup \dots \cup U_n$. Let us recall that for any i there is a bijection $\varphi_i : U_i \to \mathbb{A}^n$ such that $\varphi_i([x_0, \dots, x_i, \dots, x_n]) = (\frac{x_0}{x_i}, \dots, \hat{1}, \dots, \frac{x_n}{x_i})$. The inverse map is $j_i : \mathbb{A}^n \to U_i$ such that $j_i(y_1, \dots, y_n) = [y_1, \dots, 1, \dots, y_n]$.

Proposition 2.6.1. *The map φ_i is a homeomorphism, for $i = 0, \dots, n$.*

Proof. Assume $i = 0$ (the other cases are similar).

We introduce two maps:

(i) *dehomogeneization* of polynomials with respect to x_0.

It is a map $^a : K[x_0, x_1, \dots, x_n] \to K[y_1, \dots, y_n]$ such that

$$^a(F(x_0, \dots, x_n)) = {}^aF(y_1, \dots, y_n) := F(1, y_1, \dots, y_n).$$

Note that ^{a} is a ring homomorphism.

(ii) *homogeneization* of polynomials with respect to x_0.

It is a map $^h : K[y_1, \dots, y_n] \to K[x_0, x_1, \dots, x_n]$ defined by

$$^h(G(y_1, \dots, y_n)) = {}^hG(x_0, \dots, x_n) := x_0^{\deg G} G(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}).$$

${}^hG$ is always a homogeneous polynomial of the same degree as G. The map ^{h} is clearly not a ring homomorphism. Note that always ${}^a({}^hG) = G$ but in general ${}^h({}^aF) \neq F$; what we can say is that, if $F(x_0, \dots, x_n)$ is homogeneous, then there exists $r \geq 0$ such that $F = x_0^r({}^h({}^aF))$.

Let $X \subset U_0$ be closed in the topology induced by the Zariski topology of the projective space, i.e. $X = U_0 \cap V_P(I)$ where I is a homogeneous ideal of $K[x_0, x_1, \dots, x_n]$. Define ${}^aI = \{{}^aF \mid F \in I\}$: it is an ideal of $K[y_1, \dots, y_n]$ (because ^{a} is a ring homomorphism). We prove that $\varphi_0(X) = V({}^aI)$. Indeed, let $P[x_0, \dots, x_n]$ be a point of U_0; then $\varphi_0(P) = (\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}) \in \varphi_0(X) \iff P[x_0, \dots, x_n] = [1, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}] \in X = V_P(I) \iff F(1, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}) = 0 \ \forall \ {}^aF \in {}^aI \iff \varphi_0(P) \in V({}^aI)$.

Conversely: let $Y = V(\alpha)$ be a Zariski closed subset of $\mathbb{A}^n$, where α is an ideal of $K[y_1, \dots, y_n]$. Let ${}^h\alpha$ be the homogeneous ideal of $K[x_0, x_1, \dots, x_n]$ generated by the set $\{{}^hG \mid G \in \alpha\}$. We prove that $\varphi_0^{-1}(Y) = V_P({}^h\alpha) \cap U_0$. Indeed $[1, x_1, \dots, x_n] \in \varphi_0^{-1}(Y) \iff (x_1, \dots, x_n) \in Y \iff G(x_1, \dots, x_n) = {}^hG(1, x_1, \dots, x_n) = 0 \ \forall \ G \in \alpha \iff [1, x_1, \dots, x_n] \in V_P({}^h\alpha)$. $\square$

From now on we will often identify $\mathbb{A}^n$ with U_0 via φ_0 (and similarly with U_i via φ_i). So if $P[x_0, \dots, x_n] \in U_0$, we will refer to $x_0, \dots, x_n$ as the homogeneous coordinates of P and to $\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}$ as the non–homogeneous or affine coordinates of P.

Exercises 2.6.2. It will be useful to remember that any algebraically closed field is infinite.

1. Assume that K is an infinite field.

   a) Prove that, if $n \geq 1$, then in $\mathbb{A}^n_K$ the complementar set of any hypersurface has infinitely many points.

   b) Prove that, if K is algebraically closed and $n \geq 2$, then also any hypersurface has infinitely many points.

2. Prove that the Zariski topology on $\mathbb{A}^n$ is T_1.

3. Let $F \in K[x_0, x_1, \dots, x_n]$ be a homogeneous polynomial. Check that its irreducible factors are homogeneous. (Hint: prove that a product of two polynomials not both homogeneous is not homogeneous.)

Solution of Exercise 1.

Let the hypersurface in question be defined by $F(x_1,\dots,x_n)=0$, F non constant. We can assume that the variable x_n occurs in F. So we have an expression

$$F = f_0 + f_1 x_n + \cdots + f_d x_n^d,$$

with $f_i \in K[x_1,\dots,x_{n-1}]$ $\forall i$, $d>0$ and $f_d \neq 0$.

a) We proceed by induction on the number of variables. If $n=1$, the statement is true because K is infinite. Let $n>1$: by the inductive assumption, there exist infinitely many $(a_1,\dots,a_{n-1}) \in K^{n-1}$ such that $f_d(a_1,\dots,a_{n-1}) \neq 0$. Then for any such $(n-1)$-tuple $F(a_1,\dots,a_{n-1},x_n)$ is a non-zero polynomial of degree $d>0$ in $K[x_n]$: it has finitely many zeros, so there are infinitely many $a_n \in K$ such that $F(a_1,\dots,a_{n-1},a_n) \neq 0$.

b) As in a), there exist infinitely many $(a_1,\dots,a_{n-1}) \in K^{n-1}$ such that $f_d(a_1,\dots,a_{n-1}) \neq 0$. Since K is algebraically closed, for each of these $(a_1,\dots,a_{n-1})$ there is at least one $a_n \in K$ such that $F(a_1,\dots,a_{n-1},a_n)=0$.

Chapter 3

The ideal of an algebraic set and the Hilbert Nullstellensatz.

3.1 The ideal of an algebraic set

Let $X \subset \mathbb{A}^n$ be an affine variety, $X = V(\alpha)$, where $\alpha \subset K[x_1, \dots, x_n]$ is an ideal.

The ideal α defining X is not unique. We have already made this observation in the case of the hypersurfaces (Section 2.3). For another example, let $O = \{(0,0)\} \subset \mathbb{A}^2$ be the origin; then $O = V(x_1, x_2) = V(x_1^2, x_2) = V(x_1^2, x_2^3) = V(x_1^2, x_1x_2, x_2^2) = \dots$ Nevertheless, there is an ideal we can canonically associate to X: the biggest one among the ideals defining it.

We give the following definition:

Definition 3.1.1. Let $Y \subset \mathbb{A}^n$ be any set. The *ideal of* Y is

$$I(Y) = \{F \in K[x_1, \dots, x_n] \mid F(P) = 0 \text{ for any } P \in Y\} = \{F \in K[x_1, \dots, x_n] \mid Y \subset V(F)\} :$$

it is the set of **all** polynomials vanishing on Y. Note that $I(Y)$ is in fact an ideal, because the sum of two polynomials vanishing along Y also vanishes along Y, and the product of any polynomial by a polynomial vanishing along Y again vanishes along Y.

Example 3.1.2. Maximal ideal of a point. If $P(a_1, \dots, a_n)$ is a point, then $I(P) = \langle x_1 - a_1, \dots, x_n - a_n \rangle$. Indeed all the polynomials of $\langle x_1 - a_1, \dots, x_n - a_n \rangle$ vanish on P, and moreover it is a maximal ideal.

The fact that $\langle x_1 - a_1, \dots, x_n - a_n \rangle$ is maximal can be understood looking at the quotient ring $K[x_1, \dots, x_n]/\langle x_1 - a_1, \dots, x_n - a_n \rangle$: the idea is that in the quotient the variables $x_1, \dots, x_n$ are replaced by the constants $a_1, \dots, a_n$, so it has to be $K[a_1, \dots, a_n] = K$. Since the quotient is a field, the ideal is maximal.

Another proof of the maximality of $\langle x_1 - a_1, \ldots, x_n - a_n\rangle$ can be given by exploiting the expansion in power series around $\underline{a} := (a_1, \ldots, a_n)$ of any polynomial $F(x_1, \ldots, x_n)$. I first recall that this expansion is possible for polynomials over any field, without involving any differentiation process, but using only the formal definition of derivative for polynomials. See for instance [W], pp. 21-23.

The proof goes as follows. Assume that $F(a_1, \ldots, a_n) = 0$ and use the Taylor expansion:

$$F(x_1, \ldots, x_n) = F(\underline{a}) + \sum_{i=1}^{n}(x_i - a_i)F_{x_i}(\underline{a}) + \sum_{i,j=1}^{n}(x_i - a_i)(x_j - a_j)F_{x_i x_j}(\underline{a}) + \ldots$$

It follows that $F \in \langle x_1 - a_1, \ldots, x_n - a_n\rangle$.

Remark 2. The following relations follow immediately by the definition:

(i) if $Y \subset Y'$, then $I(Y) \supset I(Y')$;

(ii) $I(Y \cup Y') = I(Y) \cap I(Y')$;

(iii) $I(Y \cap Y') \supset I(Y) + I(Y')$.

In the projective ambient, we have an analogous situation.

Definition 3.1.3. If $Z \subset \mathbb{P}^n$ is any set, the *homogeneous ideal of* Z is, by definition, the homogeneous ideal of $K[x_0, x_1, \ldots, x_n]$ generated by the set

$$\{G \in K[x_0, x_1, \ldots, x_n] \mid G \text{ is homogeneous and } V_P(G) \supset Z\}.$$

It is denoted $I_h(Z)$.

Relations similar to (i),(ii),(iii) of Remark 2 are satisfied. $I_h(Z)$ is also the set of polynomials $F(x_0, \ldots, x_n)$ such that every point of Z is a projective zero of F.

If $X = V(\alpha)$ we want to understand the relation between α and $I(X)$. Let $\alpha \subset K[x_1, \ldots, x_n]$ be an ideal. Let $\sqrt{\alpha}$ denote the radical of α:

$$\sqrt{\alpha} =: \{F \in K[x_1, \ldots, x_n] \mid \exists r \geq 1 \ s.t. \ F^r \in \alpha\}.$$

Note that $\sqrt{\alpha}$ is an ideal (why?) and that always $\alpha \subset \sqrt{\alpha}$; if equality holds, then α is called a *radical ideal*.

Proposition 3.1.4. *The ideal of a subset of the affine space is radical. More precisely:*

1. *for any* $X \subset \mathbb{A}^n$, $I(X)$ *is a radical ideal;*

2. *for any $Z \subset \mathbb{P}^n$, $I_h(Z)$ is a homogeneous radical ideal.*

Proof. 1. If $F \in \sqrt{I(X)}$, let $r \geq 1$ such that $F^r \in I(X)$: if $P \in X$, then $(F^r)(P) = 0 = (F(P))^r$ in the base field K. Therefore $F(P) = 0$.

2. is similar, taking into account that $I_h(Z)$ is a homogeneous ideal (see Exercise 6).

□

We can interpret I as a map from $\mathcal{P}(\mathbb{A}^n)$, the power set of the affine space, to $\mathcal{P}(K[x_1, \ldots, x_n])$, the power set of the polynomial ring. On the other hand, V can be seen as a map in the opposite sense. We have:

Proposition 3.1.5. *Let $\alpha \subset K[x_1, \ldots, x_n]$ be an ideal, let $Y \subset \mathbb{A}^n$ be any subset. Then:*

(i) $\alpha \subset I(V(\alpha))$;

(ii) $Y \subset V(I(Y))$;

(iii) $V(I(Y)) = \overline{Y}$: the closure of Y in the Zariski topology of $\mathbb{A}^n$.

Proof. (i) If $F \in \alpha$ and $P \in V(\alpha)$, then $F(P) = 0$, so $F \in I(V(\alpha))$.

(ii) If $P \in Y$ and $F \in I(Y)$, then $F(P) = 0$, so $P \in V(I(Y))$.

(iii) Taking closures in (ii), we get: $\overline{Y} \subset \overline{V(I(Y))} = V(I(Y))$, because it is already closed. Conversely, let $X = V(\beta)$ be any closed set containing Y: $X = V(\beta) \supset Y$. Then $I(Y) \supset I(V(\beta)) \supset \beta$ by (i); we apply V again: $V(\beta) = X \supset V(I(Y))$ so any closed set containing Y contains $V(I(Y))$ so $\overline{Y} \supset V(I(Y))$.

□

Similar properties relate homogeneous ideals of $K[x_0, x_1, \ldots, x_n]$ and subsets of $\mathbb{P}^n$; in particular, if $Z \subset \mathbb{P}^n$, then $V_P(I_h(Z)) = \overline{Z}$, the closure of Z in the Zariski topology of $\mathbb{P}^n$. In the projective case, one has to take care of the fact that any homogeneous ideal is generated by the set of its homogeneous elements, and so, to prove an inclusion between homogeneous ideals, it is enough to check it on the homogeneous elements.

3.2 Nullstellensatz

There is no characterization of $I(V(\alpha))$ **in general**. We can only say that it is a radical ideal containing α, so it contains also $\sqrt{\alpha}$. To characterise $I(V(\alpha))$ we have to put the properties of the base field K into play.

The following celebrated theorem gives the answer for algebraically closed fields.

Theorem 3.2.1 (Hilbert's Nullstellensatz - Theorem of zeros). *Let K be an algebraically closed field. Let $\alpha \subset K[x_1, \dots, x_n]$ be an ideal. Then $I(V(\alpha)) = \sqrt{\alpha}$.*

Remark 3. The assumption on K is necessary. Let me recall that K is algebraically closed if any non–constant polynomial of $K[x]$ has at least one root in K, or, equivalently, if any irreducible polynomial of $K[x]$ has degree 1. So if K is not algebraically closed, there exists an irreducible polynomial $F \in K[x]$ of degree $d > 1$. Therefore F has no zeros in K, hence $V(F) \subset \mathbb{A}^1_K$ is empty. So $I(V(F)) = I(\emptyset) = \{G \in K[x] \mid \emptyset \subset V(G)\} = K[x]$. But $\langle F \rangle$ is a maximal ideal in $K[x]$, and $\langle F \rangle \subset \sqrt{\langle F \rangle}$. If $\langle F \rangle \neq \sqrt{\langle F \rangle}$, by the maximality $\sqrt{\langle F \rangle} = \langle 1 \rangle$, so $\exists r \geq 1$ such that $1^r = 1 \in \langle F \rangle$, which is false. Hence $\sqrt{\langle F \rangle} = \langle F \rangle \neq K[x] = I(V(F))$.

We will deduce the proof of Hilbert Nullstellensatz, after several steps, from another very important theorem, known as "Emmy Noether normalization Lemma".

We start with some definitions.

Let $K \subset E$ be fields, K subfield of E. Let $\{z_i\}_{i \in I}$ be a family of elements of E.

Definition 3.2.2. The family $\{z_i\}_{i \in I}$ is *algebraically free* over K or, equivalently, the elements z_i's are *algebraically independent* over K if there is no non–zero polynomial $F \in K[x_i]_{i \in I}$, the polynomial ring in a set of variables indexed on I, that vanishes in the elements of the family $\{z_i\}$.

For example: if the family consists of only one element z, $\{z\}$ is algebraically free over K if and only if z is transcendental over K. The family $\{\pi, \sqrt{\pi}\}$ is not algebraically free over $\mathbb{Q}$: it satisfies the non–trivial relation $x_1^2 - x_2 = 0$.

By convention, the empty family is free over any field K.

Let $\mathcal{S}$ be the set of the families of elements of E, that are algebraically free over K. $\mathcal{S}$ is a non–empty set, partially ordered by inclusion and inductive. By Zorn's lemma, $\mathcal{S}$ has maximal elements, i.e. algebraically free families that do not remain free if any element of E is added. Any such maximal algebraically free family is called a *transcendence basis* of E over K. It can be proved that, if B, B' are two transcendence bases, then they have the same cardinality, called the *transcendence degree* of E over K. It is denoted $tr.d.E/K$.

Definition 3.2.3. A *K–algebra* is a ring A containing (a subfield isomorphic to) K.

Let $y_1, \dots, y_n$ be elements of E: the *K–algebra generated by* $y_1, \dots, y_n$ is, by definition, the minimum subring of E containing $K, y_1, \dots, y_n$: it is denoted $K[y_1, \dots, y_n]$ and its elements are polynomials in the elements $y_1, \dots, y_n$ with coefficients in K. Its quotient field $K(y_1, \dots, y_n)$ is the minimum subfield of E containing $K, y_1, \dots, y_n$.

A *finitely generated K–algebra* A is a K–algebra containing finitely many elements $y_1, \dots, y_n$ such that $A = K[y_1, \dots, y_n]$.

Given elements $y_1, \dots, y_n$ in an extension E of K, we can consider the **evaluation homomorphism** from the polynomial ring in n variables to the K-algebra generated by $y_1, \dots, y_n$

$$\varphi : K[x_1, \dots, x_n] \to K[y_1, \dots, y_n] \text{ such that } F(x_1, \dots, x_n) \to F(y_1, \dots, y_n). \tag{3.1}$$

The kernel of φ is formed by the polynomials vanishing at the n-tuple $(y_1, \dots, y_n)$. Therefore φ is injective if and only if $y_1, \dots, y_n$ are algebraically independent over K, if and only if φ gives an isomorphism between the K-algebra $K[y_1, \dots, y_n]$ and the polynomial ring in n variables.

Remark 4. A K-algebra A is finitely generated if and only if A is isomorphic to a quotient of a polynomial ring in finitely many variables over K. Indeed, if $A = K[y_1, \dots, y_n]$, considering the evaluation map φ (3.1), from the homomorphism theorem it follows that $A \simeq K[x_1, \dots, x_n]/\ker\varphi$. Conversely, given a quotient $A = K[x_1, \dots, x_n]/\alpha$, let $\xi = [x_i]$ be the equivalence class of the variable x_i in A. Then any element of A can be written as a polynomial $F(\xi_1, \dots, \xi_n)$, therefore A is the K-algebra generated by $\xi_1, \dots, \xi_n$.

Proposition 3.2.4. *$K(y_1, \dots, y_n)$ has a transcendence basis over K contained in the set $\{y_1, \dots, y_n\}$.*

Proof. Let $\mathcal{S}$ be the set of all subfamilies of $\{y_1, \dots, y_n\}$ formed by algebraically independent elements: $\mathcal{S}$ is a finite set so it has maximal elements with respect to the inclusion. We can assume that $\{y_1, \dots, y_r\}$ is such a maximal family. Then $y_{r+1}, \dots, y_n$ are all algebraic over $K(y_1, \dots, y_r)$ so $K(y_1, \dots, y_n)$ is an algebraic extension of $K(y_1, \dots, y_r)$. If $z \in K(y_1, \dots, y_n)$ is any element, then z is algebraic over $K(y_1, \dots, y_r)$, so the family $\{y_1, \dots, y_r, z\}$ is not algebraically free. □

Corollary 3.2.5. *$tr.d.K(y_1, \dots, y_n)/K \leq n$.*

Let now $A \subset B$ be rings, A a subring of B.

Definition 3.2.6. Let $b \in B$: b is *integral over* A if it is a root of a monic polynomial of $A[x]$, i.e., there exist $a_1, \ldots, a_n \in A$ such that

$$b^n + a_1 b^{n-1} + a_2 b^{n-2} + \cdots + a_n = 0.$$

Such a relation is called an integral equation, or an equation of integral dependence, for b over A.

Note that, if A is a field, then b is integral over A if and only if b is algebraic over A.

Definition 3.2.7. B is called *integral over* A, or, B is an integral extension of A, if any $b \in B$ is integral over A.

We can state now the

Theorem 3.2.8. Normalization Lemma. *Let A be a finitely generated K–algebra and an integral domain. Let $r := tr.d.K(y_1, \ldots, y_n)/K$. Then there exist elements $z_1, \ldots, z_r \in A$, algebraically independent over K, such that A is integral over $K[z_1, \ldots, z_r]$.*

Proof. We postpone the proof to Chapter 4 . □

We start now the proof of the Nullstellensatz.

1^{st} **Step.**

Let K be an algebraically closed field, let $\mathcal{M} \subset K[x_1, \ldots, x_n]$ be a maximal ideal. Then, there exist $a_1, \ldots, a_n \in K$ such that $\mathcal{M} = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$.

Proof. Let K' be the quotient ring $K[x_1, \ldots, x_n]/\mathcal{M}$: it is a field because $\mathcal{M}$ is maximal, and it is a K–algebra finitely generated by the residues in K' of $x_1, \ldots, x_n$. By the Normalization Lemma, there exist $z_1, \ldots, z_r \in K'$, algebraically independent over K, such that K' is integral over $A := K[z_1, \ldots, z_r]$. We claim that A *is a field*: let $f \in A$, $f \neq 0$; $f \in K'$ so there exists $f^{-1} \in K'$, and f^{-1} is integral over A; we fix an integral equation for f^{-1} over A:

$$(f^{-1})^s + a_{s-1}(f^{-1})^{s-1} + \cdots + a_0 = 0$$

where $a_0, \ldots, a_{s-1} \in A$. We multiply this equation by f^{s-1}:

$$f^{-1} + a_{s-1} + \cdots + a_0 f^{s-1} = 0$$

hence $f^{-1} \in A$. So A is both a field and a polynomial ring over K, so $r = 0$ and $A = K$. Therefore K' is an algebraic extension of K, which is algebraically closed, so $K' \simeq K$. Let us fix an isomorphism $\psi : K' = K[x_1, \ldots, x_n]/\mathcal{M} \xrightarrow{\sim} K$ and let $p : K[x_1, \ldots, x_n] \to K' = K[x_1, \ldots, x_n]/\mathcal{M}$ be the canonical epimorphism.

Let $a_i = \psi(p(x_i))$, $i = 1, \ldots, n$. The kernel of $\psi \circ p$ is $\mathcal{M}$, and $x_i - a_i \in \ker(\psi \circ p)$ for any i. So $\langle x_1 - a_1, \ldots, x_n - a_n \rangle \subset \ker(\psi \circ p) = \mathcal{M}$. Since $\langle x_1 - a_1, \ldots, x_n - a_n \rangle$ is maximal (see Example 3.1.2), we conclude the proof of the 1^{st} Step.

2^{nd} **Step** (Weak Nullstellensatz).

Let K be an algebraically closed field, let $\alpha \subsetneq K[x_1, \ldots, x_n]$ be a *proper* ideal. Then $V(\alpha) \neq \emptyset$ i.e. the polynomials of α have at least one common zero in $\mathbb{A}^n_K$.

Proof. Since α is proper, there exists a maximal ideal $\mathcal{M}$ containing α. Then $V(\alpha) \supset V(\mathcal{M})$. By 1^{st} Step, $\mathcal{M} = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$, so $V(\mathcal{M}) = \{P\}$ with $P(a_1, \ldots, a_n)$, hence $P \in V(\alpha)$. For any maximal ideal containing α we get a point in $V(\alpha)$.

3^{rd} **Step** (Rabinowitch method or Rabinowitch trick).

Let K be an algebraically closed field: we will prove that $I(V(\alpha)) \subset \sqrt{\alpha}$. Since the reverse inclusion always holds, this will conclude the proof.

Let $F \in I(V(\alpha))$, $F \neq 0$ (if $F = 0$ the conclusion is clear, because each ideal contains 0), and let $\alpha = \langle G_1, \ldots, G_r \rangle$. The assumption on F means: if P is a point such that $G_1(P) = \cdots = G_r(P) = 0$, then $F(P) = 0$. The Rabinowitch trick consists in introducing an extra variable, and then specializing it. Let us consider the polynomial ring in $n+1$ variables $K[x_1, \ldots, x_{n+1}]$ and let β be the ideal $\beta = \langle G_1, \ldots, G_r, x_{n+1}F - 1 \rangle$: clearly by assumption β has no zeros in $\mathbb{A}^{n+1}$, hence, by Step 2, $1 \in \beta$, i.e. there exist $H_1, \ldots, H_{r+1} \in K[x_1, \ldots, x_{n+1}]$ such that

$$1 = H_1G_1 + \cdots + H_rG_r + H_{r+1}(x_{n+1}F - 1).$$

This is an equality of polynomials, so equality still holds if we give to some of the variables a special value. In particular we can specialize the new variable x_{n+1} replacing it with $\frac{1}{F}$. More formally, we introduce the K-homomorphism $\psi : K[x_1, \ldots, x_{n+1}] \to K(x_1, \ldots, x_n)$ defined by $H(x_1, \ldots, x_{n+1}) \to H(x_1, \ldots, x_n, \frac{1}{F})$.

The polynomials $G_1, \ldots, G_r$ do not contain x_{n+1} so $\psi(G_i) = G_i \ \forall \ i = 1, \ldots, r$. Moreover $\psi(x_{n+1}F - 1) = 0$, $\psi(1) = 1$. Therefore

$$1 = \psi(H_1G_1 + \cdots + H_rG_r + H_{r+1}(x_{n+1}F - 1)) = \psi(H_1)G_1 + \cdots + \psi(H_r)G_r$$

where $\psi(H_i)$ is a rational function with denominator a power of F. By multiplying this relation by a common denominator, that is a power of F, we get an expression of the form:

$$F^m = H'_1G_1 + \cdots + H'_rG_r,$$

so $F \in \sqrt{\alpha}$. □

Corollary 3.2.9. *Let K be an algebraically closed field.*

1. *There is a bijection between the algebraic subsets of $\mathbb{A}^n$ and the radical ideals of $K[x_1, \dots, x_n]$. The bijection is given by $\alpha \to V(\alpha)$ and $X \to I(X)$. In fact, if X is closed in the Zariski topology, then $V(I(X)) = X$; if α is a radical ideal, then $I(V(\alpha)) = \alpha$.*

2. *Let $X, Y \subset \mathbb{A}^n$ be Zariski closed sets. Then*

   (i) $I(X \cap Y) = \sqrt{I(X) + I(Y)}$;

   (ii) $I(X \cup Y) = I(X) \cap I(Y) = \sqrt{I(X)I(Y)}$.

3. *The points of a hypersurface determine its reduced equation.*

Proof. 1. is clear. 2. follows from next Lemma 3.2.10, using the Nullstellensatz. To prove 3., assume that F, G are square-free polynomials in $K[x_1 \dots, x_n]$ such that $V(F) = V(G)$. Notice that if F is square-free, the $\langle F \rangle = \sqrt{F}$. By the Nullstellensatz it follows that $\sqrt{F} = I(V(F)) = I(V(G)) = \sqrt{G}$, so $\langle F \rangle = \langle G \rangle$, which means that F, G differ at most by units. □

Lemma 3.2.10. *Let α, β be ideals of $K[x_1, \dots, x_n]$. Then*

a) $\sqrt{\sqrt{\alpha}} = \sqrt{\alpha}$;

b) $\sqrt{\alpha + \beta} = \sqrt{\sqrt{\alpha} + \sqrt{\beta}}$;

c) $\sqrt{\alpha \cap \beta} = \sqrt{\alpha\beta} = \sqrt{\alpha} \cap \sqrt{\beta}$.

Proof. a) if $F \in \sqrt{\sqrt{\alpha}}$, there exists $r \geq 1$ such that $F^r \in \sqrt{\alpha}$, hence there exists $s \geq 1$ such that $F^{rs} \in \alpha$.

b) $\alpha \subset \sqrt{\alpha}$, $\beta \subset \sqrt{\beta}$ imply $\alpha + \beta \subset \sqrt{\alpha} + \sqrt{\beta}$ hence $\sqrt{\alpha + \beta} \subset \sqrt{\sqrt{\alpha} + \sqrt{\beta}}$.

Conversely, $\alpha \subset \alpha + \beta$, $\beta \subset \alpha + \beta$ imply $\sqrt{\alpha} \subset \sqrt{\alpha + \beta}$, $\sqrt{\beta} \subset \sqrt{\alpha + \beta}$, hence $\sqrt{\alpha} + \sqrt{\beta} \subset \sqrt{\alpha + \beta}$ so $\sqrt{\sqrt{\alpha} + \sqrt{\beta}} \subset \sqrt{\sqrt{\alpha + \beta}} = \sqrt{\alpha + \beta}$.

c) $\alpha\beta \subset \alpha \cap \beta \subset \alpha$ (resp. $\subset \beta$) therefore $\sqrt{\alpha\beta} \subset \sqrt{\alpha \cap \beta} \subset \sqrt{\alpha} \cap \sqrt{\beta}$. If $F \in \sqrt{\alpha} \cap \sqrt{\beta}$, then $F^r \in \alpha$, $F^s \in \beta$ for suitable $r, s \geq 1$, hence $F^{r+s} \in \alpha\beta$, so $F \in \sqrt{\alpha\beta}$.

□

Part 2.*(i)* of Corollary 3.2.9 implies that $I(X \cap Y) = I(X) + I(Y)$ if and only if $I(X) + I(Y)$ is a radical ideal (see Remark 2 (iii)).

Remark 5. The weak form of the Nullstellensatz says that a system of algebraic equations has at least one solution over an algebraically closed field if, and only if, the ideal generated by the corresponding polynomials is proper, or, equivalently, if it is impossible to find a linear combination of them, with coefficients in the polynomial ring, equal to the constant 1. The proof of Nullstellensatz we have given is not constructive, in the sense that, given polynomials $F_1, \dots, F_r$, it does not say how to check if 1 belongs or not to the ideal $\langle F_1, \dots, F_r \rangle$.

The problem of making the proof constructive is connected to the more general "ideal membership problem", which asks, given an ideal $\alpha \subset K[x_1 \dots, x_n]$ and a polynomial $G \in K[x_1 \dots, x_n]$, to decide if $G \in \alpha$ or not.

Answers to these problems can be given with the tools of computational algebra, in particular using the theory of Gröbner bases. There are effective versions of the Nullstellensatz that allow to bound the degrees of the coefficients in a possible expression $1 = H_1F_1 + \cdots + H_rF_r$, depending on the degrees of $F_1, \dots, F_r$, and hence to reduce the question to a problem in linear algebra.

3.3 Homogeneous Nullstellensatz

We move now to the projective space. There exist *proper* homogeneous ideals of $K[x_0, x_1, \dots, x_n]$ without zeros in $\mathbb{P}^n$, even assuming K algebraically closed: for example the maximal ideal $\langle x_0, x_1, \dots, x_n \rangle$. For such an ideal I, the Nullstellensatz fails, indeed $I_h(V_P(I)) = I_h(\emptyset) = K[x_0, \dots, x_n]$, but $\sqrt{I} \neq K]x_0, \dots, x_n]$, because $1 \in I$ if and only if $1 \in \sqrt{I}$.

The following characterization holds:

Proposition 3.3.1. *Let K be an algebraically closed field and let I be a homogeneous ideal of $K[x_0, x_1, \dots, x_n]$.*

The following are equivalent:
(i) $V_P(I) = \emptyset$;
(ii) either $I = K[x_0, x_1, \dots, x_n]$ or $\sqrt{I} = \langle x_0, x_1, \dots, x_n \rangle$;
(iii) there exists $d \geq 1$ such that $I \supset K[x_0, x_1, \dots, x_n]_d$, the homogeneous component of $K[x_0, x_1, \dots, x_n]$ of degree d.

Proof. (i)⇒(ii) Let $p : \mathbb{A}^{n+1} - \{0\} \to \mathbb{P}^n$ be the canonical surjection. We have: $V_P(I) = p(V(I) - \{0\})$, where $V(I) \subset \mathbb{A}^{n+1}$. So if $V_P(I) = \emptyset$, then either $V(I) = \emptyset$ or $V(I) = \{0\}$. If $V(I) = \emptyset$ then $I(V(I)) = I(\emptyset) = K[x_0, x_1, \dots, x_n]$; if $V(I) = \{0\}$, then $I(V(I)) = \langle x_0, x_1, \dots, x_n \rangle = \sqrt{I}$ by the Nullstellensatz.

(ii)⇒(iii) Let $\sqrt{I} = K[x_0, x_1, \dots, x_n]$, then $1 \in \sqrt{I}$ so $1^r = 1 \in I (r \geq 1)$. If $\sqrt{I} = \langle x_0, x_1, \dots, x_n \rangle$, then for any variable x_k there exists an index $i_k \geq 1$ such that $x_k^{i_k} \in I$. If $d \geq i_0 + i_1 + \cdots + i_n - n$, then any monomial of degree d is in I, so $K[x_0, x_1, \dots, x_n]_d \subset I$.

(iii)⇒(i) because no point in $\mathbb{P}^n$ has all coordinates equal to 0. □

Theorem 3.3.2. *Let K be an algebraically closed field and I be a homogeneous ideal of $K[x_0, x_1, \dots, x_n]$. If F is a homogeneous non–constant polynomial such that $V_P(F) \supset V_P(I)$ (i.e. F vanishes on $V_P(I)$, or $F \in I_h(V_P(I))$), then $F \in \sqrt{I}$.*

Proof. We have $p(V(I) - \{0\}) = V_P(I) \subset V_P(F)$. Since F is non–constant, we have also $V(F) = p^{-1}(V_P(F)) \cup \{0\}$, so $V(F) \supset V(I)$; by the Nullstellensatz $I(V(I)) = \sqrt{I} \supset I(V(F)) = \sqrt{(F)} \ni F$. □

Corollary 3.3.3 (homogeneous Nullstellensatz)**.** *Let I be a homogeneous ideal of $K[x_0, x_1, \dots, x_n]$ such that $V_P(I) \neq \emptyset$, K algebraically closed. Then $\sqrt{I} = I_h(V_P(I))$.*

Definition 3.3.4. A homogeneous ideal of $K[x_0, x_1, \dots, x_n]$ such that $\sqrt{I} = \langle x_0, x_1, \dots, x_n \rangle$ is called *irrelevant*.

Corollary 3.3.5. *Let K be an algebraically closed field. There is a bijection between the set of projective algebraic subsets of $\mathbb{P}^n$ and the set of radical homogeneous non–irrelevant ideals of $K[x_0, x_1, \dots, x_n]$.*

Remark 6. Let $X \subset \mathbb{P}^n$ be an algebraic set, $X \neq \emptyset$. The affine cone of X, denoted by $C(X)$, is the following subset of $\mathbb{A}^{n+1}$: $C(X) = p^{-1}(X) \cup \{0\}$, where $p : (K^{n+1})^* \to \mathbb{P}^n$ is the canonical projection (see Section 1.2). If $X = V_P(F_1, \dots, F_r)$, with $F_1, \dots, F_r$ homogeneous, then clearly $C(X) = V(F_1, \dots, F_r)$.

Note that, if K is infinite, then $I(C(X))$ is a homogeneous ideal. Indeed, assuming $X \neq \emptyset$, if $F = F_0 + \cdots + F_d \in I(C(X))$ and $P(a_0, \dots, a_n) \in C(X)$, then for any $\lambda \in K \setminus \{0\}$

$$F(\lambda a_0, \dots, \lambda a_n) = 0 = F_0 + \lambda F_1(a_0, \dots, a_n) + \cdots + \lambda^d F_d(a_0, \dots, a_n);$$

hence $F_0 = F_1(a_0, \dots, a_n) = \cdots = F_d(a_0, \dots, a_n) = 0$, which means that $F_1, \dots, F_d \in I(C(X))$.

Note also that any homogeneous polynomial vanishes on X if and only if it vanishes on $C(X)$, therefore $I(C(X)) = I_h(X)$.

Exercises 3.3.6.

1. Give a non-trivial example of an ideal α of $K[x_1, \dots, x_n]$ such that $\alpha \neq \sqrt{\alpha}$.

2. Let K be an algebraically closed with char $K \neq 2$. Show that the following closed subsets of the affine plane are such that equality does not hold in the relation $I(Y \cap Y') \supset I(Y) + I(Y')$: $Y = V(x^2 + y^2 - 1)$ and $Y' = V(y - 1)$.

3. Let $\alpha \subset K[x_1, \ldots, x_n]$ be an ideal. Prove that $\alpha = \sqrt{\alpha}$ if and only if the quotient ring $K[x_1, \ldots, x_n]/\alpha$ does not contain any non–zero nilpotent.

4. Consider $\mathbb{Z} \subset \mathbb{Q}$. Prove that if an element $y \in \mathbb{Q}$ is integral over $\mathbb{Z}$, then $y \in \mathbb{Z}$. (Hint: fixed $y = a/b \in \mathbb{Q}$ integral over $\mathbb{Z}$, write an integral equation for y, then use the unique factorization in $\mathbb{Z}$.)

5. Let us recall that a prime ideal of a ring R is an ideal $\mathcal{P}$ such that $a \notin \mathcal{P}$, $b \notin \mathcal{P}$ implies $ab \notin \mathcal{P}$. Prove that any prime ideal is a radical ideal.

6. * Let I be a homogeneous ideal of $K[x_1, \ldots, x_n]$ satisfying the following condition: *if F is a homogeneous polynomial such that $F^r \in I$ for some positive integer r, then $F \in I$.* Prove that I is a radical ideal. (Hint: take F non homogeneous such that for some $r \geq 1$ $F^r \in I$, then use induction on the number of non-zero homogeneous components of F to prove that $F \in I$.)

Chapter 4

The Normalization Lemma

Even if it is known as Normalization "Lemma", this is a deep theorem in algebra, with many applications, not merely a lemma to prove the Nullstellensatz. Later we will see how it is used to study the dimension of K-algebras (Chapter 8) and its interesting geometric interpretation (Theorem 17.1.3).

It takes its name from Emmy Noether, who in 1926 proved it under the hypothesis that K is infinite. The case where K is a finite field was proved by Oscar Zariski in 1943. To prove the Normalization Lemma, we will first see a couple of results about integral elements over a ring. Then we will see a proof over an **infinite field**, rather similar to the original one. It is less technical than any proof of the general case. For other proofs see [AM] or [L].

Let $A \subseteq B$ be rings, where A is a subring of B. In this case we also say that B is an A-algebra. Note that B has a natural structure of A-module. If B is finitely generated **as A-module**, then B is called a **finite A-algebra**. This means that there exist elements $b_1, \dots, b_r \in B$ such that $B = b_1A + b_2A + \dots + b_rA$, i.e. any element of B is a linear combination with coefficients in A of the generators $b_1, \dots, b_r$: if $b \in B$, then there is an expression $b = a_1b_1 + \cdots + a_rb_r$, with $a_1, \dots, a_r \in A$.

If B is finitely generated **as a ring containing** A, then B is called a **finitely generated A-algebra**. In this case there exists a finite number of elements of B, $b_1, \dots, b_r$, such that $B = A[b_1, \dots, b_r]$, i.e., B is the minimal ring containing A and the elements $b_1, \dots, b_r$. For any element of B there is an expression as polynomial with coefficients in A in the elements $b_1, \dots, b_r$. Another way to express that B is a finitely generated A-algebra is saying that B is (isomorphic to) a quotient of a polynomial ring in a finite number of variables with coefficients in A. Indeed, if $B = A[b_1, \dots, b_r]$, we can define a surjective ring homomorphism φ mapping any polynomial $f(x_1, \dots, x_r) \in A[x_1, \dots, x_r]$ to $f(b_1, \dots, b_r)$. So, by the homomorphism theorem, $B \simeq A[x_1, \dots, x_r]/\ker\varphi$.

Theorem 4.0.1. *Let $b \in B$, let $A[b] \subseteq B$ be the A-algebra generated by b: $A \subseteq A[b] \subseteq B$. The following are equivalent:*

1. *b is integral over A;*
2. *$A[b]$ is a finite A-algebra;*
3. *there exists a subring C of B, with $A[b] \subseteq C \subseteq B$, such that C is a finite A-algebra.*

Proof. 1) $\Rightarrow$ 2) $A[b]$ is generated by all the powers of b as A-module; we will prove that it is generated by finitely many powers of b. By assumption there is a relation $b^n + a_1 b^{n-1} + \dots + a_n = 0$, with $a_1, \dots, a_n \in A$. Therefore, for any $r \geq 0$, $b^{n+r} = -(a_1 b^{n+r-1} + \dots + a_n b^r)$. By induction on r it follows that all positive powers of b belong to the A-module generated by $1, b, \dots, b^{n-1}$.

2) $\Rightarrow$ 3) It is enough to take $C = A[b]$.

3) $\Rightarrow$ 1) Let $c_1, \dots, c_r$ be generators of C as A-module: $C = c_1 A + \dots + c_r A$. Then, for any $i = 1, \dots, r$, bc_i is a linear combination of $c_1, \dots, c_r$ with coefficients in A. So there exists an $r \times r$ matrix $M = (m_{ij})_{i,j=1\dots,r}$ with entries in A such that

$$bc_i = \sum_{j=1}^{r} m_{ij} c_j, \tag{4.1}$$

i.e. $(bE_r - M)\underline{c} = 0$, where $\underline{c} = {}^t(c_1 \dots c_r)$ and E_r is the identity matrix. Multiplying both members of equation (4.1) at the left by the adjoint matrix ${}^{ad}(bE_r - M)$, we get $\det(bE_r - M)c_i = 0$ for any i. Since $c_1, \dots, c_r$ generate C, there is an expression $1 = c_1\alpha_1 + \dots + c_r\alpha_r$. Therefore $\det(bE_r - M) = \det(bE_r - M)\cdot 1 = \det(bE_r - M)c_1\alpha_1 + \dots + \det(bE_r - M)c_r\alpha_r = 0$. The expansion of $\det(bE_r - M)$ gives a relation of integral dependence of b over A. □

Corollary 4.0.2. *If $b \in B$ is integral over A, then $A[b]$ is integral extension of A.*

Proof. If $y \in A[b]$, then $A[y] \subset A[b] \subset B$, where $A[b]$ is a finite A-algebra by 2. of Theorem 4.0.1. The conclusion follows from the characterization 3. of integral elements of the same Theorem. □

Remark 7. Equation (4.1) says that b is an eigenvalue of the matrix M. The conclusion is that b is a root of the characteristic polynomial of M. But, since we work over a ring not over a field, we cannot jump straight to the conclusion. In fact we have to use the assumption that $c_1, \dots, c_r$ generate C as A-module.

Remark 8. We will need also the following easy property, known as "**Transitivity of finiteness**". Let $A \subseteq B$. Suppose that N is a finitely generated B-module. Then N is also an A-module, by restriction of the scalars. Assume also that B is finitely generated as an A-module. Then N is finitely generated as an A-module. Indeed if $y_1, \ldots, y_n$ generate N over B and $x_1, \ldots, x_m$ generate B as A-module, then the mn products $x_i y_j$ generate N over A.

Corollary 4.0.3. *Let $A \subseteq B$.*

1. *Let $b_1, \ldots, b_n \in B$ be integral over A. Then $A[b_1, \ldots, b_n]$ is a finite A-module.*

2. **Transitivity of integral dependence**. *Let $A \subset B \subset C$ be rings. If B is integral extension of A and C is integral extension of B, then C is integral extension of A.*

Proof. 1. By induction on n. The case $n = 1$ is part of Theorem 4.0.1. Assume $n > 1$, let $A_r = A[b_1, \ldots, b_r]$; then by inductive hypothesis A_{n-1} is a finitely generated A-module. $A_n = A_{n-1}[b_n]$ is a finitely generated A_{n-1}-module by the case $n = 1$, since b_n is integral over A and hence also over A_{n-1}. Then the thesis follows by the transitivity of finiteness (Remark 8).

2. Let $c \in C$, then we have an equation $c^n + b_1 c^{n-1} + \cdots + b_n = 0$, with $b_i \in B$ for any index i. The ring $B' = A[b_1, \ldots, b_n]$ is a finitely generated A-module by part 1., and $B'[c]$ is a finitely generated B'-module, since c is integral over B'. Hence $B'[c]$ is a finite A-module, by transitivity of finiteness (Remark 8), and therefore c is integral over A by Theorem 4.0.1 3). □

We are now ready to prove

Theorem 4.0.4. Normalization Lemma. *Let $A = K[y_1, \ldots, y_n]$ be a finitely generated K–algebra and an integral domain. Let $r :=$ tr.d. $Q(A)/K =$ tr.d. $K(y_1, \ldots, y_n)/K$. Then there exist elements $z_1, \ldots, z_r \in A$, algebraically independent over K, such that A is integral over the K-algebra $B = K[z_1, \ldots, z_r]$.*

Proof. We give a proof by induction on n, assuming that K is infinite.

If $n = 1$, then $A = K[y]$. There are two possibilities, either $r = 1$ or $r = 0$; $r = 1$ if and only if y is transcendental over K, in this case $A = B$; $r = 0$, if and only if y is algebraic over K, in which case A is an algebraic extension of finite degree of K and $B = K$.

Let $n \geq 2$ and assume the theorem is true for K-algebras with $n - 1$ generators. Let $\varphi : K[x_1, \ldots, x_n] \to A$ be the surjective homomorphism mapping a polynomial $f(x_1, \ldots, x_n)$ to $f(y_1, \ldots, y_n)$. If φ is an isomorphism, then $r = n$ and $B = A$. So we assume that

$ker\ \varphi \neq (0)$ and $r < n$: there exists a non-zero polynomial f such that $f(y_1, \ldots, y_n) = 0$. Possibly renaming the variables, we can assume that x_n appears explicitly in f.

Assume first that f is monic of degree d with respect to x_n; then $f(y_1, \ldots, y_n) = 0$ is a relation of integral dependence for y_n over $K[y_1, \ldots, y_{n-1}]$, which implies that $A = K[y_1, \ldots, y_n]$ is a finite module over $K[y_1, \ldots, y_{n-1}]$, generated by $1, y_n, \ldots, y_n^{d-1}$. By Theorem 4.0.1, every element of A is integral over $K[y_1, \ldots, y_{n-1}]$. By inductive assumption, there exists $B = K[z_1, \ldots, z_r]$ with $z_1, \ldots, z_r$ algebraically independent over K, such that $K[y_1, \ldots, y_{n-1}]$ is integral over B. By Transitivity of integral dependence (Corollary 4.0.3 2.), also A is integral over B.

It remains the case where in the kernel of φ there is no monic polynomial in x_n. We claim that we can "change coordinates" **linearly** in $K[x_1, \ldots, x_n]$ in such a way that the polynomial f becomes monic. This means that there is another surjection $K[x_1, \ldots, x_n] \to A$ such that some element of the kernel is monic in x_n.

We consider the linear change of coordinates $x_i \to x_i + a_i x_n$, for $1 \leq i \leq n-1$ and $x_n \to x_n$, where the a_i's are suitable elements of K we are going to choose. Write f as sum of its homogeneous components $f = f_d + \text{lower degree terms}$, where $d = \deg f$. Under this transformation, $f \to f(x_1 + a_1 x_n, \ldots, x_{n-1} + a_{n-1}x_n, x_n)$. We claim it is possible to choose the coefficients a_i so that in this new polynomial the coefficient of x_n^d is non-zero, so that f has degree d also in the variable x_n. Just replacing we get $f(x_1 + a_1 x_n, \ldots, x_{n-1} + a_{n-1}x_n, x_n) = f_d(x_1 + a_1 x_n, \ldots, x_{n-1} + a_{n-1}x_n, x_n) + \text{lower degree terms}$. Then we expand the top degree term and we get $f_d(x_1 + a_1 x_n, \ldots, x_{n-1} + a_{n-1}x_n, x_n) = f_d(a_1, \ldots, a_{n-1}, 1)x_n^d + \text{lower degree terms in } x_n$. Adding gives

$$f(x_1 + a_1 x_n, \ldots, x_{n-1} + a_{n-1}x_n, x_n) = f_d(a_1, \ldots, a_{n-1}, 1)x_n^d + \text{lower degree terms in } x_n.$$

Thus we just have to choose the a_i's so that $f_d(a_1, \ldots, a_{n-1}, 1) \neq 0$. Since f_d is a non-zero homogeneous polynomial of degree $d \geq 1$, $f_d(x_1, \ldots, x_{n-1}, 1)$ is a non-zero polynomial of degree less than or equal to d in $x_1, \ldots, x_{n-1}$. Since the field K is infinite, we are done thanks to Exercise 1 in Chapter 2. $\square$

Remark 9. This proof has been adapted from MathOverflow, a "question and answer site for professional mathematicians": https://mathoverflow.net/questions/92354/noether-normalization

The same proof can be found in the book [R]. The original article of Emmy Noether is unfortunately in German [N].

A nice article on Normalization Lemma, by Judith Sally, can be found in the book "Emmy Noether in Bryn Mawr", published in the occasion of her 100th birthday ([jS]).

Emmy Noether (1882-1935) is the founder of modern algebra; her story is very interesting and in some aspects symbolic of the difficulties encountered by women mathematicians. As quoted in Wikipedia "In a letter to The New York Times, Albert Einstein wrote:

In the judgment of the most competent living mathematicians, Fräulein Noether was the most significant creative mathematical genius thus far produced since the higher education of women began. In the realm of algebra, in which the most gifted mathematicians have been busy for centuries, she discovered methods which have proved of enormous importance in the development of the present-day younger generation of mathematicians.

On 2 January 1935, a few months before her death, mathematician Norbert Wiener wrote *"Miss Noether is ... the greatest woman mathematician who has ever lived; and the greatest woman scientist of any sort now living, and a scholar at least on the plane of Madame Curie. "*

See also http://www.enciclopediadelledonne.it/biografie/emmy-noether/

Chapter 5

The projective closure

5.1 Projective closure and its ideal

In this chapter we will identify the affine space $\mathbb{A}^n$ with the open subet $U_0 \subset \mathbb{P}^n$. As we have seen in Section 2.6, this is possible via the homeomorphisms, inverse each other, $\varphi_0 : U_0 \to \mathbb{A}^n$ and $j_0 : \mathbb{A}^n \to U_0$. Similar considerations hold for any index $i = 0, \dots, n$.

Given an affine variety $X \subset \mathbb{A}^n = U_0 \subset \mathbb{P}^n$, in this way it becomes a subet of $\mathbb{P}^n$ and it makes sense to consider its closure in the Zariski topology of the projective space.

Definition 5.1.1. The **projective closure** of X, $\overline{X}$, is the closure of X in the Zariski topology of $\mathbb{P}^n$.

Since the map φ_0 is a homeomorphism, we have: $\overline{X} \cap \mathbb{A}^n = X$ because X is closed in $\mathbb{A}^n$. The points of $\overline{X} \cap H_0$, where H_0 is the hyperplane at infinity $V_P(x_0)$, are called the "points at infinity" of X in the fixed embedding.

Remark 10. Note that, if K is an infinite field, then the projective closure of $\mathbb{A}^n$ is $\mathbb{P}^n$, i.e. the affine space is dense in the projective space.

Indeed, let F be a homogeneous polynomial of degree d vanishing along $\mathbb{A}^n = U_0$. We can write $F = F_0 x_0^d + F_1 x_0^{d-1} + \dots + F_d$, where F_i is a homogeneous polynomial of degree i in $x_1, \dots, x_n$ for any i. By assumption, for every $P(a_1, \dots, a_n) \in \mathbb{A}^n$, $P \in V_P(F)$, i.e. $F(1, a_1, \dots, a_n) = 0 = {}^aF(a_1, \dots, a_n)$. So ${}^aF \in I(\mathbb{A}^n)$. We claim that $I(\mathbb{A}^n) = (0)$: if $n = 1$, this follows from the principle of identity of polynomials, because K is infinite. If $n \geq 2$, assume that $F(a_1, ..., a_n) = 0$ for all $(a_1, ..., a_n) \in K^n$ and consider $F(a_1, ..., a_{n-1}, x)$: either it has positive degree in x for some choice of $(a_1, ..., a_n)$, but then it has finitely many zeros against the assumption; or it is constant in x for any choice of $(a_1, \dots, a_n)$, so F belongs to

$K[x_1, ..., x_{n-1}]$ and we can conclude by induction. So the claim is proved. We get therefore that $F_0 = F_1 = \ldots = F_d = 0$ and $F = 0$.

We want to find the relation between the equations of $X \subset \mathbb{A}^n$ and those of its projective closure $\overline{X} \subset \mathbb{P}^n$.

Proposition 5.1.2. *Let $X \subset \mathbb{A}^n$ be an affine variety, $\overline{X}$ be its projective closure. Then*

$$I_h(\overline{X}) = {}^hI(X) := \langle {}^hF | F \in I(X) \rangle.$$

Proof. Let $F \in I_h(\overline{X})$ be a homogeneous polynomial. If $P(a_1, \ldots, a_n) \in X$, then $[1, a_1, \ldots, a_n] \in \overline{X}$, so $F(1, a_1, \ldots, a_n) = 0 = {}^aF(a_1, \ldots, a_n)$. Hence ${}^aF \in I(X)$. There exists $k \geq 0$ such that $F = (x_0^k)^h({}^aF)$ (see proof of Proposition 2.6.1), so $F \in {}^hI(X)$. Hence $I_h(\overline{X}) \subset {}^hI(X)$.

Conversely, if $G \in I(X)$ and $P(a_1, \ldots, a_n) \in X$, then $G(a_1, \ldots, a_n) = 0 = {}^hG(1, a_1, \ldots, a_n)$, so ${}^hG \in I_h(X)$ (here X is seen as a subset of $\mathbb{P}^n$). So ${}^hI(X) \subset I_h(X)$. Since $I_h(X) = I_h(\overline{X})$ (see Exercise 1), we have the claim. □

In particular, if X is a hypersurface and $I(X) = \langle F \rangle$, then $I_h(\overline{X}) = \langle {}^hF \rangle$.

Next example, that will occupy the rest of this Chapter, will show that, **in general**, from $I(X) = \langle F_1, \ldots, F_r \rangle$, it does not follow ${}^hI(X) = \langle {}^hF_1, \ldots, {}^hF_r \rangle$. Only in the last thirty years, thanks to the development of symbolic algebra and in particular of the theory of Gröbner bases, the problem of characterizing the systems of generators of $I(X)$, whose homogeneization generates ${}^hI(X)$, has been solved.

5.2 An extended example: the skew cubic

The example of the skew cubic is of fundamental importance in algebraic geometry, because of the many geometrical phenomena that appear, and are developed in different classes of varieties of which the skew cubic is the first case.

Example 5.2.1 (The skew cubic)**.** In this example we assume that K is infinite. The affine skew cubic is the following closed subset X of $\mathbb{A}^3$: $X = V(y - x^2, z - x^3)$ (we use variables x, y, z). X is the image of the map $\varphi : \mathbb{A}^1 \to \mathbb{A}^3$ such that $\varphi(t) = (t, t^2, t^3)$. Note that $\varphi : \mathbb{A}^1 \to X$ is a homeomorphism (see Exercise 3, Chapter 1). Let α be the ideal $\langle y - x^2, z - x^3 \rangle$. Note that $X = V(\alpha)$. We claim that $\alpha = I(X) = \{F \in K[x, y, z] \mid F(x, x^2, x^3) = 0$ for any $x \in K\}$. Proceeding as in Chapter 3, Example 3.1.2, we consider the development of any polynomial $G \in K[x, y, z]$ in Taylor series around (x, x^2, x^3), and we get the claim. We observe also that α is a prime ideal; to see this, we consider the ring homomorphism $K[x,y,z] \to K[x]$ such that $F(x,y,z) \to F(x,x^2,x^3)$: it is surjective and its kernel is α, therefore the quotient ring $K[x,y,z]/\alpha$ is isomorphic to $K[x]$, which is an integral domain. Therefore α is prime.

Let $\overline{X}$ be the projective closure of X in $\mathbb{P}^3$. First we will study $\overline{X}$ geometrically, then we will determine its homogeneous ideal. We claim that it is the image of the map $\psi : \mathbb{P}^1 \to \mathbb{P}^3$ such that $\psi([\lambda,\mu]) = [\lambda^3, \lambda^2\mu, \lambda\mu^2, \mu^3]$. We identify $\mathbb{A}^1$ with the open subset of $\mathbb{P}^1$ defined by $\lambda \neq 0$ i.e. U_0, and $\mathbb{A}^3$ with the open subset of $\mathbb{P}^3$ defined by $x_0 \neq 0$ (U_0 again). Note that $\psi|_{\mathbb{A}^1} = \varphi$, because $\psi([1,t]) = [1,t,t^2,t^3] =$ via the identification of $\mathbb{A}^3$ with $U_0 = (t,t^2,t^3) = \varphi(t)$. Moreover $\psi([0,1]) = [0,0,0,1]$. So $\psi(\mathbb{P}^1) = X \cup \{[0,0,0,1]\}$.

Let G be a homogeneous polynomial of $K[x_0,x_1,x_2,x_3]$ such that $X \subset V_P(G)$. Then $G(1,t,t^2,t^3) = 0\ \forall t \in K$, so $G(\lambda^3, \lambda^2\mu, \lambda\mu^2, \mu^3) = 0\ \forall \mu \in K,\ \forall \lambda \in K^*$. Since K is infinite, then $G(\lambda^3, \lambda^2\mu, \lambda\mu^2, \mu^3)$ is the zero polynomial in λ and μ, so $G(0,0,0,1) = 0$ and $V_P(G) \supset \psi(\mathbb{P}^1)$, therefore $\overline{X} \supset \psi(\mathbb{P}^1)$.

Conversely, we prove that $\psi(\mathbb{P}^1)$ is Zariski closed, more precisely

$$\psi(\mathbb{P}^1) = V_P(F_0,F_1,F_2) \text{ where } F_0 := x_1x_3 - x_2^2, F_1 := x_1x_2 - x_0x_3, F_2 := x_0x_2 - x_1^2.$$

One inclusion is clear: every point of $\mathbb{P}^3$ of coordinates $[\lambda^3, \lambda^2\mu, \lambda\mu^2, \mu^3]$ satisfies the three quadratic equations $F_0 = F_1 = F_2 = 0$. Conversely, let $F_i(y_0,\dots,y_3) = 0\ \forall i = 1,\dots,3$, i.e. $y_1y_3 = y_2^2$, $y_1y_2 = y_0y_3$, $y_0y_2 = y_1^2$. We observe that either $y_0 \neq 0$ or $y_3 \neq 0$, otherwise also $y_1 = y_2 = 0$.

Assume $y_0 \neq 0$, then, using the three equations, we get

$$[y_0,y_1,y_2,y_3] = [y_0^3, y_0^2y_1, y_0^2y_2, y_0^2y_3] = [y_0^3, y_0^2y_1, y_0y_1^2, y_0y_1y_2] = [y_0^3, y_0^2y_1, y_0y_1^2, y_1^3] = \psi([y_0,y_1]).$$

Similarly, if $y_3 \neq 0$, $[y_0,y_1,y_2,y_3] = \psi([y_2,y_3])$. So $\psi(\mathbb{P}^1) = \overline{X}$.

The three polynomials F_0,F_1,F_2 are the 2×2 minors of the matrix

$$M = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$$

with entries in $K[x_0,x_1,x_2,x_3]$. Let $F = y - x^2$, $G = z - x^3$ be the two generators of $I(X)$; ${}^hF = x_0x_2 - x_1^2$, ${}^hG = x_0^2x_3 - x_1^3$, hence $V_P({}^hF, {}^hG) = V_P(x_0x_2 - x_1^2, x_0^2x_3 - x_1^3) \neq \overline{X}$, because $V_P({}^hF, {}^hG)$ contains the whole line "at infinity" $V_P(x_0,x_1)$, which is not contained in $\overline{X}$.

We have seen that the projective closure of the affine skew cubic X is $\overline{X} = V_P(F_0,F_1,F_2)$; we shall prove now the non-trivial fact:

Proposition 5.2.2. $I_h(\overline{X}) = \langle F_0, F_1, F_2\rangle$.

Proof. For any integer number $d \geq 0$, let $I_h(\overline{X})_d := I_h(\overline{X}) \cap K[x_0, x_1, x_2, x_3]_d$: it is a K-vector space of dimension $\leq \binom{d+3}{3}$. We define a K-linear map ρ_d having $I_h(\overline{X})_d$ as kernel:

$$\rho_d : K[x_0, x_1, x_2, x_3]_d \to K[\lambda, \mu]_{3d}$$

such that $\rho_d(F) = F(\lambda^3, \lambda^2\mu, \lambda\mu^2, \mu^3)$. Since ρ_d is clearly surjective, we compute

$$\dim I_h(\overline{X})_d = \binom{d+3}{3} - (3d+1) = (d^3 + 6d^2 - 7d)/6.$$

For $d \geq 2$, we define now a second K-linear map

$$\varphi_d : K[x_0, x_1, x_2, x_3]_{d-2}^{\oplus 3} \to I_h(\overline{X})_d$$

such that $\varphi_d(G_0, G_1, G_2) = G_0F_0 + G_1F_1 + G_2F_2$. Our aim is to prove that φ_d is surjective. The elements of its kernel are called the *syzygies of degree* d among the polynomials F_0, F_1, F_2. Two obvious syzygies of degree 3 are constructed by developing, according to the Laplace rule, the determinant of the matrix obtained repeating one of the rows of M, for example

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}.$$

It gives $x_0F_0 + x_1F_1 + x_2F_2 = 0$, so (x_0, x_1, x_2) is a syzygy of degree 3. Similarly (x_1, x_2, x_3).

We put $H_1 = (x_0, x_1, x_2)$ and $H_2 = (x_1, x_2, x_3)$, they both belong to $\ker \varphi_3$. Note that H_1 and H_2 give rise to syzygies of all degrees ≥ 3, in fact we can construct a third linear map

$$\psi_d : K[x_0, x_1, x_2, x_3]_{d-3}^{\oplus 2} \to \ker \varphi_d$$

putting $\psi_d(A, B) = H_1A + H_2B = (x_0, x_1, x_2)A + (x_1, x_2, x_3)B = (x_0A + x_1B, x_1A + x_2B, x_2A + x_3B)$.

Claim. ψ_d is an isomorphism.

Assuming the claim, we are able to compute $\dim \ker \varphi_d = 2\binom{d}{3}$, therefore

$$\dim Im\ \varphi_d = 3\binom{d+1}{3} - 2\binom{d}{3}$$

which coincides with the dimension of $I_h(\overline{X})_d$ previously computed. This proves that φ_d is surjective for all d and concludes the proof of the Proposition.

Proof of the Claim. Let (G_0, G_1, G_2) belong to $\ker \varphi_d$. This means that the following matrix N with entries in $K[x_0, x_1, x_2, x_3]$ is non-invertible:

$$N := \begin{pmatrix} G_0 & G_1 & G_2 \\ x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}.$$

Therefore, the rows of N are linearly dependent over the quotient field of the polynomial ring $K(x_0, \ldots, x_3)$. Since the last two rows are linearly independent, there exist reduced rational functions $\frac{a_1}{a_0}, \frac{b_1}{b_0} \in K(x_0, x_1, x_2, x_3)$, such that

$$G_0 = \frac{a_1}{a_0}x_0 + \frac{b_1}{b_0}x_1 = \frac{a_1b_0x_0 + a_0b_1x_1}{a_0b_0}$$

and similarly

$$G_1 = \frac{a_1b_0x_1 + a_0b_1x_2}{a_0b_0}, G_2 = \frac{a_1b_0x_2 + a_0b_1x_3}{a_0b_0}$$

The G_i's are polynomials, therefore the denominator a_0b_0 divides the numerator in each of the three expressions on the right hand side. Moreover, if p is a prime factor of a_0, then p divides the three products b_0x_0, b_0x_1, b_0x_2, hence p divides b_0. We can repeat the reasoning for a prime divisor of b_0, so obtaining that $a_0 = b_0$ (up to invertible constants). We get:

$$G_0 = \frac{a_1x_0 + b_1x_1}{b_0}, G_1 = \frac{a_1x_1 + b_1x_2}{b_0}, G_2 = \frac{a_1x_2 + b_1x_3}{b_0},$$

therefore b_0 divides the numerators

$$c_0 := a_1x_0 + b_1x_1, c_1 := a_1x_1 + b_1x_2, c_2 := a_1x_2 + b_1x_3.$$

Hence b_0 divides also $x_1c_0 - x_0c_1 = b_1(x_1^2 - x_0x_1) = -b_1F_2$, and similarly $x_2c_0 - x_0c_2 = b_1F_1$, $x_2c_1 - x_1c_2 = -b_1F_0$. But F_0, F_1, F_2 are irreducible and coprime, so we conclude that $b_0 \mid b_1$. But b_0 and b_1 are coprime, so finally we get $b_0 = a_0 = 1$. □

As an important by-product of the proof of Proposition 5.2.2 we have the **minimal free resolution** of the R-module $I_h(\overline{X})$, where $R = K[x_0, x_1, x_2, x_3]$:

$$0 \to R^{\oplus 2} \xrightarrow{\psi} R^{\oplus 3} \xrightarrow{\varphi} I_h(\overline{X}) \to 0$$

where ψ is represented by the transposed of the matrix M and φ by the triple of polynomials (F_0, F_1, F_2).

Exercises 5.2.3.

1. Let $X \subset \mathbb{A}^n$ be a closed subset, $\overline{X}$ be its projective closure in $\mathbb{P}^n$. Prove that $I_h(X) = I_h(\overline{X})$.

2. Find a system of generators of the ideal of the affine skew cubic X, such that, if you homogeneize them, you get a system of generators for $I_h(\overline{X})$.

Chapter 6

Irreducible components

6.1 Irreducible topological spaces

The aim of this chapter is to introduce the irreducible components of the affine varieties, the "building blocks" of the algebraic varieties. The idea is that the irreducible varieties are a generalization in any dimensions of the irreducible hypersurfaces: any hypersurface is a finite union of irreducible hypersurfaces, similarly any algebraic variety (affine or projective) is a finite unione of irreducible varieties. The notion of irreducible topological space is typical of algebraic geometry and is interesting in this context, although it is not so for Hausdorff topological spaces.

Definition 6.1.1. Let X be a topological space. X is *irreducible* if it is not empty and the following condition holds: if $X = X_1 \cup X_2$ with X_1, X_2 closed subsets of X, then either $X = X_1$ or $X = X_2$.

Equivalently, passing to the complementar sets, X is irreducible if it is non empty and, for all pair of non–empty open subsets U, V, we have $U \cap V \neq \emptyset$.

Note that, by definition, $\emptyset$ is not irreducible.

Proposition 6.1.2. *X is irreducible if and only if any non–empty open subset U of X is dense in X.*

Proof. Let X be irreducible, let P be a point of X and let I_P be an open neighbourhood of P in X. I_P and U are non–empty and open, so $I_P \cap U \neq \emptyset$, therefore $P \in \overline{U}$. This proves that $\overline{U} = X$.

Conversely, assume that all open subsets are dense. Let U, $V \neq \emptyset$ be open subsets. Let $P \in U$ be a point. By assumption $P \in \overline{V} = X$, so $V \cap U \neq \emptyset$ (U is an open neighbourhood of P). □

Example 6.1.3. *1. If $X = \{P\}$ is a unique point, then X is irreducible.*

2. Let K be an infinite field. Then $\mathbb{A}^1$ is irreducible, because proper closed subsets are finite sets. The same holds for $\mathbb{P}^1$.

3. Let $f : X \to Y$ be a continuous map of topological spaces. If X is irreducible and f is surjective, then Y is irreducible.

4. Let $Y \subset X$, $Y \neq \emptyset$, be a subset endowed with the induced topology. Then Y is irreducible if and only if the following holds: if $Y \subset Z_1 \cup Z_2$, with Z_1 and Z_2 closed in X, then either $Y \subset Z_1$ or $Y \subset Z_2$; equivalently: if $Y \cap U \neq \emptyset$, $Y \cap V \neq \emptyset$, with U, V open subsets of X, then $Y \cap U \cap V \neq \emptyset$.

Proposition 6.1.4. *Let X be a topological space, Y a subset of X. Y is irreducible if and only if $\overline{Y}$ is irreducible.*

Proof. Note first that if $U \subset X$ is open and $U \cap Y = \emptyset$ then $U \cap \overline{Y} = \emptyset$. Otherwise, if $P \in U \cap \overline{Y}$, let A be an open neighbourhood of P: then $A \cap Y \neq \emptyset$. In particular, U is an open neighbourhood of P so $U \cap Y \neq \emptyset$.

Let Y be irreducible. If U and V are open subsets of X such that $U \cap \overline{Y} \neq \emptyset$, $V \cap \overline{Y} \neq \emptyset$, then $U \cap Y \neq \emptyset$ and $V \cap Y \neq \emptyset$ so $Y \cap U \cap V \neq \emptyset$ by the irreducibility of Y. Hence $\overline{Y} \cap (U \cap V) \neq \emptyset$. So $\overline{Y}$ is irreducible. If $\overline{Y}$ is irreducible, we get the irreducibility of Y in a completely analogous way. □

Corollary 6.1.5. *Let X be an irreducible topological space and let U be a non–empty open subset of X. Then U is irreducible.*

Proof. By Proposition 6.1.2 $\overline{U} = X$, which is irreducible. By Proposition 6.1.4 U is irreducible. □

6.2 Irreducible algebraic varieties

For algebraic sets (both affine and projective) irreducibility can be expressed in a purely algebraic way.

Proposition 6.2.1. *Let $X \subset \mathbb{A}^n$ (resp. $\mathbb{P}^n$) be an algebraic variety equipped with the Zariski topology, i.e. the induced topology by the Zariski topology of the affine (or projective) space. X is irreducible if and only if $I(X)$ (resp. $I_h(X)$) is prime.*

Proof. Assume first that X is irreducible, $X \subset \mathbb{A}^n$. Let F, G be polynomials in $K[x_1, \dots, x_n]$ such that $FG \in I(X)$: then

$$V(F) \cup V(G) = V(FG) \supset V(I(X)) = X,$$

hence either $X \subset V(F)$ or $X \subset V(G)$. In the former case, if $P \in X$ then $F(P) = 0$, so $F \in I(X)$, in the second case $G \in I(X)$; hence $I(X)$ is prime.

Assume now that $I(X)$ is prime. Let $X = X_1 \cup X_2$ be the union of two closed subsets. Then $I(X) = I(X_1) \cap I(X_2)$ (see Lesson 4). Assume that $X_1 \neq X$, then $I(X_1)$ strictly contains $I(X)$, otherwise, if $I(X) = I(X_1)$, it would follow $X_1 = V(I(X_1)) = V(I(X)) = X$ because both are closed. So there exists $F \in I(X_1)$ such that $F \notin I(X)$. But for every $G \in I(X_2)$, $FG \in I(X_1) \cap I(X_2) = I(X)$, which is prime: since $F \notin I(X)$, then $G \in I(X)$. So $I(X_2) \subset I(X)$, and we conclude that $I(X_2) = I(X)$, so $X_2 = X$.

If $X \subset \mathbb{P}^n$, the proof is similar, taking into account the following Lemma.

Lemma 6.2.2. *Let $\mathcal{P} \subset K[x_0, x_1, \dots, x_n]$ be a homogeneous ideal. Then $\mathcal{P}$ is prime if and only if, for every pair of homogeneous polynomials F, G such that $FG \in \mathcal{P}$, either $F \in \mathcal{P}$ or $G \in \mathcal{P}$.*

Proof of the Lemma. Let H, K be any polynomials such that $HK \in \mathcal{P}$. Let $H = H_0 + H_1 + \cdots + H_d$, $K = K_0 + K_1 + \cdots + K_e$ (with $H_d \neq 0 \neq K_e$) be their expressions as sums of homogeneous polynomials. Then $HK = H_0K_0 + (H_0K_1 + H_1K_0) + \cdots + H_dK_e$: the last product is the homogeneous component of degree $d + e$ of HK. $\mathcal{P}$ being homogeneous, $H_dK_e \in \mathcal{P}$; by assumption either $H_d \in \mathcal{P}$ or $K_e \in \mathcal{P}$. In the former case, $HK - H_dK = (H - H_d)K$ belongs to $\mathcal{P}$ while in the second one $H(K - K_e) \in \mathcal{P}$. So in both cases we can proceed by induction. □

We list now some consequences of Proposition 6.2.1.

1. Let K be an infinite field. Then $\mathbb{A}^n$ and $\mathbb{P}^n$ are irreducible, because $I(\mathbb{A}^n) = I_h(\mathbb{P}^n) = (0)$, and (0) is prime because any polynomial ring with coefficients in a field is an integral domain.

2. Let $Y \subset \mathbb{P}^n$ be closed. Y is irreducible if and only if its affine cone $C(Y)$ is irreducible.

3. Let $Y = V(F) \subset \mathbb{A}^n$, be a hypersurface over an algebraically closed field K. If F is irreducible, then Y is irreducible.

4. Let K be algebraically closed. There is a bijection between prime ideals of $K[x_1, \dots, x_n]$ and irreducible algebraic subsets of $\mathbb{A}^n$. In particular, the maximal ideals correspond to the

points. Similarly, there is a bijection between homogeneous non–irrelevant prime ideals of $K[x_0, x_1, \dots, x_n]$ and irreducible algebraic subsets of $\mathbb{P}^n$.

6.3 Irreducible components

Our next task is to prove that any algebraic variety can be written as a **finite** union of irreducible varieties.

Definition 6.3.1. A topological space X is called *noetherian* if it satisfies the following equivalent conditions:

(i) the ascending chain condition for open subsets;

(ii) the descending chain condition for closed subsets;

(iii) any non–empty set of open subsets of X has maximal elements;

(iv) any non–empty set of closed subsets of X has minimal elements.

The proof of the equivalence is standard (compare with the properties defining noetherian rings).

Example 6.3.2. *$\mathbb{A}^n$ is noetherian: if the following is a descending chain of closed subsets of $\mathbb{A}^n$*

$$Y_1 \supset Y_2 \supset \cdots \supset Y_k \supset \dots,$$

then

$$I(Y_1) \subset I(Y_2) \subset \cdots \subset I(Y_k) \subset \dots$$

is an ascending chain of ideals of $K[x_1, \dots, x_n]$, hence it is stationary from a suitable m on; therefore $V(I(Y_m)) = Y_m = V(I(Y_{m+1})) = Y_{m+1} = \dots$.

Similarly $\mathbb{P}^n$ is noetherian.

Proposition 6.3.3. *Let X be a noetherian topological space and Y be a non–empty closed subset of X. Then Y can be written as a finite union $Y = Y_1 \cup \cdots \cup Y_r$ of irreducible closed subsets. The maximal Y_i's in the union are uniquely determined by Y and are called the "irreducible components" of Y. They are the maximal irreducible subsets of Y.*

Proof. By contradiction. Let S be the set of the non–empty closed subsets of X which are not a finite union of irreducible closed subsets: assume $S \neq \emptyset$. By noetherianity S has minimal elements, fix one of them Z. Z is not irreducible, so $Z = Z_1 \cup Z_2$, $Z_i \neq Z$ for $i = 1, 2$. So $Z_1, Z_2 \notin S$, hence Z_1, Z_2 are both finite unions of irreducible closed subsets, so such is Z: a contradiction.

Now assume that $Y = Y_1 \cup \cdots \cup Y_r$, with $Y_i \not\subseteq Y_j$ if $i \neq j$ and Y_i irreducible closed for all i. If there is another similar expression $Y = Y'_1 \cup \cdots \cup Y'_s$, $Y'_i \not\subseteq Y'_j$ for $i \neq j$, then $Y'_1 \subset Y_1 \cup \ldots Y_r$, so $Y'_1 = \bigcup_{i=1}^r (Y'_1 \cap Y_i)$, hence $Y'_1 \subset Y_i$ for some i, and we can assume $i = 1$. Similarly, $Y_1 \subset Y'_j$, for some j, so $Y'_1 \subset Y_1 \subset Y'_j$, so $j = 1$ and $Y_1 = Y'_1$. Now let $Z = \overline{Y - Y_1} = Y_2 \cup \cdots \cup Y_r = Y'_2 \cup \cdots \cup Y'_s$ and proceed by induction. □

Corollary 6.3.4. *Any algebraic variety in $\mathbb{A}^n$ (resp. in $\mathbb{P}^n$) can be written in a unique way as the finite union of its irreducible components.*

Note that the irreducible components of X are its maximal irreducible algebraic subsets. They correspond to the minimal prime ideals over $I(X)$. Since $I(X)$ is radical, these minimal prime ideals coincide with the primary ideals appearing in the primary decomposition of $I(X)$.

6.4 Quasi–projective varieties

Often the irreducible closed subsets of $\mathbb{A}^n$ are called *affine varieties*, i.e., the term variety is reserved to the irreducible ones. Similarly for the irreducible closed subsets of $\mathbb{P}^n$.

Definition 6.4.1. A locally closed subset in $\mathbb{P}^n$ is the intersection of an open and a closed subset. An irreducible locally closed subset of $\mathbb{P}^n$ is called a *quasi–projective variety:* it is open in an irreducible closed subset Z of $\mathbb{P}^n$, therefore it is dense in Z.

We conclude this chapter with the (non-trivial) proof of the irreducibility of the product of irreducible affine varieties.

Proposition 6.4.2. *Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be irreducible affine varieties. Then $X \times Y$ is an irreducible subvariety of $\mathbb{A}^{n+m}$.*

Proof. Let $X \times Y = W_1 \cup W_2$, with W_1, W_2 closed. For any $P \in X$, the map $\{P\} \times Y \to Y$ which takes (P, Q) to Q is a homeomorphism, so $\{P\} \times Y$ is irreducible. $\{P\} \times Y = (W_1 \cap (\{P\} \times Y)) \cup (W_2 \cap (\{P\} \times Y))$, so $\exists i \in \{1, 2\}$ such that $\{P\} \times Y \subset W_i$. Let $X_i = \{P \in X \mid \{P\} \times Y \subset W_i\}$, $i = 1, 2$. Note that $X = X_1 \cup X_2$.

Claim. X_i is closed in X.
Let $X^i(Q) = \{P \in X \mid (P, Q) \in W_i\}$, $Q \in Y$. We have: $(X \times \{Q\}) \cap W_i = X^i(Q) \times \{Q\} \simeq X^i(Q)$; $X \times \{Q\}$ and W_i are closed in $X \times Y$, so $X^i(Q) \times \{Q\}$ is closed in $X \times Y$ and also in $X \times \{Q\}$, so $X^i(Q)$ is closed in X. Note that $X_i = \bigcap_{Q \in Y} X^i(Q)$, hence X_i is closed, which proves the Claim.

Since X is irreducible, $X = X_1 \cup X_2$ implies that either $X = X_1$ or $X = X_2$, so either $X \times Y = W_1$ or $X \times Y = W_2$. $\square$

Exercises 6.4.3.

1. Let $X \neq \emptyset$ be a topological space. Prove that X is irreducible if and only if all non–empty open subsets of X are connected.

2. Prove that the *cuspidal cubic* $Y \subset \mathbb{A}^2_{\mathbb{C}}$ of equation $x^3 - y^2 = 0$ is irreducible. (Hint: express Y as image of $\mathbb{A}^1$ in a continuous map...)

3. Give an example of two irreducible subvarieties of $\mathbb{P}^3$ whose intersection is reducible.

4. Find the irreducible components of the following algebraic sets over the complex field:

   a) $V(y^4 - x^2, y^4 - x^2y^2 + xy^2 - x^3) \subset \mathbb{A}^2$;

   b) $V(y^2 - xz, z^2 - y^3) \subset \mathbb{A}^3$.

5. Let Z be a topological space and let $\{U_\alpha\}_{\alpha \in I}$ be an open covering of Z such that $U_\alpha \cap U_\beta \neq \emptyset$ for $\alpha \neq \beta$ and that all U_α's are irreducible. Prove that Z is irreducible.

Chapter 7

Dimension

7.1 Topological dimension

There are a few equivalent ways to give the definition of dimension for algebraic varieties. In this section we will first see a topological definition, then an algebraic characterization. In a later lesson (Theorem 17.1.3), we will see a more geometrical interpretation.

Let X be a topological space.

Definition 7.1.1. The *topological dimension* of X is the supremum of the lengths of finite chains of distinct irreducible closed subsets of X, where by definiton the following chain has length n:

$$X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_n.$$

The topological dimension of X is denoted by $\dim X$. It is also called combinatorial or Krull dimension.

Example 7.1.2.

1. $\dim \mathbb{A}^1 = 1$: the maximal length chains of irreducible closed subsets all have the form $\{P\} \subset \mathbb{A}^1$.

2. $\dim \mathbb{A}^n$: a chain of length n is

$$\{0\} = V(x_1, \dots, x_n) \subset V(x_1, \dots, x_{n-1}) \subset \cdots \subset V(x_1) \subset \mathbb{A}^n.$$

   Note that $V(x_1, \dots, x_i)$ is irreducible for any $i \leq n$, because the ideal $\langle x_1, \dots, x_i \rangle$ is prime. Indeed $K[x_1, \dots, x_n]/\langle x_1, \dots, x_i \rangle \simeq K[x_{i+1}, \dots, x_n]$, which is an integral domain. Therefore we get that $\dim \mathbb{A}^n \geq n$. We will see shortly that proving equality is non trivial. We note also that, from every chain of irreducible closed subsets of $\mathbb{A}^n$, passing to their ideals, we get a chain of the same length of prime ideals in $K[x_1, \dots, x_n]$.

3. Let X be irreducible. Then $\dim X = 0$ if and only if X is the closure of every point of it.

We prove now some useful relations between the dimensions of X and of its subspaces.

Proposition 7.1.3. *1. If $Y \subset X$ is a subspace of the topological space X with the induced topology, then $\dim Y \leq \dim X$. In particular, if $\dim X$ is finite, then also $\dim Y$ is finite. In this case, the number $\dim X - \dim Y$ is called the* **codimension** *of Y in X.*

2. If $X = \bigcup_{i \in I} U_i$ is an open covering, then $\dim X = \sup_i \{\dim U_i\}$.

3. If X is noetherian and $X_1, \dots, X_s$ are its irreducible components, then $\dim X = \sup_i \dim X_i$.

4. If $Y \subset X$ is closed, X is irreducible, $\dim X$ is finite and $\dim X = \dim Y$, then $Y = X$.

Proof. 1. Let $Y_0 \subset Y_1 \subset \cdots \subset Y_n$ be a chain of irreducible closed subsets of Y. Then taking closures we get the following chain of irreducible closed subsets of X: $\overline{Y_0} \subseteq \overline{Y_1} \subseteq \cdots \subseteq \overline{Y_n}$. Note that, for any index i, $\overline{Y_i} \cap Y = Y_i$, because Y_i is closed into Y, so if $\overline{Y_i} = \overline{Y_{i+1}}$, then $Y_i = Y_{i+1}$. Therefore the two chains have the same length and we can conclude that $\dim Y \leq \dim X$.

2. Let $X_0 \subset X_1 \subset \cdots \subset X_n$ be a chain of irreducible closed subsets of X. Let $P \in X_0$ be a point: there exists an index $i \in I$ such that $P \in U_i$. So $\forall k = 0, \dots, n$ $X_k \cap U_i \neq \emptyset$: it is an irreducible closed subset of U_i, irreducible because open in X_k which is irreducible. Consider

$$X_0 \cap U_i \subset X_1 \cap U_i \subset \cdots \subset X_n \cap U_i;$$

it is a chain of length n, because $\overline{X_k \cap U_i} = X_k$: in fact $X_k \cap U_i$ is open in X_k hence dense. Therefore, for any chain of irreducible closed subsets of X, there exists a chain of the same length of irreducible closed subsets of some U_i. So $\dim X \leq \sup \dim U_i$. By 1., equality holds.

3. Any chain of irreducible closed subsets of X is completely contained in an irreducible component of X. The conclusion follows as in 2.

4. If $Y_0 \subset Y_1 \subset \cdots \subset Y_n$ is a chain of irreducible closed subsets of Y of maximal length, then it is also a maximal length chain in X, because $\dim X = \dim Y$. Hence $X = Y_n$, because X is irreducible, and we conclude that $X \subset Y$. □

Corollary 7.1.4. $\dim \mathbb{P}^n = \dim \mathbb{A}^n$.

Proof. The equality follows from $\mathbb{P}^n = U_0 \cup \cdots \cup U_n$, and the homeomorphism of U_i with $\mathbb{A}^n$ for all i. □

If X is noetherian and all its irreducible components have the same dimension r, then X is said to have *pure dimension* r. Note that the topological dimension is invariant by homeomorphism. By definition, a *curve* is an algebraic set of pure dimension 1; a *surface* is an algebraic set of pure dimension 2.

7.2 Dimension of algebraic varieties

We want to study the dimensions of affine algebraic sets. The following definition results to be very important.

Definition 7.2.1. Let $X \subset \mathbb{A}^n$ be an algebraic set. The *coordinate ring* of X is

$$K[X] := K[x_1, \dots, x_n]/I(X).$$

It is a finitely generated reduced K–algebra, i.e. there are no non–zero nilpotents, because $I(X)$ is radical (see Exercise 3, Chapter 3).

There is the canonical epimorphism $K[x_1, \dots, x_n] \to K[X]$ such that $F \to [F]$. The elements of $K[X]$ can be interpreted as *polynomial functions* on X: to a polynomial F, we can associate the function $f : X \to K$ such that $P(a_1, \dots, a_n) \to F(a_1, \dots, a_n)$.

Two polynomials F, G define the same function on X if, and only if, $F(P) = G(P)$ for every point $P \in X$, i.e. if $F - G \in I(X)$, which means exactly that F and G have the same image in $K[X]$.

$K[X]$ is generated as K–algebra by $[x_1], \dots, [x_n]$: they can be interpreted as *coordinate functions* on X. We will denote them by $t_1, \dots, t_n$. In fact $t_i : X \to K$ is the function which associates to $P(a_1, \dots, a_n)$ the coordinate a_i. Note that the function f can be interpreted as $F(t_1, \dots, t_n)$: the polynomial F evaluated at the n– tuple of the coordinate functions.

In the projective space we can do an analogous construction. If $Y \subset \mathbb{P}^n$ is closed, then by definition the *homogeneous coordinate ring* of Y is

$$S(Y) := K[x_0, x_1, \dots, x_n]/I_h(Y).$$

Also $S(Y)$ is a finitely generated reduced K–algebra, but its elements cannot be interpreted as functions on Y. They are functions on the cone $C(Y)$.

We note that, from the fact that $I_h(Y)$ is homogeneous it follows that also $S(Y)$ is a graded ring, with the graduation induced by the polynomial ring. Indeed, if $F - G \in I_h(Y)$, and $F = F_0 + \ldots + F_d$, $G = G_0 + \ldots + G_e$ are their decompositions in homogeneous components, then it follows that $F_0 - G_0 \in I_h(Y)$, $F_1 - G_1 \in I_h(Y)$, and so on. Therefore $S(Y) = \oplus_{d \geq 0} S(Y)_d$, where $S(Y)_d$ is the subgroup of the classes of homogeneous polynomials of degree d. $S(Y)_d$ is also a K-vector space of finite dimension. The function that associates to any integer d the dimension of $S(Y)_d$ as K-vector space is called the *Hilbert function* of Y.

Definition 7.2.2. Let R be a ring. The *Krull dimension* of R is the supremum of the lengths of the chains of prime ideals of R

$$\mathcal{P}_0 \subset \mathcal{P}_1 \subset \cdots \subset \mathcal{P}_r.$$

Similarly, the *heigth* of a prime ideal $\mathcal{P}$ is the sup of the lengths of the chains of prime ideals contained in $\mathcal{P}$: it is denoted $\text{ht}\mathcal{P}$.

Proposition 7.2.3. *Let K be an algebraically closed field. Let X be an affine algebraic set contained in $\mathbb{A}^n$. Then $\dim X = \dim K[X]$. In particular $\dim \mathbb{A}^n = \dim K[x_1, \ldots, x_n]$.*

Proof. By the Nullstellensatz and its Corollary 3.2.9 the chains of irreducible closed subsets of X correspond bijectively to the chains of prime ideals of $K[x_1, \ldots, x_n]$ containing $I(X)$, and therefore also to the chains of prime ideals of the quotient ring $K[X] = K[x_1, \ldots, x_n]/I(X)$. □

The dimension theory for commutative rings contains some important theorems about the dimension of K–algebras. The following theorem states the basic properties in the case of integral domains and the algebraic characterization of dimension for affine varieties.

Theorem 7.2.4. *Let K be any field. Let A be a finitely generated K–algebra and an integral domain.*

1. $\dim A = tr.d.Q(A)/K$*, where $Q(A)$ is the quotient field of A. In particular* $\dim A$ *is finite.*

2. *Let $\mathcal{P} \subset A$ be any prime ideal. Then* $\dim A = \text{ht}\mathcal{P} + \dim A/\mathcal{P}$.

Proof. We postpone the proof to next chapter. It relies on the Normalization Lemma and on the Cohen-Seidenberg theorems about the structure of prime ideals for integral extensions of K-algebras. □

Corollary 7.2.5. *Let K be an algebraically closed field.*

1. $\dim \mathbb{A}^n = \dim \mathbb{P}^n = n$.

2. *If X is an irreducible affine variety, then* $\dim X = tr.d.K(X)/K$*, where $K(X)$ denotes the quotient field of $K[X]$.*

3. *If $X \subset \mathbb{A}^n$ is an irreducible affine variety, then* $\dim X = n - \mathrm{ht} I(X)$.

Proof. 1. $\dim K[x_1, \dots, x_n] = tr.d.K(x_1, \dots, x_n)/K = n$.

2. follows immediately from Theorem 7.2.4, 1.

3. is Theorem 7.2.4, 2, applied to the case $A = K[x_1, \dots, x_n]$ and $\mathcal{P} = I(X)$. □

Note that the homogeneous coordinate ring of $\mathbb{P}^n$ is $K[x_0, \dots, x_n]$, whose dimension is $n + 1$, strictly bigger than the dimension of $\mathbb{P}^n$. Similarly, if Y is a projective algebraic variety, then $\dim S(Y) = \dim C(Y)$, the affine cone over Y.

Corollary 7.2.5 tells us how to compute the dimension of an affine irreducible variety over an algebraically closed field K. If X is a reducible affine variety, and $X = X_1 \cup \dots \cup X_r$ is its decomposition as union of irreducible components, then Proposition 7.1.3 tells us that $\dim X$ is the maximum of the dimensions $\dim X_i$.

The following is the characterization of the algebraic varieties of codimension 1 in $\mathbb{A}^n$.

Proposition 7.2.6. *Let $X \subset \mathbb{A}^n$ be an affine variety over an algebraically closed field. Then X is a hypersurface if and only if X is of pure dimension $n - 1$.*

Proof. Let $X \subset \mathbb{A}^n$ be a hypersurface, with $I(X) = (F) = (F_1 \dots F_s)$, where $F_1, \dots, F_s$ are the (distinct) irreducible factors of F all of multiplicity one. Then $X = V(F_1 \dots F_s) = V(F_1) \cup \dots \cup V(F_s)$; therefore $V(F_1)$,..., $V(F_s)$ are the irreducible components of X, whose ideals are (F_1), ..., (F_s). So by Corollary 7.2.5 it is enough to prove that $\mathrm{ht}(F_i) = 1$, for $i = 1, \dots, s$.

If $\mathcal{P} \subset (F_i)$ is a prime ideal, then either $\mathcal{P} = (0)$ or there exists $G \in \mathcal{P}$, $G \neq 0$. In the second case, let A be an irreducible factor of G belonging to $\mathcal{P}$: $A \in (F_i)$ so $A = HF_i$. Since A is irreducible, either H or F_i is invertible; but F_i is irreducible, so H is invertible, hence $(A) = (F_i) \subset \mathcal{P}$. Therefore either $\mathcal{P} = (0)$ or $\mathcal{P} = (F_i)$, and it follows that $\mathrm{ht}(F_i) = 1$.

Conversely, assume that X is irreducible of dimension $n - 1$. Since $X \neq \mathbb{A}^n$, there exists $F \in I(X)$, $F \neq 0$, with irreducible factorization $F = F_1 \dots F_s$. Hence $X \subset V(F) = V(F_1) \cup \dots \cup V(F_s)$. By the irreducibility of X, $X \subset V(F_i)$, which is irreducible of dimension $n - 1$, by the first part. So $X = V(F_i)$ (by Proposition 7.1.3, 4). □

This proposition does not generalise to higher codimension. There exist codimension 2 algebraic subsets of $\mathbb{A}^n$ whose ideal is not generated by two polynomials. An example in $\mathbb{A}^3$ is the curve X parametrised by (t^3, t^4, t^5). It is possible to show that a minimal system of generators of $I(X)$ is formed by the three polynomials $x^3 - yz, y^2 - xz, z^2 - x^2y$. One can easily show that $I(X)$ cannot be generated by two polynomials. For a proof and a discussion of this example, and more generally of the ideals of the curves admitting a parametrization of the form $x = t^{n_1}, y = t^{n_2}, z = t^{n_3}$, see [K], Chapter V.

Proposition 7.2.7. *Let* $X \subset \mathbb{A}^n_K$*,* $Y \subset \mathbb{A}^m_K$ *be irreducible closed subsets, over an algebraically closed field* K*. Then* $\dim X \times Y = \dim X + \dim Y$*.*

Proof. Let $r = \dim X$, $s = \dim Y$; let $t_1, \dots, t_n$ (resp. $u_1, \dots, u_m$) be coordinate functions on $\mathbb{A}^n$ (resp. $\mathbb{A}^m$). We can assume that $t_1, \dots, t_r$ is a transcendence basis of $Q(K[X])$ and $u_1, \dots, u_s$ a transcendence basis of $Q(K[Y])$. By definition, $K[X \times Y]$ is generated as K–algebra by $t_1, \dots, t_n, u_1, \dots, u_m$: we want to show that $t_1, \dots, t_r, u_1, \dots, u_s$ is a transcendence basis of $Q(K[X \times Y])$ over K. Assume that $F(x_1, \dots, x_r, y_1, \dots, y_s)$ is a polynomial which vanishes on $t_1, \dots, t_r, u_1, \dots, u_s$, i.e. F defines the zero function on $X \times Y$. Then, $\forall\ P \in X$, $F(P; y_1, \dots, y_s)$ is zero on Y, i.e. $F(P; u_1, \dots, u_s) = 0$. Since $u_1, \dots, u_s$ are algebraically independent, every coefficient $a_i(P)$ of $F(P; y_1, \dots, y_s)$ is zero, $\forall\ P \in X$. Since $t_1, \dots, t_r$ are algebraically independent, the polynomials $a_i(x_1, \dots, x_r)$ are zero, so $F(x_1, \dots, x_r, y_1, \dots, y_s) = 0$. So $t_1, \dots, t_r, u_1, \dots, u_s$ are algebraically independent. Since this is certainly a maximal algebraically free set, it is a transcendence basis. □

Exercises 7.2.8.

1. Prove that a proper closed subset of an irreducible curve is a finite set. Deduce that any bijection between irreducible curves is a homeomorphism.

2. Let $X \subset \mathbb{A}^2$ be the cuspidal cubic of equation: $x^3 - y^2 = 0$, let $K[X]$ be its coordinate ring. Prove that all elements of $K[X]$ can be written in a unique way in the form $f(x) + yg(x)$, where f, g are polynomials in the variable x. Deduce that $K[X]$ is not isomorphic to a polynomial ring.

Chapter 8

Dimension of K-algebras.

The purpose of this chapter is to prove Theorem 7.2.4. In reality we will not give a complete proof of it, but we will only enunciate the Cohen-Seidenberg theorems and then we will see how, from them and from the Normalization Lemma, the theorem follows.

8.1 Prime ideals of integral extensions

Let $R \subset T$ be rings, R subring of T. We are interested in relations between the prime ideals of R and those of T. We are principally concerned with the case where T is integral over R, but we formulate the definitions in greater generality.

It is easily seen that if $\mathcal{Q}$ is a prime ideal of T, then $\mathcal{Q} \cap R$ is a prime ideal of R, called contraction of $\mathcal{Q}$ in R. We list four properties that might hold for a pair $R \subset T$.

(LO) *Lying over.* For any prime ideal $\mathcal{P}$ in R there exists a prime ideal $\mathcal{Q}$ in T with $\mathcal{Q} \cap R = \mathcal{P}$.

(GU) *Going up.* Given prime ideals $\mathcal{P} \subset \mathcal{P}_0$ in R and $\mathcal{Q}$ in T with $\mathcal{Q} \cap R = \mathcal{P}$, there exists a prime ideal $\mathcal{Q}_0$ in T satisfying $\mathcal{Q} \subset \mathcal{Q}_0$ and $\mathcal{Q}_0 \cap R = \mathcal{P}_0$.

(GD) *Going down.* Given prime ideals $\mathcal{P} \subset \mathcal{P}_0$ in R and $\mathcal{Q}_0$ in T with $\mathcal{Q}_0 \cap R = \mathcal{P}_0$, there exists a prime ideal $\mathcal{Q}$ in T satisfying $\mathcal{Q} \subset \mathcal{Q}_0$ and $\mathcal{Q} \cap R = \mathcal{P}$.

(INC) *Incomparable.* Two different prime ideals in T with the same contraction in R cannot be comparable: if $\mathcal{Q} \subsetneq \mathcal{Q}_0$ are prime ideals of T, then $\mathcal{Q} \cap R \subsetneq \mathcal{Q}_0 \cap R$.

Next Theorem 8.1.4 states conditions on the pair of rings that ensure the validity of the above properties. We first need some definitions.

Proposition 8.1.1. *Let $R \subset T$. The set $\overline{R}$ of all elements of T that are integral over R is a subring of T.*

Proof. It relies on Theorem 4.0.1. If $x, y \in \overline{R}$, $R[x, y]$ is a finite R-module. Therefore $x+y, x-y, xy$ are integral over R, because they all belong to $R[x, y]$. □

Definition 8.1.2. $\overline{R}$ is called *the integral closure of R in T*. R is called *integrally closed* in T if $\overline{R} = R$. An integral domain that is integrally closed in its field of fractions is called *normal*.

Next Proposition extends Exercise 4 of Chapter 3.

Proposition 8.1.3. *If A is a UFD, then it is a normal ring. In particular, any polynomial ring with coefficients in a field is normal.*

Proof. Indeed, let $f/g \in Q(A)$ be an element of the quotient field of A, with f, g coprime. Assume that f/g is integral over A: we have an equation of integral dependence

$$(f/g)^r + a_1(f/g)^{r-1} + \cdots + a_r = 0,$$

with coefficients $a_1, \ldots, a_r \in A$. Multiplying everything by g^r we get:

$$f^r = -a_1 f^{r-1} g - \cdots - a_r g^r = g(-a_1 f^{r-1} - \cdots - a_r g^{r-1},$$

therefore $g | f^r$. So each irreducible factor of g divides f. Since f, g are coprime, we conclude that $g = \pm 1$ and $f/g \in A$. □

Theorem 8.1.4. *Let $R \subset T$ be rings with T integral over R. Then:*

1. *the pair $R \subset T$ satisfies (LO), (INC) and (GU);*
2. *if moreover R and T are integral domains and R is normal, then also (GD) is satisfied.*

Proof. For a proof, see for instance [AM] or [P]. □

8.2 Length of chains of prime ideals in K-algebras

Next Theorem 8.2.2 is the key to prove Theorem 7.2.4. First we need to state one more property of integral extensions. It extends what we proved in the first step of the proof of the Nullstellensatz.

Proposition 8.2.1. *Let $R \subset T$ be integral domains, T integral over R. Then T is a field if and only if R is a field.*

Proof. Suppose R is a field, let $y \in T, y \neq 0$. Let

$$y^n + r_1 y^{n-1} + \cdots + r_n = 0,\ \ r_i \in R$$

be an equation of integral dependence for y of smallest possible degree. Since T is an integral domain we have $r_n \neq 0$, otherwise we simplify and get an equation of integral dependence of lower degree, so $y^{-1} = -r_n^{-1}(y^{n-1} + r_1 y^{n-2} + \cdots + r_{n-1}) \in T$. Hence T is a field.

Conversely, suppose that T is a field; let $x \in R,\ x \neq 0$. Then $x^{-1} \in T$, so it is integral over R, so that we have an equation

$$x^{-m} + s_1 x^{-m+1} + \cdots + s_m = 0,\ \ s_i \in R.$$

It follows that $x^{-1} = -(s_1 + s_2 x + \cdots + s_m x^{m-1}) \in R$, therefore R is a field. □

Theorem 8.2.2. *Let K be a field, let A be a finitely generated K-algebra, integral extension of $K[z_1, \dots, z_n]$, with $z_1, \dots, z_n$ algebraically independent over K. Then:*

a) Every chain of prime ideals of A: $\mathcal{P}_0 \subset \mathcal{P}_1 \subset \cdots \subset \mathcal{P}_l$ has length $l \leq n$;

b) Assume that the chain is non-extendable, then $l = n$ if and only if

$$\mathcal{P}_0 \cap K[z_1, \dots, z_n] = (0).$$

Proof. By induction on n.

If $n = 0$, then A is integral extension of K. We claim that every prime ideal $\mathcal{P}$ of A is maximal; indeed, first observe that also $A/\mathcal{P}$ is integral extension of K, because, if $a \in A$, from an equation of algebraic dependence for a over K, passing to the quotient we get a similar equation for $[a]$ over K. So by Proposition 8.2.1 it follows that $A/\mathcal{P}$ is a field, and we conclude that $\mathcal{P}$ is maximal. So $l = 0$. Moreover $\mathcal{P} \cap K = (0)$.

Let $n \geq 1$, and let $\mathcal{P}_0 \subset \mathcal{P}_1 \subset \cdots \subset \mathcal{P}_l$ be a chain of prime ideals in A. Let $\mathcal{Q}_i = \mathcal{P}_i \cap K[z_1, \dots, z_n]$. Then, by Theorem 8.1.4, (INC), $\mathcal{Q}_0 \subset \cdots \subset \mathcal{Q}_l$ is a chain of prime ideals in $K[z_1, \dots, z_n]$. If $l = 0$ we are done, so assume $l \geq 1$. Then $\mathcal{Q}_1$ contains a non-zero element, and, since $\mathcal{Q}_1$ is prime and $K[z_1, \dots, z_n]$ is a UFD, there exists $f \in \mathcal{Q}_1$ irreducible. We pass to the quotient with respect to (f), that is contained in $\mathcal{Q}_i$ for any $i \geq 1$. So we get a chain of length $l - 1$ in $K[z_1, \dots, z_n]/(f)$, which is an integral domain:

$$\mathcal{Q}_1/(f) \subset \cdots \subset \mathcal{Q}_l/(f).$$

By the Normalization Lemma, $K[z_1, \dots, z_n]/(f)$ is an integral extension of a polynomial ring $K[y_1, \dots, y_{n-1}]$. Hence, by the induction hypothesis, we have $l - 1 \leq n - 1$, i.e. $l \leq n$. This proves part a).

Assume now that the chain $\mathcal{P}_0 \subset \mathcal{P}_1 \subset \cdots \subset \mathcal{P}_l$ is not extendable. Assume $\mathcal{Q}_0 = \mathcal{P}_0 \cap K[z_1, \dots, z_n] = (0)$. Let $A' = A/\mathcal{P}_0$, $\mathcal{P}'_i = \mathcal{P}_i/\mathcal{P}_0$ for any i. The composite map $K[z_1, \dots, z_n] \hookrightarrow A \to A/\mathcal{P}_0$ is injective because $\mathcal{Q}_0 = (0)$, so $A/\mathcal{P}_0$ is integral over $K[z_1, \dots, z_n]$. We have that $K[z_1, \dots, z_n]$ is a normal ring (see Proposition 8.1.3). Hence, we can apply Theorem 8.1.4 (GD) to this extension of rings, as follows. We have $\mathcal{Q}_0 \subsetneq \mathcal{Q}_1$. As before there exists $f \in \mathcal{Q}_1$ irreducible, generating a prime ideal with $(f) \subset \mathcal{Q}_1$. We have also $\mathcal{Q}_1 = \mathcal{P}'_1 \cap K[z_1, \dots, z_n]$, so by (GD) property there exists a prime ideal $\mathcal{N} \subset \mathcal{P}'_1$ of A' such that $\mathcal{N} \cap K[z_1, \dots, z_n] = (f)$. But the chain $\mathcal{P}'_0 \subset \mathcal{P}'_1$ is not extendable and $\mathcal{P}'_0 = (0)$, hence $\mathcal{N} = \mathcal{P}'_1$, and $(f) = \mathcal{Q}_1$. It follows that $K[z_1, \dots, z_n]/(f)$ is a subring of $A/\mathcal{P}_1$ (in the sense that the induced map $K[z_1, \dots, z_n]/(f) \to A/\mathcal{P}_1$ is injective) and this is an integral extension. Again by Normalization Lemma, $K[z_1, \dots, z_n]/(f)$ is integral over a polynomial ring $K[y_1, \dots, y_{n-1}]$. Since $(0) = \mathcal{P}_1/\mathcal{P}_1 \subset \cdots \subset \mathcal{P}_l/\mathcal{P}_1$ is a non-extendable chain of prime ideals of $A/\mathcal{P}_1$, such that $(0) \cap K[y_1, \dots, y_{n-1}] = (0)$, by inductive assumption we conclude that $l - 1 = n - 1$.

If $\mathcal{Q}_0 \neq 0$, let $g \in \mathcal{Q}_0$ non 0. The ring $K[z_1, \dots, z_n]/(g)$ is integral over a polynomial ring in $n-1$ variables, so the chain $\mathcal{Q}_0/(g) \subset \cdots \subset \mathcal{Q}_l/(g)$ has length at most $n-1$ and $l < n$.

□

8.3 Consequences

The following series of Corollaries of Theorem 8.2.2 proves the desired results and more.

Corollary 8.3.1. *Let A be an integral domain finitely generated as K-algebra. Let $n = tr.d.Q(A)/K$. Then*

1. **all** *non-extendable chains of prime ideals of A have length n.*

2. *The Krull dimension of A is finite and equal to n.*

3. *Let $\mathcal{Q} \subset \mathcal{P}$ be two prime ideals of A. If*

$$\mathcal{Q} = \mathcal{P}_0 \subset \mathcal{P}_1 \subset \cdots \subset \mathcal{P}_l = \mathcal{P}$$

   is a non-extendable chain of prime ideals between $\mathcal{Q}$ and $\mathcal{P}$, then $l = tr.d.Q(A/\mathcal{Q})/K - tr.d.Q(A/\mathcal{P})/K$.

4. *Every maximal ideal of A has height n.*

Proof. By the Normalization Lemma there exist n algebraically independent elements $z_1, \dots, z_n \in A$, such that A is integral over $K[z_1, \dots, z_n]$. Since A is a domain, for any non-extendable chain of prime ideals $\mathcal{P}_0 \subset \mathcal{P}_1 \subset \cdots \subset \mathcal{P}_l$, we have $\mathcal{P}_0 = (0)$, because (0) is prime and contained in any ideal, hence $\mathcal{Q}_0 = \mathcal{P}_0 \cap K[z_1, \dots z_n] = (0)$. The proof of (1) follows by Theorem 8.2.2. (2) follows from (1).

To prove (3), note that, by (1), we can extend $\mathcal{Q} = \mathcal{P}_0 \subset \mathcal{P}_1 \subset \cdots \subset \mathcal{P}_l = \mathcal{P}$ to a non-extendable chain of prime ideals of A of length n:

$$(0) \subset \cdots \subset \mathcal{Q} = \mathcal{P}_0 \subset \mathcal{P}_1 \subset \cdots \subset \mathcal{P}_l = \mathcal{P} \subset \mathcal{P}_{l+1} \subset \dots .$$

The part of the chain from $\mathcal{Q}$ up has length equal to $\dim A/\mathcal{Q} = tr.d.Q(A/\mathcal{Q})/K$, because there is a natural bijection between the set of prime ideals of $A/\mathcal{Q}$ and that of prime ideals of A containing $\mathcal{Q}$. Similarly the part from $\mathcal{P}$ up has length equal to $\dim A/\mathcal{P} = tr.d.Q(A/\mathcal{P})/K$. So (3) follows.

(4) follows because the last ideal in a non-extendable chain of prime ideals of A must be a maximal ideal. □

Corollary 8.3.2. *Let $\mathcal{P} \subset K[x_1, \dots, x_n]$ be a prime ideal of the polynomial ring in n variables. Then* $\dim A/\mathcal{P} = n - ht(\mathcal{P})$.

Proof. Let

$$(0) = \mathcal{P}_0 \subset \cdots \subset \mathcal{P} \subset \cdots \subset \mathcal{P}_n \tag{8.1}$$

be a non-extendable chain of length n of prime ideals in $K[x_1, \dots, x_n]$ passing through $\mathcal{P}$. The subchain $(0) = \mathcal{P}_0 \subset \cdots \subset \mathcal{P}$ is a non-extendable chain of prime ideals contained in $\mathcal{P}$, so it has length $ht\mathcal{P}$, whereas the subchain $\mathcal{P} \subset \cdots \subset \mathcal{P}_n$ has length $\dim A/\mathcal{P}$, so the thesis follows. □

Note that the first part of Theorem 7.2.4 follows from Corollary 8.3.1, 2. and the second part is Corollary 8.3.2.

If A is any integral domain, the property that all non-extendable chains of prime ideals of A have the same length does not hold in general. There are even examples (not easy to construct) of noetherian domains whose Krull dimension is not finite or where there are non-extendable chains of prime ideals of different lengths. The rings where the property in Corollary 8.3.1 (3) holds are called *catenary* rings.

Chapter 9

Regular and rational functions.

9.1 Regular functions

In this chapter, we will define the regular functions on algebraic varieties, not only on closed subsets of affine or projective space, but more in general on locally closed subsets. This will allow to associate to any algebraic variety an algebraic invariant, its ring of regular functions. An analogous construction will be given also for a more general class of functions, the rational functions, that will bring to a second invariant, the field of rational functions.

Let $X \subset \mathbb{P}^n$ be a locally closed subset and P be a point of X. Let $\varphi : X \to K$ be a function.

Definition 9.1.1. 1. φ is *regular at* P if there exists a suitable neighbourhood of P in which φ can be expressed as a quotient of homogeneous polynomials of the same degree; more precisely, if there exist an open neighbourhood U of P in X and homogeneous polynomials F, $G \in K[x_0, x_1, \ldots, x_n]$ with $\deg F = \deg G$, such that $U \cap V_P(G) = \emptyset$ and $\varphi(Q) = F(Q)/G(Q)$, for all $Q \in U$. Note that the quotient $F(Q)/G(Q)$ is well defined.

2. φ is *regular on* X if φ is regular at every point P of X.

This definition of regular function is of *local* character; we can express it saying that φ is regular if it can locally be expressed by quotients of homogeneous polynomials of the same degree.

The set of regular functions on X is denoted by $\mathcal{O}(X)$: it contains K (identified with the set of constant functions), and can be given the structure of a K–algebra, by the definitions:

$$(\varphi + \psi)(P) = \varphi(P) + \psi(P)$$

$$(\varphi\psi)(P) = \varphi(P)\psi(P),$$

for $P \in X$. (Check that $\varphi + \psi$ and $\varphi\psi$ are indeed regular on X.)

Proposition 9.1.2. *Let $\varphi : X \to K$ be a regular function. Let K be identified with $\mathbb{A}^1$ with Zariski topology. Then φ is continuous.*

Proof. It is enough to prove that $\varphi^{-1}(c)$ is closed in X, $\forall\ c \in K$. For all $P \in X$, choose an open neighbourhood U_P and homogeneous polynomials F_P, G_P such that $\varphi|_{U_P} = F_P/G_P$. Then

$$\varphi^{-1}(c) \cap U_P = \{Q \in U_P | F_P(Q) - cG_P(Q) = 0\} = U_P \cap V_P(F_P - cG_P)$$

is closed in U_P. The proposition then follows from:

Lemma 9.1.3. *Let T be a topological space, $T = \cup_{i \in I} U_i$ be an open covering of T, $Z \subset T$ be a subset. Then Z is closed if and only if $Z \cap U_i$ is closed in U_i for all i.*

Proof. Assume that $U_i = X \setminus C_i$ and $Z \cap U_i = Z_i \cap U_i$, with C_i and Z_i closed in X.

Claim: $Z = \bigcap_{i \in I}(Z_i \cup C_i)$, hence it is closed.

In fact: if $P \in Z$, then $P \in Z \cap U_i$ for a suitable i. Therefore $P \in Z_i \cap U_i$, so $P \in Z_i \cup C_i$. If $P \notin Z_j \cap U_j$ for some j, then $P \notin U_j$ so $P \in C_j$ and therefore $P \in Z_j \cup C_j$.

Conversely, if $P \in \bigcap_{i \in I}(Z_i \cup C_i)$, then $\forall\ i$, either $P \in Z_i$ or $P \in C_i$. Since $\exists j$ such that $P \in U_j$, hence $P \notin C_j$, so $P \in Z_j$, so $P \in Z_j \cap U_j = Z \cap U_j$. □

□

Corollary 9.1.4. *1. Let $\varphi \in \mathcal{O}(X)$: then $\varphi^{-1}(0)$ is closed. It is denoted $V(\varphi)$ and called the set of zeros of φ.*

2. Let X be a quasi–projective (irreducible) variety and φ, $\psi \in \mathcal{O}(X)$. Assume that there exists U, open non –empty subset such that $\varphi|_U = \psi|_U$. Then $\varphi = \psi$.

Proof. 1. is clear. To prove 2. we note that $\varphi - \psi \in \mathcal{O}(X)$ so $V(\varphi - \psi)$ is closed. By assumption $V(\varphi - \psi) \supset U$, which is dense, because X is irreducible. So $V(\varphi - \psi) = X$.

□

If $X \subset \mathbb{A}^n$ is locally closed in an affine space, we can use on X both homogeneous and non–homogeneous coordinates. If φ is a regular function according to Definition 9.1.1, from a local expression of φ of the form F/G, with F, G homogeneous of the same degree on an open subset of X, we pass to the expression ${}^aF/{}^aG$ for the same function in non-homogeneous coordinates. Note that now ${}^aF, {}^aG$ are no longer homogeneous nor of the same degree, in general.

On the other hand, assume we have a function on X locally represented by quotients of polynomials in n variables; if A/B is such a local expression, with $\deg A = a, \deg = b$, $a \leq b$, the same function is represented in homogeneous coordinates by the following quotient of homogeneous polynomials of the same degree: $(x_0^{a-b})^h A/{}^h B$. Similarly if $a \geq b$.

From this discussion it follows that all polynomial functions are regular: for instance, if $F(x_1, \dots, x_n)$ is a polynomial of degree d, the polynomial function defined by F can be expressed in homogeneous coordinates in the form $\frac{{}^hF(x_0,\dots,x_n)}{x_0^d}$. In particular, if X is an affine variety, $K[X] \subset \mathcal{O}(X)$.

If $\alpha \subset K[X]$ is an ideal, we can consider $V(\alpha) := \bigcap_{\varphi \in \alpha} V(\varphi)$: it is closed into X. Note that α is of the form $\alpha = \overline{\alpha}/I(X)$, where $\overline{\alpha}$ is the inverse image of α in the canonical epimorphism, it is an ideal of $K[x_1, \dots, x_n]$ containing $I(X)$, hence $V(\alpha) = V(\overline{\alpha}) \cap X = V(\overline{\alpha})$.

If K is algebraically closed, the following relative form follows immediately from the Nullstellensatz.

Proposition 9.1.5 (Relative Nullstellensatz)**.** *Let K be an algebraically closed field, let X be an affine variety closed in $\mathbb{A}^n_K$ and $K[X]$ its coordinate ring.*

1. *If $\alpha \subset K[X]$ is a proper ideal then $V(\alpha) \neq \emptyset$.*
2. *If $f \in K[X]$ and f vanishes at all points $P \in X$ such that $g_1(P) = \cdots = g_m(P) = 0$ ($g_1, \dots, g_m \in K[X]$), then $f^r \in \langle g_1, \dots, g_m \rangle \subset K[X]$, for some $r \geq 1$.*

Theorem 9.1.6. *Let K be an algebraically closed field. Let $X \subset \mathbb{A}^n_K$ be closed in the Zariski topology. Then $\mathcal{O}(X) \simeq K[X]$. It is an integral domain if and only if X is irreducible.*

Proof. We have already noticed that $K[X] \subset \mathcal{O}(X)$. It remains to prove the opposite inclusion. So let $f \in \mathcal{O}(X)$.

(i) Assume first that X is irreducible. For all $P \in X$ fix an open neighbourhood U_P of P and polynomials F_P, G_P such that $V_P(G_P) \cap U_P = \emptyset$ and $f|_{U_P} = F_P/G_P$. Let f_P, g_P be the functions in $K[X]$ defined by F_P and G_P. Then $g_P f = f_P$ holds on U_P, so it holds on X (by Corollary 9.1.4 (2), because X is irreducible). Let $\alpha \subset K[X]$ be the ideal $\alpha = \langle g_P \rangle_{P \in X}$, generated by all denominators of the various local expressions of φ; α has no zeros on X, because for any P $g_P(P) \neq 0$, so $\alpha = K[X]$. Therefore there exist suitable polynomial functions $h_P \in K[X]$ such that $1 = \sum_{P \in X} h_P g_P$ (sum with finite support). Hence in $\mathcal{O}(X)$ we have the relation: $f = f \sum h_P g_P = \sum h_P(g_P f) = \sum h_P f_P \in K[X]$.

(ii) Let X be reducible: from $g_P f = f_P$ on U_P, we cannot deduce that the same equality holds on X. The idea is to change suitably the local expressions. For any $P \in X$, there exists $R \in K[x_1, \dots, x_n]$ such that $R(P) \neq 0$ and $R \in I(X \setminus U_P)$, so $r \in \mathcal{O}(X)$ is zero outside U_P. So $r g_P f = f_P r$ on X and we conclude as above, after replacing g_P with $g_P r$ and f_P with $f_P r$. □

The characterization of regular functions on projective varieties is completely different: we will see in Theorem 15.2.2 that, if X is an irreducible projective variety, then $\mathcal{O}(X) \simeq K$, i.e. the unique regular functions are constant.

This gives the motivation for introducing the following weaker concept of rational function.

9.2 Rational functions

Definition 9.2.1. Let X be a quasi–projective variety. A *rational function* on X is a germ of regular functions on some open non–empty subset of X.

Precisely, let $\mathcal{K}$ be the set $\{(U, f) | \ U \neq \emptyset, \text{open subset of } X, f \in \mathcal{O}(U)\}$. The following relation on $\mathcal{K}$ is an equivalence relation:

$$(U, f) \sim (U', f') \text{ if and only if } f|_{U \cap U'} = f'|_{U \cap U'}.$$

Reflexive and symmetric properties are quite obvious. Transitive property: let $(U, f) \sim (U', f')$ and $(U', f') \sim (U'', f'')$. Then $f|_{U \cap U'} = f'|_{U \cap U'}$ and $f'|_{U' \cap U''} = f''|_{U' \cap U''}$, hence $f|_{U \cap U' \cap U''} = f''|_{U \cap U' \cap U''}$. $U \cap U' \cap U''$ is a non–empty open subset of $U \cap U''$, which is irreducible and quasi–projective, so by Corollary 9.1.4 $f|_{U \cap U''} = f''|_{U \cap U''}$.

Let $K(X) := \mathcal{K}/ \sim$: its elements are by definition the rational functions on X. $K(X)$ can be given the structure of a field in the following natural way.

Let $\langle U, f \rangle$ denote the class of (U, f) in $K(X)$. We define:

$$\langle U, f \rangle + \langle U', f' \rangle = \langle U \cap U', f + f' \rangle,$$

$$\langle U, f \rangle \langle U', f' \rangle = \langle U \cap U', f f' \rangle$$

(check that the definitions are well posed!).

There is a natural inclusion: $K \to K(X)$ such that $c \to \langle X, c \rangle$. Moreover, if $\langle U, f \rangle \neq 0 = \langle X, 0 \rangle$, then $U \setminus V(f)$ is not empty, so there exists $\langle U, f \rangle^{-1} = \langle U \setminus V(f), f^{-1} \rangle$: the axioms of a field are all satisfied.

There is also a natural injective map: $\mathcal{O}(X) \to K(X)$ such that $\varphi \to \langle X, \varphi \rangle$.

Summarizing, $K(X)$ is a field, called the **field of rational functions** of the quasi-projective variety X. It is an extension of the base field K, and contains the ring of regular functions $\mathcal{O}(X)$.

Proposition 9.2.2. *If $X \subset \mathbb{A}^n$ is an irreducible affine variety over an algebraically closed field, then $K(X) \simeq Q(\mathcal{O}(X)) = Q(K[X]) = K(t_1, \dots, t_n)$, where $t_1, \dots, t_n$ are the coordinate functions on X.*

Proof. The isomorphism is as follows:

(i) $\psi : K(X) \to Q(\mathcal{O}(X))$

If $\langle U, \varphi \rangle \in K(X)$, then there exists $V \subset U$, open and non–empty, such that $\varphi \mid_V = F/G$, where $F, G \in K[x_1, \dots, x_n]$ and $V(G) \cap V = \emptyset$. We set $\psi(\langle U, \varphi \rangle) = f/g$.

(ii) $\psi' : Q(\mathcal{O}(X)) \to K(X)$

If $f/g \in Q(\mathcal{O}(X))$, we set $\psi'(f/g) = \langle X \setminus V(g), f/g \rangle$.

It is easy to check that ψ and ψ' are well defined and inverse each other. □

Corollary 9.2.3. *If X is an irreducible affine variety over an algebraically closed field, then $\dim X$ is equal to the transcendence degree over K of its field of rational functions.*

Proof. It follows from Corollary 7.2.5. □

Proposition 9.2.4. *If X is a quasi–projective variety and $U \neq \emptyset$ is an open subset, then $K(X) \simeq K(U)$.*

Proof. We have the maps: $K(U) \to K(X)$ such that $\langle V, \varphi \rangle \to \langle V, \varphi \rangle$, and $K(X) \to K(U)$ such that $\langle A, \psi \rangle \to \langle A \cap U, \psi \mid_{A \cap U} \rangle$: they are K–homomorphisms inverse each other. □

Note. The term K-homomorphism means that the elements of K remain fixed.

Corollary 9.2.5. *If X is an irreducible projective variety contained in $\mathbb{P}^n$, if i is an index such that $X \cap U_i \neq \emptyset$ (where U_i is the open subset where $x_i \neq 0$), then $\dim X = \dim X \cap U_i = tr.d.K(X)/K$.*

Proof. By Proposition 7.1.3, $\dim X = \sup_i \dim(X \cap U_i)$. By Corollary 9.2.3 and Proposition 9.2.4, if $X \cap U_i$ is non–empty, $\dim(X \cap U_i) = tr.d.K(X \cap U_i)/K = tr.d.K(X)/K$: it is independent of i. □

If $\langle U, \varphi\rangle \in K(X)$, we can consider all possible representatives of it, i.e. all pairs $\langle U_i, \varphi_i\rangle$ such that $\langle U, \varphi\rangle = \langle U_i, \varphi_i\rangle$. Then $\tilde{U} = \bigcup_i U_i$ is the maximum open subset of X on which φ can be seen as a function: it is called the *domain of definition* (or of regularity) of $\langle U, \varphi\rangle$, or simply of φ. It is sometimes denoted domφ. If $P \in \tilde{U}$, we say that φ *is regular at* P.

We can consider the set of all rational functions on X which are regular at P: it is denoted by $\mathcal{O}_{P,X}$. It is a subring of $K(X)$ containing $\mathcal{O}(X)$, called the *local ring of* X *at* P. In fact, $\mathcal{O}_{P,X}$ is a local ring, whose maximal ideal, denoted $\mathcal{M}_{P,X}$, is the set of rational functions φ such that $\varphi(P)$ is defined and $\varphi(P) = 0$. To see this, observe that an element of $\mathcal{O}_{P,X}$ can be represented as $\langle U, F/G\rangle$: its inverse in $K(X)$ is $\langle U \setminus V_P(F), G/F\rangle$, which belongs to $\mathcal{O}_{P,X}$ if and only if $F(P) \neq 0$. We will see in Section 9.3 that $\mathcal{O}_{P,X}$ is the localization $K[X]_{I_X(P)}$.

As in Proposition 9.2.4 for the fields of rational functions, also for the local rings of points it can easily be proved that, if $U \neq \emptyset$ is an open subset of X containing P, then $\mathcal{O}_{P,X} \simeq \mathcal{O}_{P,U}$. So the ring $\mathcal{O}_{P,X}$ only depends on the local behaviour of X in the neighbourhood of P.

The *residue field* of $\mathcal{O}_{P,X}$ is the quotient $\mathcal{O}_{P,X}/\mathcal{M}_{P,X}$. This field results to be naturally isomorphic to the base field K. In fact consider the evaluation map $\mathcal{O}_{P,X} \to K$ such that φ goes to $\varphi(P)$: it is surjective with kernel $\mathcal{M}_{P,X}$, so $\mathcal{O}_{P,X}/\mathcal{M}_{P,X} \simeq K$.

Example 9.2.6. 1. The cuspidal cubic.

Let $X \subset \mathbb{A}^2$ be the curve $V(x_1^3 - x_2^2)$. Then $F = x_2$, $G = x_1$ define the function $\varphi = x_2/x_1$ which is regular at the points $P(a_1, a_2)$ such that $a_1 \neq 0$. Another representation of the same function is $\varphi = x_1^2/x_2$, which shows that φ is regular at P if $a_2 \neq 0$. If φ admits another representation F'/G', then $G'x_2 - F'x_1$ vanishes on an open subset of X, which is irreducible (see Exercise 2, Chapter 6), hence $G'x_2 - F'x_1$ vanishes on X, and therefore $G'x_2 - F'x_1 \in \langle x_1^3 - x_2^2\rangle$. We can write $G'x_2 - F'x_1 = H(x_1, x_2)(x_1^3 - x_2^2)$, for a suitable H, so $(G' + Hx_2)x_2 = (F' + Hx_1^2)x_1$. By the UFD property, it follows that there exists $A(x_1, x_2)$ such that $G' + Hx_2 = x_1 A$, $F' + Hx_1^2 = x_2 A$, so $(F', G') = (x_2 A - x_1^2 H, x_1 A - x_2 H) = A(x_2, x_1) - H(x_1^2, x_2)$.

This shows that there are essentially only the above two representations of φ. So $\varphi \in K(X)$ and its domain of regularity is $X \setminus \{0, 0\}$. We will see later (Example 10.1.2) another way to explain why the domain of definition cannot be all X.

2. The stereographic projection.

Let $X \subset \mathbb{P}^2$ be the curve $V_P(x_1^2 + x_2^2 - x_0^2)$. Let $f := x_1/(x_0 - x_2)$ denote the germ of the regular function defined by $x_1/(x_0 - x_2)$ on $X \setminus V_P(x_0 - x_2) = X \setminus \{[1, 0, 1]\} = X \setminus \{P\}$. On X we have $x_1^2 = (x_0 - x_2)(x_0 + x_2)$ so f is represented also as $(x_0 + x_2)/x_1$ on

$X \setminus V_P(x_1) = X \setminus \{P, Q\}$, where $Q = [1, 0, -1]$. If we identify K with the affine line $V_P(x_2) \setminus V_P(x_0)$ (the points of the x_1–axis lying in the affine plane U_0), then f can be interpreted as the stereographic projection of X centered at P, which takes $A[a_0, a_1, a_2]$ to the intersection of the line AP with the line $V_P(x_2)$. To see this, observe that AP has equation $a_1x_0 + (a_2 - a_0)x_1 - a_1x_2 = 0$; and $AP \cap V_P(x_2)$ is the point $[a_0 - a_2, a_1, 0]$.

9.3 Algebraic characterization of the local ring $\mathcal{O}_{P,X}$.

Let us recall the construction of the *ring of fractions of a ring* A with respect to a multiplicative subset S.

Let A be a ring and $S \subset A$ be a multiplicative subset. The following relation in $A \times S$ is an equivalence relation:

$$(a, s) \simeq (b, t) \text{ if and only if } \exists\ u \in S \text{ such that } u(at - bs) = 0.$$

Then the quotient $A \times S/_{\simeq}$ is denoted $S^{-1}A$ or A_S and $[(a, s)]$ is denoted $\frac{a}{s}$. A_S becomes a commutative ring with unit with operations $\frac{a}{s} + \frac{b}{t} = \frac{at+bs}{st}$ and $\frac{a}{s}\frac{b}{t} = \frac{ab}{st}$ (check that they are well–defined). With these operations, A_S is called the ring of fractions of A with respect to S, or the *localization* of A in S.

There is a natural homomorphism $j : A \to S^{-1}A$ such that $j(a) = \frac{a}{1}$, which makes $S^{-1}A$ an A–algebra (in the sense that it contains a homomorphic image of A). Note that j is the zero map if and only if $0 \in S$. More precisely if $0 \in S$ then $S^{-1}A$ is the zero ring: this case will always be excluded in what follows. Moreover j is injective if and only if every element in S is not a zero divisor. In this case $j(A)$ will be identified with A.

Example 9.3.1.

1. Let A be an integral domain and set $S = A \setminus \{0\}$. Then $A_S = Q(A)$: the quotient field of A.

2. If $\mathcal{P} \subset A$ is a prime ideal, then $S = A \setminus \mathcal{P}$ is a multiplicative set and A_S is denoted $A_{\mathcal{P}}$ and called the localization of A at $\mathcal{P}$.

3. If $f \in A$, then the multiplicative set generated by f is

$$S = \{1, f, f^2, \dots, f^n, \dots\} :$$

A_S is denoted A_f.

4. If $S = \{x \in A \mid x \text{ is regular}\}$, then A_S is called the total ring of fractions of A: it is the maximum ring in which A can be canonically embedded.

It is easy to verify that the ring A_S enjoys the following *universal property*:

(i) if $s \in S$, then $j(s)$ is invertible;

(ii) if B is a ring with a given homomorphism $f : A \to B$ such that for any $s \in S$ $f(s)$ is invertible, then f factorizes through A_S, i.e. there exists a unique homomorphism $\overline{f}$ such that $\overline{f} \circ j = f$.

We will see now the relations between ideals of A_S and ideals of A.

If $\alpha \subset A$ is any ideal, then $\alpha A_S = \{\frac{a}{s} \mid a \in \alpha\}$ is called the *extension of* α in A_S and denoted also α^e. It is an ideal, precisely the ideal generated by the set $\{\frac{a}{1} \mid a \in \alpha\}$.

If $\beta \subset A_S$ is an ideal, then $j^{-1}(\beta) =: \beta^c$ is called the contraction of β and is clearly an ideal.

The following Proposition gives the complete picture.

Proposition 9.3.2. *1. For any ideal $\alpha \subset A : \alpha^{ec} \supset \alpha$;*

2. for any ideal $\beta \subset A_S : \beta = \beta^{ce}$;

3. α^e is proper if and only if $\alpha \cap S = \emptyset$;

4. $\alpha^{ec} = \{x \in A \mid \exists s \in S$ such that $sx \in \alpha\}$.

Proof. 1. and 2. are straightforward.

3. if $1 = \frac{a}{s} \in \alpha^e$, then there exists $u \in S$ such that $u(s - a) = 0$, i.e. $us = ua \in S \cap \alpha$. Conversely, if $s \in S \cap \alpha$ then $1 = \frac{s}{s} \in \alpha^e$.

4.

$$\alpha^{ec} = \{x \in A \mid j(x) = \frac{x}{1} \in \alpha^e\} =$$

$$= \{x \in A \mid \exists a \in \alpha, t \in S \text{ such that } \frac{x}{1} = \frac{a}{t}\} =$$

$$= \{x \in A \mid \exists a \in \alpha, t, u \in S \text{ such that } u(xt - a) = 0\}.$$

Hence, if $x \in \alpha^{ec}$, then: $(ut)x = ua \in \alpha$. Conversely: if there exists $s \in S$ such that $sx = a \in \alpha$, then $\frac{x}{1} = \frac{a}{s}$, i.e. $j(x) \in \alpha^e$. □

If α is an ideal of A such that $\alpha = \alpha^{ec}$, α is called *saturated* with S. For example, if $\mathcal{P}$ is a prime ideal and $S \cap \mathcal{P} = \emptyset$, then $\mathcal{P}$ is saturated and $\mathcal{P}^e$ is prime. Conversely, if $\mathcal{Q} \subset A_S$ is a prime ideal, then $\mathcal{Q}^c$ is prime in A.

Corollary 9.3.3. *There is a bijection between the set of prime ideals of A_S and the set of prime ideals of A not intersecting S. In particular, if $S = A \setminus \mathcal{P}$, $\mathcal{P}$ prime, the prime ideals of $A_\mathcal{P}$ correspond bijectively to the prime ideals of A contained in $\mathcal{P}$, hence $A_\mathcal{P}$ is a local ring with maximal ideal $\mathcal{P}^e$, denoted $\mathcal{P}A_\mathcal{P}$, and residue field $A_\mathcal{P}/\mathcal{P}A_\mathcal{P}$. Moreover* $\dim A_\mathcal{P} = \mathrm{ht}\mathcal{P}$.

In particular we get the characterization of $\mathcal{O}_{P,X}$. Let $X \subset \mathbb{A}^n$ be an affine variety, let P be a point of X and $I(P) \subset K[x_1, \dots, x_n]$ be the ideal of P. Let $I_X(P) := I(P)/I(X)$ be the ideal of $K[X]$ formed by the regular functions on X vanishing at P. Then we can construct the localization

$$\mathcal{O}(X)_{I_X(P)} = \left\{ \frac{f}{g} \mid f, g \in \mathcal{O}(X), g(P) \neq 0 \right\} \subset K(X).$$

It is canonically identified with $\mathcal{O}_{P,X}$. In particular, if K is algebraically closed:

$$\dim \mathcal{O}_{P,X} = \text{ht } I_X(P) = \dim \mathcal{O}(X) = \dim X.$$

There is a bijection between prime ideals of $\mathcal{O}_{P,X}$ and prime ideals of $\mathcal{O}(X)$ contained in $I_X(P)$; they also correspond to prime ideals of $K[x_1, \dots, x_n]$ contained in $I(P)$ and containing $I(X)$.

If X is an affine variety, it is possible to define the local ring $\mathcal{O}_{P,X}$ also if X is reducible, in a purely algebraic way, simply as localization of $K[X]$ at the maximal ideal $I_X(P)$. The natural map j from $K[X]$ to $\mathcal{O}_{P,X}$ is injective if and only if $K[X] \setminus I_X(P)$ does not contain any zero divisor. A non-zero function f is a zero divisor in $K[X]$ if there exists a non-zero g such that $fg = 0$, i.e. $X = V(f) \cup V(g)$ is an expression of X as union of proper closed subsets. For j to be injective it is required that every zero divisor f belongs to $I_X(P)$, which means that all the irreducible components of X pass through P.

Exercises 9.3.4. 1. Prove that the irreducible affine varieties and the open subsets of irreducible affine varieties are quasi–projective varieties.

2. Let $X = \{P, Q\}$ be the union of two points in an affine space over K. Prove that $\mathcal{O}(X)$ is isomorphic to $K \times K$.

Chapter 10

Regular maps

10.1 Regular maps or morphisms

Let X, Y be quasi–projective varieties (or more generally locally closed sets). Let $\varphi : X \to Y$ be a map.

Definition 10.1.1. φ is a *regular map* or a *morphism* if

(i) φ is continuous for the Zariski topology;

(ii) φ preserves regular functions, i.e. for all $U \subset Y$ (U open and non–empty) and for all $f \in \mathcal{O}(U)$, then $f \circ \varphi \in \mathcal{O}(\varphi^{-1}(U))$:

$$\begin{array}{ccccc} X & \stackrel{\varphi}{\longrightarrow} & Y & & \\ \uparrow & & \uparrow & & \\ \varphi^{-1}(U) & \stackrel{\varphi|}{\longrightarrow} & U & \stackrel{f}{\to} & K \end{array}$$

Note that:
a) for all X the identity map $1_X : X \to X$ is regular;
b) for all X, Y, Z and regular maps $X \stackrel{\varphi}{\to} Y$, $Y \stackrel{\psi}{\to} Z$, the composite map $\psi \circ \varphi$ is regular.

An *isomorphism* of varieties is a regular map which possesses regular inverse, i.e. a regular map $\varphi : X \to Y$ such that there exists a regular map $\psi : Y \to X$ verifying the conditions $\psi \circ \varphi = 1_X$ and $\varphi \circ \psi = 1_Y$. In this case X and Y are said to be isomorphic, and we write: $X \simeq Y$.

If $\varphi : X \to Y$ is regular, there is a natural K–homomorphism $\varphi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$, called the *comorphism associated to* φ, defined by: $f \to \varphi^*(f) := f \circ \varphi$.

The construction of the comorphism is *functorial*, which means that:

a) $1_X^* = 1_{\mathcal{O}(X)}$;
b) $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.

This implies that, if $X \simeq Y$, then $\mathcal{O}(X) \simeq \mathcal{O}(Y)$. In fact, if $\varphi : X \to Y$ is an isomorphism and ψ is its inverse, then $\varphi \circ \psi = 1_Y$, so $(\varphi \circ \psi)^* = \psi^* \circ \varphi^* = (1_Y)^* = 1_{\mathcal{O}(Y)}$ and similarly $\psi \circ \varphi = 1_X$ implies $\varphi^* \circ \psi^* = 1_{\mathcal{O}(X)}$.

Example 10.1.2.

1) The homeomorphism $\varphi_i : U_i \to \mathbb{A}^n$ of Section 2.6 is an isomorphism.

2) There exist homeomorphisms which are not isomorphisms. Let $Y = V(x^3 - y^2) \subset \mathbb{A}^2$, the cuspidal cubic. We have seen (see Exercise 2, Chapter 7) that $K[Y] \not\simeq K[\mathbb{A}^1]$, hence Y is not isomorphic to the affine line $\mathbb{A}^1$. Nevertheless, the map

$$\varphi : \mathbb{A}^1 \to Y \text{ such that } t \to (t^2, t^3)$$

is regular, bijective and also a homeomorphism (see Exercise 1, Lesson 7).

Its inverse $\varphi^{-1} : Y \to \mathbb{A}^1$ is defined by

$$(x, y) \to \begin{cases} \frac{y}{x} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad (x, y) = (0, 0). \end{cases}$$

Note that φ^{-1} cannot be regular at the point $(0, 0)$.

10.2 Affine case

Next Proposition tells us how a morphism is given in practice, when the codomain is contained in an affine space.

Proposition 10.2.1. *Let $\varphi : X \to Y \subset \mathbb{A}^n$ be a map. Then φ is regular if and only if $\varphi_i := t_i \circ \varphi$ is a regular function on X, for all $i = 1, \dots, n$, where $t_1, \dots, t_n$ are the coordinate functions on Y.*

Proof. If φ is regular, then $\varphi_i = \varphi^*(t_i)$ is regular by definition.

Conversely, assume that φ_i is a regular function on X for all i. Let $Z \subset Y$ be a closed subset and we have to prove that $\varphi^{-1}(Z)$ is closed in X. Since any closed subset of $\mathbb{A}^n$ is an intersection of hypersurfaces, it is enough to consider $\varphi^{-1}(Y \cap V(F))$ with $F \in K[x_1, \dots, x_n]$:

$$\varphi^{-1}(Y \cap V(F)) = \{P \in X | F(\varphi(P)) = F(\varphi_1, \dots, \varphi_n)(P) = 0\} = V(F(\varphi_1, \dots, \varphi_n)).$$

But note that $F(\varphi_1, \dots, \varphi_n) \in \mathcal{O}(X)$: it is the composition of F with the regular functions $\varphi_1, \dots, \varphi_n$. Hence $\varphi^{-1}(Y \cap V(F))$ is closed, so we can conclude that φ is continuous. If $U \subset Y$ and $f \in \mathcal{O}(U)$, for any point P of U choose an open neighbourhood U_P such that $f = F_P/G_P$ on U_P. So $f \circ \varphi = F_P(\varphi_1, \dots, \varphi_n)/G_P(\varphi_1, \dots, \varphi_n)$ on $\varphi^{-1}(U_P)$, hence it is regular on each $\varphi^{-1}(U_P)$ and by consequence on $\varphi^{-1}(U)$. □

Remark 11. If $\varphi : X \to Y$ is a regular map and $Y \subset \mathbb{A}^n$, by Proposition 10.2.1 we can represent φ in the form $\varphi = (\varphi_1, \dots, \varphi_n)$, where $\varphi_1, \dots, \varphi_n \in \mathcal{O}(X)$ and $\varphi_i = \varphi^*(t_i)$. Note that $\varphi_1, \dots, \varphi_n$ are not arbitrary in $\mathcal{O}(X)$ but such that Im $\varphi \subset Y$.

Let us recall that, if Y is closed in $\mathbb{A}^n$ and K is algebraically closed, then $t_1, \dots, t_n$ generate $\mathcal{O}(Y)$, hence $\varphi_1, \dots, \varphi_n$ generate $\varphi^*(\mathcal{O}(Y))$ as K-algebra. This observation is the key for the following important result.

Theorem 10.2.2. *Let K be an algebraically closed field. Let X be a locally closed algebraic set and Y be an affine algebraic set. Let $Hom(X, Y)$ denote the set of regular maps from X to Y and $Hom(\mathcal{O}(Y), \mathcal{O}(X))$ denote the set of K– homomorphisms from $\mathcal{O}(Y)$ to $\mathcal{O}(X)$.*

Then the map $Hom(X, Y) \to Hom(\mathcal{O}(Y), \mathcal{O}(X))$, such that $\varphi : X \to Y$ goes to $\varphi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$, is bijective.

Proof. Let $Y \subset \mathbb{A}^n$ and let $t_1, \dots, t_n$ be the coordinate functions on Y, so $\mathcal{O}(Y) = K[t_1, \dots, t_n]$. Let $u : \mathcal{O}(Y) \to \mathcal{O}(X)$ be a K–homomorphism: we want to define a morphism $u^\sharp : X \to Y$ whose associated comorphism is u. By Remark 11, if $u^\sharp$ exists, its components have to be $u(t_1), \dots, u(t_n)$. So we define

$$\begin{array}{rccc} u^\sharp : & X & \to & \mathbb{A}^n \\ & P & \to & (u(t_1)(P)), \dots, u(t_n)(P)). \end{array}$$

This is a morphism by Proposition 10.2.1. We claim that $u^\sharp(X) \subset Y$. Let $F \in I(Y)$ and $P \in X$: then

$$\begin{aligned} F(u^\sharp(P)) = F(u(t_1)(P), \dots, u(t_n)(P)) = \\ = F(u(t_1), \dots, u(t_n))(P) = \\ = u(F(t_1, \dots, t_n))(P) \text{ because } u \text{ is } K\text{-homomorphism} = \\ = u(0)(P) = \\ = 0(P) = 0. \end{aligned} \tag{10.1}$$

So $u^\sharp$ is a regular map from X to Y.

We consider now $(u^\sharp)^* : \mathcal{O}(Y) \to \mathcal{O}(X)$: it maps a function f to $f \circ u^\sharp = f(u(t_1), \dots, u(t_n)) = u(f)$, so $(u^\sharp)^* = u$. Conversely, if $\varphi : X \to Y$ is regular, then $(\varphi^*)^\sharp$ maps P to

$$(\varphi^*(t_1)(P), \dots, \varphi^*(t_n)(P)) = (\varphi_1(P), \dots, \varphi_n(P)),$$

so $(\varphi^*)^\sharp = \varphi$. □

Note that, by definition, $1^\sharp_{\mathcal{O}(X)} = 1_X$, for all affine X; moreover $(v \circ u)^\sharp = u^\sharp \circ v^\sharp$ for all $u : \mathcal{O}(Z) \to \mathcal{O}(Y)$, $v : \mathcal{O}(Y) \to \mathcal{O}(X)$, K–homomorphisms of rings of regular functions of affine algebraic sets: this means that also this construction is functorial.

The construction of the comorphism associated to a regular function and the result of Theorem 10.2.2 can be rephrased using the language of categories. We will see it in Chapter 11.

If X and Y are quasi–projective varieties and $\varphi : X \to Y$ is a regular map, it is not always possible to define a comorphism $K(Y) \to K(X)$. If f is a rational function on Y with $\operatorname{dom} f = U$, it can happen that $\varphi(X) \cap \operatorname{dom} f = \emptyset$, in which case $f \circ \varphi$ does not exist. Nevertheless, if we assume that φ is **dominant**, i.e. $\overline{\varphi(X)} = Y$, then certainly $\varphi(X) \cap U \neq \emptyset$, hence $\langle \varphi^{-1}(U), f \circ \varphi \rangle \in K(X)$. We obtain a K–homomorphism, which is necessarily injective, $K(Y) \to K(X)$, also denoted by φ^*. Note that in this case, we have: $\dim X \geq \dim Y$. As above, it is possible to check that, if $X \simeq Y$, then $K(X) \simeq K(Y)$, hence $\dim X = \dim Y$. Moreover, if $P \in X$ and $Q = \varphi(P)$, then φ^* induces a map $\mathcal{O}_{Q,Y} \to \mathcal{O}_{P,X}$, such that $\varphi^* \mathcal{M}_{Q,Y} \subset \mathcal{M}_{P,X}$. Also in this case, if φ is an isomorphism, then $\mathcal{O}_{Q,Y} \simeq \mathcal{O}_{P,X}$.

10.3 Projective case

We will see now how to express in practice a regular map when the target is contained in a projective space. Let $X \subset \mathbb{P}^n$ be a quasi–projective variety and $\varphi : X \to \mathbb{P}^m$ be a map.

Proposition 10.3.1. *φ is a morphism if and only if, for any $P \in X$, there exist an open neighbourhood U_P of P and $n+1$ homogeneous polynomials $F_0, \dots, F_m$ of the same degree in $K[x_0, x_1, \dots, x_n]$, such that, if $Q \in U_P$, then $\varphi(Q) = [F_0(Q), \dots, F_m(Q)]$. In particular, for any $Q \in U_P$, there exists an index i such that $F_i(Q) \neq 0$.*

Proof. "⇒" Let $P \in X$, $Q = \varphi(P)$ and assume that $Q \in U_0$. Then $U := \varphi^{-1}(U_0)$ is an open neighbourhood of P and we can consider the restriction $\varphi|_U : U \to U_0$, which is regular.

Possibly after restricting U, using non–homogeneous coordinates on U_0, we can assume that $\varphi|_U = (F_1/G_1, \dots, F_m/G_m)$, where (F_1, G_1), ..., (F_m, G_m) are pairs of homogeneous polynomials of the same degree such that $V_P(G_i) \cap U = \emptyset$ for all index i. We can reduce the fractions F_i/G_i to a common denominator F_0, so that $\deg F_0 = \deg F_1 = \dots = \deg F_m$ and $\varphi|_U = (F_1/F_0, \dots, F_m/F_0) = [F_0, F_1, \dots, F_m]$, with $F_0(Q) \neq 0$ for $Q \in U$.

"$\Leftarrow$" Possibly after restricting U_P, we can assume $F_i(Q) \neq 0$ for all $Q \in U_P$ and suitable i. Let $i = 0$: then $\varphi|_{U_P} : U_P \to U_0$ operates as follows:

$$\varphi|_{U_P}(Q) = (F_1(Q)/F_0(Q), \dots, F_m(Q)/F_0(Q)),$$

so it is a morphism by Proposition 10.2.1. From this remark, one deduces that also φ is a morphism. □

10.4 Examples of morphisms

Example 10.4.1 (Stereographic projection)**.**

Let $X \subset \mathbb{P}^2$, $X = V_P(x_1^2 + x_2^2 - x_0^2)$, be the projective closure of the unitary circle. We define $\varphi : X \to \mathbb{P}^1$ by

$$[x_0, x_1, x_2] \to \begin{cases} [x_0 - x_2, x_1] & \text{if } (x_0 - x_2, x_1) \neq (0,0) \\ [x_1, x_0 + x_2] & \text{if } (x_1, x_0 + x_2) \neq (0,0). \end{cases}$$

φ is well–defined because, on X, $x_1^2 = (x_0 - x_2)(x_0 + x_2)$. Moreover

$$(x_1, x_0 - x_2) \neq (0,0) \Leftrightarrow [x_0, x_1, x_2] \in X \setminus \{[1,0,1]\},$$

$$(x_0 + x_2, x_1) \neq (0,0) \Leftrightarrow [x_0, x_1, x_2] \in X \setminus \{[1,0,-1]\}.$$

The map φ is the natural extension of the rational function $f : X \setminus \{[1,0,1]\} \to K$ such that $[x_0, x_1, x_2] \to x_1/(x_0 - x_2)$ (Example 9.2.6, 2). Now if we identify $\mathbb{P}^1$ with the line $V_P(x_2) \subset \mathbb{P}^2$, the North pole $N[1,0,1]$, centre of the stereographic projection, goes to the point at infinity of the line $\mathbb{P}^1$.

By geometric reasons φ is invertible and $\varphi^{-1} : \mathbb{P}^1 \to X$ takes $[\lambda, \mu]$ to $[\lambda^2 + \mu^2, 2\lambda\mu, \mu^2 - \lambda^2]$ (note the connection with the Pitagorean triples!). Indeed the line through N and $[\lambda, \mu, 0]$ has equation: $\mu x_0 - \lambda x_1 - \mu x_2 = 0$. Its intersections with X are represented by the system:

$$\begin{cases} \mu x_0 - \lambda x_1 - \mu x_2 = 0 \\ x_1^2 + x_2^2 - x_0^2 = 0 \end{cases}$$

Assuming $\mu \neq 0$ this system is equivalent to the following:

$$\begin{cases} \mu x_0 - \lambda x_1 - \mu x_2 = 0 \\ \mu^2 x_0^2 = \mu^2(x_1^2 + x_2^2) = (\lambda x_1 + \mu x_2)^2 \end{cases}$$

Therefore, either $x_1 = 0$ and $x_0 = x_2$, or

$$\begin{cases} (\mu^2 - \lambda^2)x_1 - 2\lambda\mu x_2 = 0 \\ \mu x_0 = \lambda x_1 + \mu x_2 \end{cases}$$

which gives the required expression.

Example 10.4.2. Affine transformations and affinities.

Let $A = (a_{ij})$ be a $n \times n$ matrix with entries in K, let $B = (b_1, \dots, b_n) \in \mathbb{A}^n$ be a point. The map $\tau_A : \mathbb{A}^n \to \mathbb{A}^n$ defined by $(x_1, \dots, x_n) \to (y_1, \dots, y_n)$, such that

$$\{y_i = \sum_j a_{ij}x_j + b_i, i = 1, \dots, n,$$

is a regular map called an affinity of $\mathbb{A}^n$. In matrix notation τ_A is $Y = AX + B$. If A is of rank n, then τ_A is said non–degenerate and is an isomorphism: the inverse map τ_A^{-1} is represented by $X = A^{-1}Y - A^{-1}B$. More in general, an affine transformation from $\mathbb{A}^n$ to $\mathbb{A}^m$ is a map represented in matrix form by $Y = AX + B$, where A is a $m \times n$ matrix and $B \in \mathbb{A}^m$. It is injective if and only if $\mathrm{rk}A = n$ and surjective if and only if $\mathrm{rk}A = m$.

The isomorphisms of an algebraic set X in itself are called **automorphisms of** X: they form a group for the usual composition of maps, denoted by $Aut\ X$. If $X = \mathbb{A}^n$, the non–degenerate affine transformations form a subgroup of $Aut\ \mathbb{A}^n$.

If $n = 1$ and the characteristic of K is 0, then $Aut\ \mathbb{A}^1$ coincides with this subgroup. In fact, let $\varphi : \mathbb{A}^1 \to \mathbb{A}^1$ be an automorphism: it is represented by a polynomial $F(x)$ such that there exists $G(x)$ satisfying the condition $G(F(t)) = t$ for all $t \in \mathbb{A}^1$, i.e. $G(F(x)) = x$ in the polynomial ring $K[x]$. Then, taking derivatives, we get $G'(F(x))F'(x) = 1$, which implies $F'(t) \neq 0$ for all $t \in K$, so $F'(x)$ is a non–zero constant. Hence, F is linear and G is linear too.

If $n \geq 2$, then $Aut\ \mathbb{A}^n$ is not completely described yet. There exist non–linear automorphisms of degree d, for all d. For example, for $n = 2$: let $\varphi : \mathbb{A}^2 \to \mathbb{A}^2$ be given by $(x, y) \to (x, y + P(x))$, where P is any polynomial of $K[x]$. Then $\varphi^{-1} : (x', y') \to (x', y' - P(x'))$. A very important and difficult open problem in Algebraic Geometry is the Jacobian conjecture, stating that, in characteristic zero, a regular map $\varphi : \mathbb{A}^n \to \mathbb{A}^n$ is an automorphism if and only if the Jacobian determinant $\mid J(\varphi) \mid$ is a non-zero constant.

Example 10.4.3. Projective transformations.

Let A be a $(n+1)\times(n+1)$–matrix with entries in K. Let $P[x_0,\dots,x_n]\in\mathbb{P}^n$: then $[a_{00}x_0+\dots+a_{0n}x_n,\dots,a_{n0}x_0+\dots+a_{nn}x_n]$ is a point of $\mathbb{P}^n$ if and only if it is different from $[0,\dots,0]$. So A defines a regular map $\tau:\mathbb{P}^n\to\mathbb{P}^n$ if and only if $\mathrm{rk}A=n+1$. If $\mathrm{rk}A=r<n+1$, then A defines a regular map whose domain is the quasi–projective variety $\mathbb{P}^n\setminus\mathbb{P}(kerA)$. If $\mathrm{rk}A=n+1$, then τ is an isomorphism, called a projective transformation or projectivity. Note that the matrices λA, $\lambda\in K^*$, all define the same projective transformation. So $PGL(n+1,K):=GL(n+1,K)/K^*$ acts on $\mathbb{P}^n$ as the group of projective transformations.

If $X,Y\subset\mathbb{P}^n$, they are called **projectively equivalent** if there exists a projective transformation $\tau:\mathbb{P}^n\to\mathbb{P}^n$ such that $\tau(X)=Y$.

Theorem 10.4.4. *Fundamental theorem on projective transformations.*

Let two $(n+2)$–tuples of points of $\mathbb{P}^n$ in general position be fixed: $P_0,\dots,P_{n+1}$ and $Q_0,\dots,Q_{n+1}$. Then there exists one, and only one, isomorphic projective transformation τ of $\mathbb{P}^n$ in itself, such that $\tau(P_i)=Q_i$ for all index i.

Proof. Put $P_i=[v_i]$, $Q_i=[w_i]$, $i=0,\dots,n+1$. So $\{v_0,\dots,v_n\}$ and $\{w_0,\dots,w_n\}$ are two bases of K^{n+1}, hence there exist scalars $\lambda_0,\dots,\lambda_n,\mu_0,\dots,\mu_n$ such that

$$v_{n+1}=\lambda_0v_0+\dots+\lambda_nv_n,\quad w_{n+1}=\mu_0w_0+\dots+\mu_nw_n,$$

where the coefficients are all different from 0, because of the general position assumption. We replace v_i with λ_iv_i and w_i with μ_iw_i and get two new bases, so there exists a unique automorphism of K^{n+1} transforming the first basis in the second one and, by consequence, also v_{n+1} in w_{n+1}. This automorphism induces the required projective transformation on $\mathbb{P}^n$. □

An immediate consequence of the above theorem is that projective subspaces of the same dimension are projectively equivalent. Also two subsets of $\mathbb{P}^n$ formed both by k points in general position are projectively equivalent if $k\le n+2$. If $k>n+2$, this is no longer true, already in the case of four points on a projective line. The problem of describing the classes of projective equivalence of k–tuples of points of $\mathbb{P}^n$, for $k>n+2$, is one the first problems of classical Invariant Theory. The solution in the case $k=4$, $n=1$ is given by the notion of *cross–ratio.*

Example 10.4.5. Affine and non-affine quasi–projective varieties.

Let $X\subset\mathbb{A}^n$ be an affine variety with $I(X)=\langle G_1,\dots,G_r\rangle$, then $X_F:=X\setminus V(F)$ is isomorphic to a closed subset of $\mathbb{A}^{n+1}$, i.e. to $Y=V(x_{n+1}F-1,G_1,\dots,G_r)$. Indeed, the following regular maps are inverse each other:

- $\varphi : X_F \to Y$ such that $(x_1, \ldots, x_n) \to (x_1, \ldots, x_n, 1/F(x_1, \ldots, x_n))$,

- $\psi : Y \to X_F$ such that $(x_1, \ldots, x_n, x_{n+1}) \to (x_1, \ldots, x_n)$.

Hence, X_F is a quasi–projective variety contained in $\mathbb{A}^n$, not closed in $\mathbb{A}^n$, but isomorphic to a closed subset of another affine space.

Definition 10.4.6. From now on, the term *affine variety* will denote a *locally closed subset of a projective space isomorphic to some affine closed set.*

If X is an affine variety and precisely $X \simeq Y$, with $Y \subset \mathbb{A}^n$ closed, then $\mathcal{O}(X) \simeq \mathcal{O}(Y) = K[t_1, \ldots, t_n]$ is a finitely generated K–algebra. In particular, if K is algebraically closed and α is an ideal strictly contained in $\mathcal{O}(X)$, then $V(\alpha) \subset X$ is non–empty, by the relative form of the Nullstellensatz (Proposition 9.1.5). From this observation, we can deduce that the quasi–projective variety of next example is not affine.

Example 10.4.7. $\mathbb{A}^2 \setminus \{(0, 0)\}$ is not affine.

Set $X = \mathbb{A}^2 \setminus \{(0, 0)\}$: first of all we will prove that $\mathcal{O}(X) \simeq K[x, y] = \mathcal{O}(\mathbb{A}^2)$, i.e. any regular function on X can be extended to a regular function on the whole plane.

Indeed: let $f \in \mathcal{O}(X)$: if $P \neq Q$ are points of X, then there exist polynomials F, G, F', G' such that $f = F/G$ on a neighbourhood U_P of P and $f = F'/G'$ on a neighbourhood U_Q of Q. So $F'G = FG'$ on $U_P \cap U_Q \neq \emptyset$, which is open also in $\mathbb{A}^2$, hence dense. Therefore $F'G = FG'$ in $K[x, y]$. We can clearly assume that F and G are coprime and similarly for F' and G'. So by the unique factorization property, it follows that $F' = F$ and $G' = G$. In particular f admits a unique representation as F/G on X therefore $G(P) \neq 0$ for all $P \in X$. Hence G has no zeros on $\mathbb{A}^2$, so $G = c \in K^*$ and $f \in \mathcal{O}(X)$.

Now, the ideal $\langle x, y\rangle$ has no zeros in X and is proper: this proves that X is not affine.

We have exploited the fact that a polynomial in more than one variables has infinitely many zeros, a fact that allows to generalise the previous observation.

10.5 Open covering with affine varieties

Affine varieties are ubiquitous in view of the following Proposition.

Proposition 10.5.1. *Let $X \subset \mathbb{P}^n$ be quasi–projective. Then X admits an open covering by affine varieties.*

Proof. Let $X = X_0 \cup \cdots \cup X_n$ be the open covering of X where $X_i = U_i \cap X = \{P \in X \mid P[a_0, \ldots, a_n], a_i \neq 0\}$. So, fixed P, there exists an index i such that $P \in X_i$. We can assume that $P \in X_0$: X_0 is open in some affine variety Y of $\mathbb{A}^n$ (identified with U_0); set $X_0 = Y \setminus Y'$, where Y, Y' are both closed in $\mathbb{A}^n$. Since $P \notin Y'$, there exists F such that $F(P) \neq 0$ and $V(F) \supset Y'$. So $P \in Y \setminus V(F) \subset Y \setminus Y'$ and $Y \setminus V(F)$ is an affine open neighbourhood of P in $Y \setminus Y' = X_0$, that is open in X. □

10.6 The Veronese maps

Let n, d be positive integers; put $N(n,d) = \binom{n+d}{d} - 1$. Note that $\binom{n+d}{d}$ is equal to the number of (monic) monomials of degree d in the variables $x_0, \ldots, x_n$, that is equal to the number of $(n+1)$–tuples $(i_0, \ldots, i_n)$ such that $i_0 + \cdots + i_n = d$, $i_j \geq 0$. Then in $\mathbb{P}^{N(n,d)}$ we can use coordinates $\{v_{i_0 \ldots i_n}\}$, where $i_0, \ldots, i_n \geq 0$ and $i_0 + \cdots + i_n = d$. For example: if $n = 2$, $d = 2$, then $N(2,2) = \binom{4}{2} - 1 = 5$. In $\mathbb{P}^5$ we can use coordinates $v_{200}, v_{110}, v_{101}, v_{020}, v_{011}, v_{002}$.

For all n, d we define the map $v_{n,d} : \mathbb{P}^n \to \mathbb{P}^{N(n,d)}$ such that

$$[x_0, \ldots, x_n] \to [v_{d00\ldots0}, v_{d-1,10\ldots0}, \ldots, v_{0\ldots00d}]$$

where $v_{i_0 \ldots i_n} = x_0^{i_0} x_1^{i_1} \ldots x_n^{i_n}$: $v_{n,d}$ is clearly a morphism, its image is denoted by $V_{n,d}$ and is called *the Veronese variety* of type (n,d). It is in fact the projective variety of equations:

$$\{v_{i_0 \ldots i_n} v_{j_0 \ldots j_n} - v_{h_0 \ldots h_n} v_{k_0 \ldots k_n}, \forall i_0 + j_0 = h_0 + k_0, i_1 + j_1 = h_1 + k_1, \ldots \tag{10.2}$$

We prove this statement in the particular case $n = d = 2$; the general case is similar.

First of all, it is clear that the points of $v_{n,d}(\mathbb{P}^n)$ satisfy the system (10.2). Conversely, assume that $P[v_{200}, v_{110}, \ldots] \in \mathbb{P}^5$ satisfies equations (10.2), which become:

$$\begin{cases} v_{200}v_{020} = v_{110}^2 \\ v_{200}v_{002} = v_{101}^2 \\ v_{002}v_{020} = v_{011}^2 \\ v_{200}v_{011} = v_{110}v_{101} \\ v_{020}v_{101} = v_{110}v_{011} \\ v_{110}v_{002} = v_{011}v_{101} \end{cases}$$

Then, at least one of the coordinates $v_{200}, v_{020}, v_{002}$ is different from 0.

Therefore, if $v_{200} \neq 0$, then $P = v_{2,2}([v_{200}, v_{110}, v_{101}])$; if $v_{020} \neq 0$, then $P = v_{2,2}([v_{110}, v_{020}, v_{011}])$; if $v_{002} \neq 0$, then $P = v_{2,2}([v_{101}, v_{011}, v_{002}])$. Note that, if two of these three coordinates are different from 0, then the points of $\mathbb{P}^2$ found in this way have proportional coordinates, so they coincide.

We have also proved in this way that $v_{2,2}$ is an isomorphism between $\mathbb{P}^2$ and $V_{2,2}$, called the Veronese surface of $\mathbb{P}^5$. The same happens in the general case.

If $n = 1$, $v_{1,d} : \mathbb{P}^1 \to \mathbb{P}^d$ maps $[x_0, x_1]$ to $[x_0^d, x_0^{d-1}x_1, \dots, x_1^d]$: the image is called the *rational normal curve* of degree d, it is isomorphic to $\mathbb{P}^1$. If $d = 3$, we find the skew cubic (Chapter 5).

Let now $X \subset \mathbb{P}^n$ be a hypersurface of degree d: $X = V_P(F)$, with

$$F = \sum_{i_0+\cdots+i_n=d} a_{i_0\dots i_n} x_0^{i_0} \dots x_n^{i_n}.$$

Then $v_{n,d}(X) \simeq X$: it is the set of points

$$\{v_{i_0\dots i_n} \in \mathbb{P}^{N(n,d)} | \sum_{i_0+\cdots+i_n=d} a_{i_0\dots i_n} v_{i_0\dots i_n} = 0 \text{ and } [v_{i_0\dots i_n}] \in V_{n,d}\}.$$

It coincides with $V_{n,d} \cap H$, where H is a hyperplane of $\mathbb{P}^{N(n,d)}$: a hyperplane section of the Veronese variety. This is called the linearisation process, allowing to "transform" a hypersurface in a hyperplane, modulo the Veronese isomorphism.

The Veronese surface $V = V_{2,2}$ of $\mathbb{P}^5$ enjoys a lot of interesting properties. Most of them follow from its property of being covered by a 2-dimensional family of conics, which are precisely the images via $v_{2,2}$ of the lines of the plane.

To see this, we will change notation and will use as coordinates in $\mathbb{P}^5$ $w_{00}, w_{01}, w_{02}, w_{11}, w_{12}, w_{22}$, so that $v_{2,2}$ maps $[x_0, x_1, x_2]$ to the point of coordinates $w_{ij} = x_i x_j$. With this choice of coordinates, the equations of V are obtained by annihilating the 2×2 minors of the symmetric matrix:

$$M = \begin{pmatrix} w_{00} & w_{01} & w_{02} \\ w_{01} & w_{11} & w_{12} \\ w_{02} & w_{12} & w_{22} \end{pmatrix}.$$

Let ℓ be a line of $\mathbb{P}^2$ of equation $b_0x_0 + b_1x_1 + b_2x_2 = 0$. Its image is the set of points of $\mathbb{P}^5$ with coordinates $w_{ij} = x_i x_j$, such that there exists a non-zero triple $[x_0, x_1, x_2]$ with $b_0x_0 + b_1x_1 + b_2x_2 = 0$. But this last equation is equivalent to the system:

$$\begin{cases} b_0x_0^2 + b_1x_0x_1 + b_2x_0x_2 = 0 \\ b_0x_0x_1 + b_1x_1^2 + b_2x_1x_2 = 0 \\ b_0x_0x_2 + b_1x_1x_2 + b_2x_2^2 = 0 \end{cases}$$

It represents the intersection of V with the plane

$$\begin{cases} b_0 w_{00} + b_1 w_{01} + b_2 w_{02} = 0 \\ b_0 w_{01} + b_1 w_{11} + b_2 w_{12} = 0 \\ b_0 w_{02} + b_1 w_{12} + b_2 w_{22} = 0 \end{cases} \tag{10.3}$$

so $v_{2,2}(\ell)$ is a plane curve. Its degree is the number of points in its intersection with a general hyperplane in $\mathbb{P}^5$: this corresponds to the intersection in $\mathbb{P}^2$ of ℓ with a conic (a hypersurface of degree 2). Therefore $v_{2,2}(\ell)$ is a conic.

So the isomorphism $v_{2,2}$ transforms the geometry of the lines in the plane in the geometry of the conics in the Veronese surface. In particular, given two distinct points on V, there is exactly one conic contained in V and passing through them.

From this observation it is easy to deduce that the *secant lines* of V, i.e. the lines meeting V at two points, are precisely the lines of the planes generated by the conics contained in V, so that the (closure of the) union of these secant lines coincides with the union of the planes of the conics of V. This union results to be the cubic hypersurface defined by the equation

$$\det M = \det \begin{pmatrix} w_{00} & w_{01} & w_{02} \\ w_{01} & w_{11} & w_{12} \\ w_{02} & w_{12} & w_{22} \end{pmatrix} = 0.$$

Indeed a point in $\mathbb{P}^5$, of coordinates $[w_{ij}]$ belongs to the plane of a conic contained in V if and only if there exists a non-zero triple $[b_0, b_1, b_2]$ which is solution of the homogeneous system (10.3).

Exercises 10.6.1. 1. Let X, Y be closed subsets of $\mathbb{A}^n$. Consider $X \times Y \subset \mathbb{A}^{2n}$ and the linear subspace, called the diagonal, $\Delta \subset \mathbb{A}^{2n}$ defined by the equations $x_i - y_i = 0$, $i = 1, \dots, n$. Prove that $(X \times Y) \cap \Delta$ is isomorphic to $X \cap Y$, constructing an explicit regular map with regular inverse.

2. Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be the map defined by $f(x, y) = (x, xy)$. Check that f is regular and find the image $f(\mathbb{A}^2)$: is it open in $\mathbb{A}^2$? Dense? Closed? Locally closed? Irreducible?

3. Let $v_{1,d} : \mathbb{P}^1 \to \mathbb{P}^d$ be the d-tuple Veronese map, such that $v_{1,d}([x_0, x_1]) = [x_0^d, x_0^{d-1}x_1, \dots, x_1^d]$).

a) Check that the image of $v_{1,d}$ is C_d, the projective algebraic set defined by the 2×2 minors of the matrix

$$A = \begin{pmatrix} x_0 & x_1 & \dots & x_{d-1} \\ x_1 & x_2 & \dots & x_d \end{pmatrix}.$$

C_d is called the rational normal curve of degree d.

b) Prove that $v_{1,d} : \mathbb{P}^1 \to C_d$ is an isomorphism, by explicitly constructing its inverse morphism.

c) Prove that any $d+1$ points on C_d are linearly independent in $\mathbb{P}^d$ (Hint: Vandermonde).

Solution of Exercise 3. This exercise generalises the example of the skew cubic.

a) Let $z_0, \dots, z_d$ be coordinates in $\mathbb{P}^d$, so that the image of the Veronese map $v_{1,d}$ is given in parametric form by $z_0 = x_0^d, \dots, z_i = x_0^{d-i} x_1^i, \dots, z_d = x_1^d$. Let I be the ideal generated by the 2×2 minors of A. It is clear that the two rows of the matrix

$$\begin{pmatrix} x_0^d & x_0^{d-1}x_1 & \dots & x_0 x_1^{d-1} \\ x_0^{d-1}x_1 & x_0^{d-2}x_1^2 & \dots & x_1^d \end{pmatrix}$$

are proportional for any x_0, x_1, so $v_{1,d}(\mathbb{P}^1) \subset V_P(I) = C_d$.

Conversely, let $[\bar{z}_0, \dots, \bar{z}_d] \in V_P(I)$. We observe that either $\bar{z}_0 \neq 0$ or $\bar{z}_d \neq 0$. If $\bar{z}_0 \neq 0$, then we can multiply all coordinates by $\bar{z}_0{}^{d-1}$ and we get:

$$[\bar{z}_0, \dots, \bar{z}_d] = [\bar{z}_0{}^d, \bar{z}_0{}^{d-1}\bar{z}_1, \dots, \bar{z}_0{}^{d-1}\bar{z}_i, \dots, \bar{z}_0{}^{d-1}\bar{z}_d].$$

If we can prove that $\bar{z}_0{}^{d-1}\bar{z}_i = \bar{z}_0{}^{d-i}\bar{z}_1{}^i$, then we conclude that our point is equal to $v_{1,d}([\bar{z}_0, \bar{z}_1])$. Note that $\bar{z}_0\bar{z}_k = \bar{z}_1\bar{z}_{k-1}$, for any $k = 1, \dots, d$. So $\bar{z}_0{}^{d-1}\bar{z}_i = \bar{z}_0{}^{d-2}(\bar{z}_1\bar{z}_{i-1}) = \bar{z}_0{}^{d-3}\bar{z}_1(\bar{z}_1\bar{z}_{i-2}) = \dots = \bar{z}_0{}^{d-i}\bar{z}_1{}^i$, as wanted.

If instead $\bar{z}_d \neq 0$, proceeding in a similar way we prove that $[\bar{z}_0, \dots, \bar{z}_d] = v_{1,d}([\bar{z}_{d-1}, \bar{z}_d])$.

b) The inverse map $\varphi : C_d \to \mathbb{P}^1$ operates in this way: $\varphi([z_0, \dots, z_d]) = [z_0, z_1] = [z_1, z_2] = \dots = [z_{d-1}, z_d]$. It is well defined because the columns of A are proportional, and it is regular because it is a projection.

c) Let $[z_0^{(k)}, \dots, z_d^{(k)}] = v_{1,d}([x_0^{(k)}, x_1^{(k)}])$, $k = 0, \dots d$, be $d+1$ points on C_d. Let $M = (z_i^{(j)})_{i,j=0,\dots d}$ be the matrix of their coordinates. If $x_0^{(k)} \neq 0$ for any k, we can assume $x_0^{(k)} = 1$ and

$$M = \begin{pmatrix} 1 & 1 & \dots & 1 \\ x_1^{(0)} & x_1^{(1)} & \dots & x_1^{(d)} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ x_1^{(0)d} & x_1^{(1)d} & \dots & x_1^{(d)d} \end{pmatrix}.$$

This is a Vandermonde matrix whose determinant is different from zero because the points are distinct.

If one of the points has the first coordinate equal to zero, then it is $[0, 0, \dots, 0, 1]$, so we can assume that it is the first point, and that all the other d points have $x_0^{(k)} = 1$. Therefore

$$M = \begin{pmatrix} 0 & 1 & \dots & 1 \\ 0 & x_1^{(1)} & \dots & x_1^{(d)} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 1 & {x_1^{(1)}}^d & \dots & {x_1^{(d)}}^d \end{pmatrix}.$$

Developing the determinant according to the first column, we find again a Vandermonde determinant, which is different from 0.

Chapter 11

The language of categories

11.1 Categories

Category theory was introduced by Samuel Eilenberg and Saunders Mac Lane in 1942-45 in their study of algebraic topology. They introduced the concepts of categories, functors, and natural transformations, with the goal of understanding the processes that preserve mathematical structures. In Algebraic Geometry it was much developed by Alexander Grothendieck, in his language of schemes.

Category theory has proven to be a powerful language for expressing some general facts and constructions that are encountered mainly in branches of algebra and geometry. Here we give an elementary introduction limiting ourselves to the simplest definitions and examples.

Definition 11.1.1. A category $\mathcal{C}$ consists of the following data:

(1) A class $ob(\mathcal{C})$ whose elements are called objects of the category;

(2) For each pair $A, B \in ob(\mathcal{C})$ of objects, a set indicated with $Hom_{\mathcal{C}}(A, B)$, or $\mathcal{C}(A, B)$, called set of morphisms or arrows from A to B. Instead of writing $f \in Hom_{\mathcal{C}}(A, B)$ it is common to use $f : A \to B$.

(3) For each triple of objects A, B, C a map of sets called composition:

$$Hom_{\mathcal{C}}(A, B) \times Hom_{\mathcal{C}}(B, C) \to Hom_{\mathcal{C}}(A, C),$$

such that

$$(f, g) \to g \circ f.$$

(4) For each object A a special element $1_A \in Hom_{\mathcal{C}}(A, A)$ called identity of A.

It is also assumed that the following axioms hold:

a) Composition is associative;

b) Identity acts as a neutral element for the composition (when it is defined).

The categories that are best known (but we will also meet others) are those in which we can interpret morphisms as particular functions between sets, their composition is the usual composition of functions, and the identity is the usual identity.

In particular we have:

(1) The category of sets, indicated with the symbol Set, in which $Hom(A, B) = Set(A, B)$ is the set of arbitrary maps from A to B.

(2) The category *Grp* of groups and homomorphisms between groups, *Ab* of abelian groups and group homomorphisms, *Rng* of rings and homomorphisms of rings, or Mod_R of modules on a ring R with homomorphisms of R-modules, etc.

(4) Top with objects the topological spaces and morphisms the continuous functions.

(5) The coverings of a given topological space and the covering maps.

(6) The notion of subcategory is rather natural: $\mathcal{C}'$ is a subcategory of $\mathcal{C}$ if the class $ob(\mathcal{C}')$ is contained in $ob(\mathcal{C})$ and, for any pair of objects A, B in $\mathcal{C}'$, $Hom_{\mathcal{C}'}(A, B) \subset Hom_{\mathcal{C}}(A, B)$. The subcategory is called full if equality holds: $Hom_{\mathcal{C}'}(A, B) = Hom_{\mathcal{C}}(A, B)$.

(7) A first example of a category where morphisms cannot be thought of as simple functions is that of a *poset*. i.e. a partially ordered set P. The objects are the elements of P and

$$Hom_P(a, b) = \begin{cases} \{*\} & \text{if } a \leq b; \\ \emptyset & \text{otherwise.} \end{cases}$$

Here $\{*\}$ denotes a set with only one element denoted by $*$, also called the singleton. A particular case of a poset category is $Op(X)$, the category of the open subsets of a topological space X.

11.2 Functors

The second notion we are going to introduce formalizes the idea of transformation of categories.

Definition 11.2.1. A (covariant) functor $F : \mathcal{A} \to \mathcal{B}$ from the category $\mathcal{A}$ to the category $\mathcal{B}$ is a law that associates to every object X of $\mathcal{A}$ an object $F(X)$ of $\mathcal{B}$ and to every morphism $f : X \to Y$ in $\mathcal{A}$ a morphism $F(f) : F(X) \to F(Y)$ in $\mathcal{B}$, in such a way that

a) $F(f \circ g) = F(f) \circ F(g)$ (when the composition is defined),

b) $F(1_X) = 1_{F(X)}$.

The composition of functors can be done as in the case of functions.

Contravariant functors are defined by imposing that to every morphism $f : X \to Y$ is associated a morphism $F(f) : F(Y) \to F(X)$ so that we have $F(f \circ g) = F(g) \circ F(f)$. In other words, contravariant functors invert the arrows.

Given a category $\mathcal{C}$, we can define the opposite category $\mathcal{C}^0$, or $\mathcal{C}^{op}$, whose objects are the same as those of $\mathcal{C}$ while $Hom_{\mathcal{C}^0}(A, B) = Hom_{\mathcal{C}}(B, A)$. It is easily seen that a contravariant functor from $\mathcal{A}$ to $\mathcal{B}$ is also a covariant functor from $\mathcal{A}$ to $\mathcal{B}^0$ (or from $\mathcal{A}^0$ to $\mathcal{B}$).

Example 11.2.2. *Examples of functors.*

1. *Forgetful functors.* The law $U : Grp \to Set$ which maps a group to its underlying set and a group homomorphism to its underlying function of sets is a functor. Functors like this, which "forget" some structure, are termed forgetful functors. Another example is the functor $Rng \to Ab$ which maps a ring to its underlying additive abelian group. Morphisms in Rng (ring homomorphisms) become morphisms in Ab (abelian group homomorphisms).

2. *Free functors.* Going in the opposite direction of forgetful functors are free functors. The free functor $F : Set \to Ab$ sends every set X to the free abelian group generated by X. Functions are mapped to group homomorphisms between free abelian groups.

3. *Representable functors.* Let $\mathcal{C}$ be a category. Each object $A \in ob(\mathcal{C})$ allows to define the following functor $h^A : \mathcal{C} \to Set$. For each object $X \in ob(\mathcal{C})$, $h^A(X) := Hom_{\mathcal{C}}(A, X) \in ob(Set)$. For each morphism $f : X \to Y$ in $\mathcal{C}$, we define $h^A(f) : Hom_{\mathcal{C}}(A, X) \to Hom_{\mathcal{C}}(A, Y)$ through the composition: $h^A(g) := f \circ g$. The functor h^A is usually denoted by $h^A := Hom_{\mathcal{C}}(A, -)$ and is a covariant functor which is said to be represented by the object A of $\mathcal{C}$.

In a completely analogous way we can define the contravariant functor $h_A := Hom_{\mathcal{C}}(-, A)$.

Among the categorical ideas there is that of isomorphism, which generalizes that of bijection between sets, of isomorphism of groups, of homeomorphism between topological spaces etc.

An isomorphism f between two objects A, B of a category $\mathcal{C}$ is a morphism $f : A \to B$ such that there exists a morphism $g : B \to A$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$.

The following property follows easily from the axioms of category.

Proposition 11.2.3. *(1) If $f : A \to B$ is an isomorphism, the morphism $g : B \to A$ such that $g \circ f = 1_A$, $f \circ g = 1_B$ is unique (and denoted f^{-1}).*

(2) If $f : A \to B$ is an isomorphism in $\mathcal{C}$ and $F : \mathcal{C} \to \mathcal{D}$ is a functor, then also $F(f) : F(A) \to F(B)$ is an isomorphism (in $\mathcal{D}$).

11.3 Natural transformations

To complete the categorical approach it is convenient to introduce the last formal definition, the one that allows to treat the functors between two given categories $A \to B$ like the objects of a new category. To do this, we must define the morphisms between two such functors, which we will call natural transformations. We give the definition for covariant functors, the contravariant case is similar.

Definition 11.3.1. Given two functors $F, G : \mathcal{A} \to \mathcal{B}$ between two categories, a natural transformation $\varphi : F \to G$ between the two functors consists in giving, for each object $A \in ob(\mathcal{A})$ a morphism $\varphi_A : F(A) \to G(A)$ (in $\mathcal{B}$) such that, for each pair of objects $A, B \in ob(\mathcal{C})$ and for each morphism $f : A \to B$ the following diagram is commutative:

$$\begin{array}{ccc} F(A) & \xrightarrow{\varphi_A} & G(A) \\ {\scriptstyle F(f)} \downarrow & & \downarrow {\scriptstyle G(f)} \\ F(B) & \xrightarrow{\varphi_B} & G(B) \end{array}$$

The class of natural transformations between two functors $F, G : \mathcal{A} \to \mathcal{B}$ is denoted by $Nat(F, G)$. Often it is a set, which can therefore be taken as the set of morphisms to define the category of functors from category $\mathcal{A}$ to category $\mathcal{B}$. We will indicate with $F(\mathcal{A}, \mathcal{B})$ this category of functors. The properties of identity and composition are easy to verify.

From the general ideas, it follows the definition of natural isomorphism between two functors: it is a natural transformation that admits an inverse, and also that of **equivalence of categories.** An equivalence between the categories $\mathcal{A}, \mathcal{B}$ is a functor $F : \mathcal{A} \to \mathcal{B}$ satisying the following two conditions:

1. for any $Y \in ob(\mathcal{B})$ there exists $X \in ob(\mathcal{A})$ such that $Y \simeq F(X)$;
2. for any pair of objects A, B in $\mathcal{A}$, F gives a bijection $Hom(A, B) \xrightarrow{F} Hom(F(A), F(B))$.

We introduce a category $\mathcal{C}$ whose objects are the affine algebraic sets over a fixed algebraically closed field K and the morphisms are the regular maps. We consider also a second category $\mathcal{C}'$ with objects the K-algebras and morphisms the K-homomorphisms. Then there is a contravariant functor that operates on the objects mapping X to $\mathcal{O}(X) = K[X]$, and on the morphisms mapping φ to the associated comorphism φ^*. Note that this functor can be interpreted as the representable functor $h_{\mathbb{A}^1}$, when $\mathbb{A}^1$ is identified with K.

If we restrict the class of objects of $\mathcal{C}'$ taking only the finitely generated reduced K-algebras (a full subcategory of the previous one), then this functor becomes an equivalence of categories. Indeed the construction of the comorphism establishes a bijection between the Hom sets $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $\mathrm{Hom}_{\mathcal{C}'}(\mathcal{O}(Y), \mathcal{O}(X))$. Moreover, for any finitely generated and reduced K-algebra A, there exists an affine algebraic set X such that A is K-isomorphic to $\mathcal{O}(X)$. To see this, we choose a finite set of generators of A, such that $A = K[\xi_1, \ldots, \xi_n]$. Then we can consider the surjective K-homomorphism Ψ from the polynomial ring $K[x_1, \ldots, x_n]$ to A sending x_i to ξ_i for any i. In view of the fundamental theorem of homomorphism, it follows that $A \simeq K[x_1, \ldots, x_n]/\ker\Psi$. The assumption that A is reduced then implies that $X := V(\ker\Psi) \subset \mathbb{A}^n$ is an affine algebraic set with $I(X) = \ker\Psi$ and $A \simeq \mathcal{O}(X)$.

We note that changing system of generators for A changes the homomorphism Ψ, and by consequence also the algebraic set X, up to isomorphism. For instance let $A = K[t]$ be a polynomial ring in one variable t: if we choose only t as system of generators, we get $X = \mathbb{A}^1$, but we can choose t, t^2, t^3, because $A = K[t, t^2, t^3]$; in this case we get the affine skew cubic in $\mathbb{A}^3$.

As a consequence of the previous discussion we have the following:

Corollary 11.3.2. *Let X, Y be affine varieties. Then $X \simeq Y$ if and only if $\mathcal{O}(X) \simeq \mathcal{O}(Y)$.*

We conclude this chapter defining an important functor. Let X be a quasi-projective algebraic variety over a field K. We consider the category $Op(X)$ of the open subsets of X, interpreted as topological space with the Zariski topology. The second category is $K - alg$, the category of K-algebras and K-homomorphisms. We define a contravariant functor $\mathcal{O}_X : Op(X) \to K - alg$ such that, for any open subset $U \subset X$, $\mathcal{O}_X(U) = \mathcal{O}(U)$, the ring of regular functions on U interpreted as quasi-projective variety. Given a morphism in $Op(X)$, this is an inclusion $U \hookrightarrow V$; this is sent by the functor $\mathcal{O}_X$ to the natural restriction map $\mathcal{O}(V) \to \mathcal{O}(U)$.

$\mathcal{O}_X$ is called the sheaf of regular functions on the variety X.

Chapter 12

Rational maps

12.1 Rational maps

Let X, Y be quasi–projective varieties over an algebraically closed field K. The idea to define rational maps is that they are to regular maps as rational functions are to regular functions.

Definition 12.1.1. The *rational maps* from X to Y are the germs of regular maps from open subsets of X to Y, i.e. they are equivalence classes of pairs (U, φ), where $U \neq \emptyset$ is open in X and $\varphi : U \to Y$ is regular. The equivalence relation is of course defined by $(U, \varphi) \sim (V, \psi)$ if and only if $\varphi|_{U \cap V} = \psi|_{U \cap V}$.

We need to prove that this is indeed an equivalence relation. The following Lemma guarantees that this is the case.

Lemma 12.1.2. *Let $\varphi, \psi : X \to Y \subset \mathbb{P}^n$ be regular maps between quasi-projective varieties. If $\varphi|_U = \psi|_U$ for $U \subset X$ open and non–empty, then $\varphi = \psi$.*

Proof. Let $P \in X$ and consider $\varphi(P)$, $\psi(P) \in Y$. There exists a hyperplane H such that $\varphi(P) \notin H$ and $\psi(P) \notin H$ (otherwise the dual projective space $\check{\mathbb{P}}^n$ would be the union of its two hyperplanes $H_{\varphi(P)}$, $H_{\psi(P)}$, defined by the conditions of containing respectively $\varphi(P)$ and $\psi(P)$).

Up to a projective transformation, we can assume that $H = V_P(x_0)$, so $\varphi(P), \psi(P) \in U_0$. Set $V = \varphi^{-1}(U_0) \cap \psi^{-1}(U_0)$: an open neighbourhood of P. Consider the restrictions of φ and ψ from V to $Y \cap U_0$: they are regular maps whose codomain is contained in $U_0 \simeq \mathbb{A}^n$. Since they coincide on $V \cap U$, their components φ_i, ψ_i, $i = 1, \dots, n$, coincide on $V \cap U$, hence on V (Corollary 9.1.4). So $\varphi_i|_V = \psi_i|_V$. In particular $\varphi(P) = \psi(P)$. □

A rational map from X to Y will be denoted by $\varphi : X \dashrightarrow Y$. As for rational functions, the domain of definition of φ, dom φ, is the maximum open subset of X such that φ is regular at the points of dom φ.

The following proposition follows from the characterization of rational functions on affine varieties.

Proposition 12.1.3. *Let X, Y be affine algebraic sets, with Y closed in $\mathbb{A}^n$. Then $\varphi : X \dashrightarrow Y$ is a rational map if and only if $\varphi = (\varphi_1, \ldots, \varphi_n)$, where $\varphi_1, \ldots, \varphi_n \in K(X)$.*

If $X \subset \mathbb{P}^n$, $Y \subset \mathbb{P}^m$, then a rational map $X \dashrightarrow Y$ is assigned by giving $m+1$ homogeneous polynomials of $K[x_0, x_1, \ldots, x_n]$ of the same degree, $F_0, \ldots, F_m$, such that *at least one* of them is not identically zero on X.

A rational map $\varphi : X \dashrightarrow Y$ is called *dominant* if the image of X via φ is dense in Y, i.e. if $\overline{\varphi(U)} = Y$, where $U =$ dom φ.

Dominant rational maps can be composed: if $\varphi : X \dashrightarrow Y$ is dominant and $\psi : Y \dashrightarrow Z$ is any rational map, then dom $\psi \cap \mathrm{Im}\varphi \neq \emptyset$, so we can define $\psi \circ \varphi : X \dashrightarrow Z$: it is the germ of the map $\psi \circ \varphi$, regular on $\varphi^{-1}(\text{dom } \psi \cap \mathrm{Im}\varphi)$. We note that also the composed rational map $\psi \circ \varphi$ is dominant.

12.2 Birational maps

Definition 12.2.1. A *birational map* from X to Y is a rational map $\varphi : X \dashrightarrow Y$ such that φ is dominant and there exists $\psi : Y \dashrightarrow X$, a dominant rational map, such that $\psi \circ \varphi = 1_X$ and $\varphi \circ \psi = 1_Y$ **as rational maps**. In this case, X and Y are called *birationally equivalent* or simply *birational*.

If $\varphi : X \dashrightarrow Y$ is a dominant rational map, then we can define the comorphism $\varphi^* : K(Y) \to K(X)$ in the usual way: it is an injective K–homomorphism.

Proposition 12.2.2. *Let X, Y be quasi–projective varieties, and let $u : K(Y) \to K(X)$ be a K–homomorphism. Then there exists a rational map $\varphi : X \dashrightarrow Y$ such that $\varphi^* = u$.*

Proof. Y is covered by open affine varieties Y_α, $\alpha \in I$ (Section 10.5); note that for any index α, $K(Y) \simeq K(Y_\alpha)$ (Proposition 9.2.4) and $K(Y_\alpha) \simeq K(t_1, \ldots, t_n)$, where $t_1, \ldots, t_n$ can be interpreted as coordinate functions on Y_α. Choose such an open subset Y_α. Then $u(t_1), \ldots, u(t_n) \in K(X)$ and there exists $U \subset X$, non–empty open subset such that $u(t_1), \ldots, u(t_n)$ are all regular on U. So $u(K[t_1, \ldots, t_n]) \subset \mathcal{O}(U)$ and we can consider the regular map

$u^\sharp : U \to Y_\alpha \hookrightarrow Y$. The germ of $u^\sharp$ gives a rational map $X \dashrightarrow Y$. It is possible to check that this rational map does not depend on the choice of Y_α and U. □

Theorem 12.2.3. *Let X, Y be quasi–projective varieties. The following are equivalent:*

(i) X is birational to Y;

(ii) $K(X) \simeq K(Y)$;

(iii) there exist non–empty open subsets $U \subset X$ and $V \subset Y$ such that $U \simeq V$.

Proof. (i) ⇔ (ii) via the construction of the comorphism φ^* associated to φ and of $u^\sharp$, associated to $u : K(Y) \to K(X)$. One checks that both constructions are functorial.

(i) ⇒ (iii) Let $\varphi : X \dashrightarrow Y$, $\psi : Y \dashrightarrow X$ be rational maps inverse each other. Put $U' = \operatorname{dom} \varphi$ and $V' = \operatorname{dom} \psi$. By assumption, $\psi \circ \varphi$ is defined on $\varphi^{-1}(V')$ and coincides with 1_X there. Similarly, $\varphi \circ \psi$ is defined on $\psi^{-1}(U')$ and equal to 1_Y. Then φ and ψ establish an isomorphism between the corresponding sets $U := \varphi^{-1}(\psi^{-1}(U'))$ and $V := \psi^{-1}(\varphi^{-1}(V'))$.

(iii) ⇒ (ii) $U \simeq V$ implies $K(U) \simeq K(V)$; but $K(U) \simeq K(X)$ and $K(V) \simeq K(Y)$ (Prop. 1.9, Lesson 10), so $K(X) \simeq K(Y)$ by transitivity. □

Corollary 12.2.4. *If X is birational to Y, then* $\dim X = \dim Y$.

Corollary 12.2.5. *The projective space $\mathbb{P}^n$ and the affine space $\mathbb{A}^n$ are birationally equivalent.*

Theorem 12.2.3 can be given an interpretation in the language of categories. We can define a category $\mathcal{C}$ whose objects are the irreducible algebraic varieties over a fixed algebraically closed field K, and the morphisms are the dominant rational maps. The isomorphisms in $\mathcal{C}$ are birational maps, so two objects are isomorphic in $\mathcal{C}$ if they are birationally equivalent. We can consider also the category $\mathcal{C}'$ with objects the fields, finitely generated extensions of K, and morphisms the K-homomorphisms. Then there is a contravariant functor $\mathcal{C} \to \mathcal{C}'$ associating to a variety X its field of rational functions $K(X)$ and to a rational map $\varphi : X \dashrightarrow Y$ its comorphism φ^*. Proposition 12.2.2 and Theorem 12.2.3 say that this functor is an equivalence of categories.

There are two classification problems for algebraic varieties, up to isomorphism and up to birational equivalence. Both are central problems of Algebraic Geometry.

12.3 Examples

Example 12.3.1. a) *The cuspidal cubic $Y = V(x^3 - y^2) \subset \mathbb{A}^2$.*

We have seen in Example 10.1.2 that Y is not isomorphic to $\mathbb{A}^1$, but Y and $\mathbb{A}^1$ are birationally equivalent. Indeed, the regular map $\varphi : \mathbb{A}^1 \to Y$, $t \to (t^2, t^3)$, admits a rational inverse $\psi : Y \dashrightarrow \mathbb{A}^1$, $(x, y) \to \frac{y}{x}$. ψ is regular on $Y \setminus \{(0, 0)\}$, ψ is dominant and $\psi \circ \varphi = 1_{\mathbb{A}^1}$, $\varphi \circ \psi = 1_Y$ as rational maps. In particular, $\varphi^* : K(Y) \to K(X)$ is a field isomorphism. Recall that $K[Y] = K[t_1, t_2]$, with $t_1^2 = t_2^3$, so $K(Y) = K(t_1, t_2) = K(t_2/t_1)$, because $t_1 = (t_2/t_1)^2 = t_2^2/t_1^2 = t_1^3/t_1^2$ and $t_2 = (t_2/t_1)^3 = t_2^3/t_1^3 = t_2^3/t_2^2$, so $K(Y)$ is generated by a unique transcendental element. Notice that φ and ψ establish isomorphisms between $\mathbb{A}^1 \setminus \{0\}$ and $Y \setminus \{(0, 0)\}$.

b) *Any rational map from $\mathbb{P}^1$ to $\mathbb{P}^n$ is regular.*

Let $\varphi : \mathbb{P}^1 \dashrightarrow \mathbb{P}^n$ be a rational map: on some open $U \subset \mathbb{P}^1$,

$$\varphi([x_0, x_1]) = [F_0(x_0, x_1), \dots, F_n(x_0, x_1)],$$

with $F_0, \dots, F_n$ homogeneous of the same degree, without non–trivial common factors. Assume that $F_i(P) = 0$ for a certain index i, with $P = [a_0, a_1]$. Then $F_i \in I_h(P) = \langle a_1 x_0 - a_0 x_1 \rangle$, i.e. $a_1 x_0 - a_0 x_1$ is a factor of F_i. This remark implies that $\forall\, Q \in \mathbb{P}^1$ there exists $i \in \{0, \dots, n\}$ such that $F_i(Q) \neq 0$, because otherwise $F_0, \dots, F_n$ would have a common factor of degree 1. Hence we conclude that φ is regular.

We have obtained that any rational map from $\mathbb{P}^1$ is in fact regular.

c) *Projections.*

Let $\varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ be a rational map, that can be represented in matrix form by $Y = AX$, where A is a $(m+1) \times (n+1)$-matrix, with entries in K. Then φ is a rational map, regular on $\mathbb{P}^n \setminus \mathbb{P}(\mathrm{Ker} A)$. Put $\Lambda := \mathbb{P}(\mathrm{Ker} A)$. If $A = (a_{ij})$, this means that Λ has cartesian equations

$$\begin{cases} a_{00}x_0 + \ldots + a_{0n}x_n = 0 \\ a_{10}x_0 + \ldots + a_{1n}x_n = 0 \\ \ldots \\ a_{m0}x_0 + \ldots + a_{mn}x_n = 0. \end{cases}$$

The map φ has a geometric interpretation: it can be seen as the *projection of centre* Λ to a complementar linear space. To see how to give this interpretation, first of all we can assume that $\mathrm{rk}\ A = m + 1$, otherwise we replace $\mathbb{P}^m$ with $\mathbb{P}(\mathrm{Im}\ A)$; hence $\dim \Lambda = (n+1) - (m+1) - 1 = n - m - 1$.

Consider first the special case in which $\Lambda : x_0 = \cdots = x_m = 0$; we can identify $\mathbb{P}^m$ with the subspace of $\mathbb{P}^n$ of equations $x_{m+1} = \cdots = x_n = 0$, so Λ and $\mathbb{P}^m$ are complementar subspaces, i.e. $\Lambda \cap \mathbb{P}^m = \emptyset$ and the linear span of Λ and $\mathbb{P}^m$ is $\mathbb{P}^n$. Then, for $Q[a_0, \ldots, a_n] \in \mathbb{P}^n \setminus \Lambda$, $\varphi(Q) = [a_0, \ldots, a_m, 0, \ldots, 0]$: it is the intersection of $\mathbb{P}^m$ with the linear span $\overline{\Lambda Q}$ of Λ and Q. In fact, $\overline{\Lambda Q}$ has equations

$$\{a_i x_j - a_j x_i = 0, i, j = 0, \ldots, m \quad \text{(check!)}$$

so $\overline{\Lambda Q} \cap \mathbb{P}^m$ has coordinates $[a_0, \ldots, a_m, 0, \ldots, 0]$.

In the general case, we perform a change of coordinates: if $\Lambda = V_P(L_0, \ldots, L_m)$, with $L_0, \ldots, L_m$ linearly independent linear forms, we can identify $\mathbb{P}^m$ with $V_P(L_{m+1}, \ldots, L_n)$, where $L_{m+1}, \ldots, L_n$ are linearly independent linear forms chosen so that $L_0, \ldots, L_m, L_{m+1}, \ldots, L_n$ is a basis of $(K^{n+1})^*$. Then $L_0, \ldots, L_m$ can be interpreted as coordinate functions on $\mathbb{P}^m$.

If $m = n - 1$, then Λ is a point P and φ, often denoted by π_P, is the projection from P to a hyperplane not containing P. Also for the projection with centre Λ often the notation π_Λ is used.

d)*Rational and unirational varieties.*

A quasi–projective variety X is called *rational* if it is birational to a projective space $\mathbb{P}^n$, or equivalently to $\mathbb{A}^n$.

By Theorem 12.2.3, X is rational if and only if $K(X) \simeq K(\mathbb{P}^n) = K(x_1, \ldots, x_n)$ for some n, i.e. $K(X)$ is an extension of K generated by a transcendence basis; this kind of extension is called a *purely transcendental extension of* K. In an equivalent way, X is rational if there exists a rational map $\varphi : \mathbb{P}^n \dashrightarrow X$ which is dominant and is an isomorphism if restricted to a suitable open subset $U \subset \mathbb{P}^n$. Hence X admits a *birational parameterization* by polynomials in n parameters.

A weaker notion is that of *unirational* variety: X is unirational if there exists a dominant rational map $\mathbb{P}^n \dashrightarrow X$ i.e. if $K(X)$ is contained in the quotient field of a polynomial ring. Hence X can be parameterized by polynomials, but not necessarily generically one–to–one.

It is clear that, if X is rational, then it is unirational. The converse implication has been an important open problem, up to 1971, when it has been solved in the negative, for varieties of dimension ≥ 3 (Clemens–Griffiths, Iskovskih–Manin, Artin-Mumford). Nevertheless rationality and unirationality are equivalent for curves (Theorem of Lüroth, 1880, over any field) and for surfaces if $\operatorname{char} K = 0$ (Theorem of Castelnuovo, 1894).

e)*Rational parameterization of a smooth quadric surface.*

As an example of rational variety with an explicit rational parameterization constructed geometrically, let us consider the Segre quadric in $\mathbb{P}^3$, of maximal rank: $X = V_P(x_0x_3 - x_1x_2)$, it is an irreducible hypersurface of degree 2. Let $\pi_P : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ be the projection of centre

$P[1,0,0,0]$, such that $\pi_P([y_0,y_1,y_2,y_3]) = [y_1,y_2,y_3]$. The restriction of π_P to X is a rational map $\tilde{\pi}_P : X \dashrightarrow \mathbb{P}^2$, regular on $X \setminus \{P\}$. $\tilde{\pi}_P$ has a rational inverse: indeed consider the rational map $\psi : \mathbb{P}^2 \dashrightarrow X$, $[y_1,y_2,y_3] \to [y_1y_2, y_1y_3, y_2y_3, y_3^2]$. The equation of X is satisfied by the points of $\psi(\mathbb{P}^2)$: $(y_1y_2)y_3^2 = (y_1y_3)(y_2y_3)$. ψ is regular on $\mathbb{P}^2 \setminus V_P(y_1y_2, y_3)$. Let us compose ψ and $\tilde{\pi}_P$:

$$[y_0,\dots,y_3] \in X \xrightarrow{\pi_P} [y_1,y_2,y_3] \xrightarrow{\psi} [y_1y_2, y_1y_3, y_2y_3, y_3^2];$$

$y_1y_2 = y_0y_3$ implies $\psi \circ \pi_P = 1_X$. In the opposite order:

$$[y_1,y_2,y_3] \xrightarrow{\psi} [y_1y_2, y_1y_3, y_2y_3, y_3^2] \xrightarrow{\pi_P} [y_1y_3, y_2y_3, y_3^2] = [y_1,y_2,y_3].$$

So X is birational to $\mathbb{P}^2$ hence it is a rational surface.

Note that if we consider another projection $\pi_{P'}$ whose centre P' is not on the quadric, we get a regular $2:1$ map to the plane, that is certainly not birational.

f) *A birational non–regular map from $\mathbb{P}^2$ to $\mathbb{P}^2$.*

The following rational map is called the *standard quadratic transformation*:

$$Q : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \quad [x_0,x_1,x_2] \to [x_1x_2, x_0x_2, x_0x_1].$$

Q is regular on $U := \mathbb{P}^2 \setminus \{A,B,C\}$, where $A[1,0,0]$, $B[0,1,0]$, $C[0,0,1]$ are the fundamental points (see Figure 1).

Let a be the line through B and C: $a = V_P(x_0)$, and similarly $b = V_P(x_1)$, $c = V_P(x_2)$. Then $Q(a) = A$, $Q(b) = B$, $Q(c) = C$. Outside these three lines Q is an isomorphism. Precisely, put $U' = \mathbb{P}^2 \setminus \{a \cup b \cup c\}$; then $Q : U' \to \mathbb{P}^2$ is regular, the image is U' and $Q^{-1} : U' \to U'$ coincides with Q. Indeed,

$$[x_0,x_1,x_2] \xrightarrow{Q} [x_1x_2, x_0x_2, x_0x_1] \xrightarrow{Q} [x_0^2x_1x_2, x_0, x_1^2x_2, x_0x_1x_2^2].$$

So $Q \circ Q = 1_{\mathbb{P}^2}$ as rational map, hence Q is birational and $Q = Q^{-1}$.

Note that another way to express Q is the following: $[x_0,x_1,x_2] \to [\frac{1}{x_0}, \frac{1}{x_1}, \frac{1}{x_2}]$.

The set of the birational maps $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is a group, called the *Cremona group*. At the end of XIX century, Max Noether proved that the Cremona group is generated by $PGL(3,K)$ and by the single standard quadratic transformation Q defined above. The analogous groups for $\mathbb{P}^n$, $n \geq 3$, are much more complicated and a complete description is still unknown.

We conclude this chapter with a theorem illustrating an application of the linearisation procedure. We shall use the following notation: given a homogeneous polynomial $F \in K[x_0,x_1,\dots,x_n]$, $D(F) := \mathbb{P}^n \setminus V_P(F)$.

Theorem 12.3.2. *Let $W \subset \mathbb{P}^n$ be a closed projective variety. Let F be a homogeneous polynomial of degree d in $K[x_0, x_1, \ldots, x_n]$ such that $W \not\subseteq V_P(F)$. Then $W \cap D(F)$ is an affine variety.*

Proof. The assumption $W \not\subseteq V_P(F)$ is equivalent to $W \cap D(F) \neq \emptyset$. Let us consider the d-tuple Veronese embedding $v_{n,d} : \mathbb{P}^n \to \mathbb{P}^{N(n,d)}$, with $N(n,d) = \binom{n+d}{d} - 1$, that gives the isomorphism $\mathbb{P}^n \simeq V_{n,d}$. In this isomorphism the hypersurface $V_P(F)$ corresponds to a hyperplane section $V_{n,d} \cap H$, for a suitable hyperplane H in $\mathbb{P}^{N(n,d)}$. Therefore we have $W \cap D(F) \simeq v_{n,d}(W \cap D(F)) = v_{n,d}(W) \setminus H = v_{n,d}(W) \cap (\mathbb{P}^{N(n,d)} \setminus H)$. There exists a projective isomorphism $\tau : \mathbb{P}^{N(n,d)} \to \mathbb{P}^{N(n,d)}$ such that $\tau(H) = H_0$, the fundamental hyperplane of equation $x_0 = 0$. Therefore, denoting $X := v_{n,d}(W)$, we get $X \cap (\mathbb{P}^{N(n,d)} \setminus H) \simeq \tau(X) \cap (\mathbb{P}^{N(n,d)} \setminus H_0) = \tau(X) \cap U_0$, which proves the theorem. □

As a consequence of Theorem 12.3.2, we get that the open subsets of the form $W \cap D(F)$ form a topology basis for W formed by affine varieties.

Exercises 12.3.3. 1. Let $\varphi : \mathbb{A}^1 \to \mathbb{A}^n$ be the map defined by $t \to (t, t^2, \ldots, t^n)$.

a) Prove that $\varphi : \mathbb{A}^1 \to \varphi(\mathbb{A}^1)$ is an isomorphism and describe $\varphi(\mathbb{A}^1)$;

b) give a description of φ^* and φ^{-1*}.

2. Prove that the Veronese variety $V_{n,d}$ is not contained in any hyperplane of $\mathbb{P}^{N(n,d)}$.

3. Let $GL_n(K)$ be the set of invertible $n \times n$ matrices with entries in K. Prove that $GL_n(K)$ can be given the structure of an affine variety.

4. Let $\varphi : X \to Y$ be a regular map and φ^* its comorphism. Prove that the kernel of φ^* is the ideal of $\varphi(X)$ in $\mathcal{O}(Y)$. In the affine case, deduce that φ is dominant if and only if φ^* is injective.

5. Prove that $\mathcal{O}(X_F)$ is isomorphic to $\mathcal{O}(X)_f$, where X is an affine algebraic variety, F a polynomial and f the regular function on X defined by F.

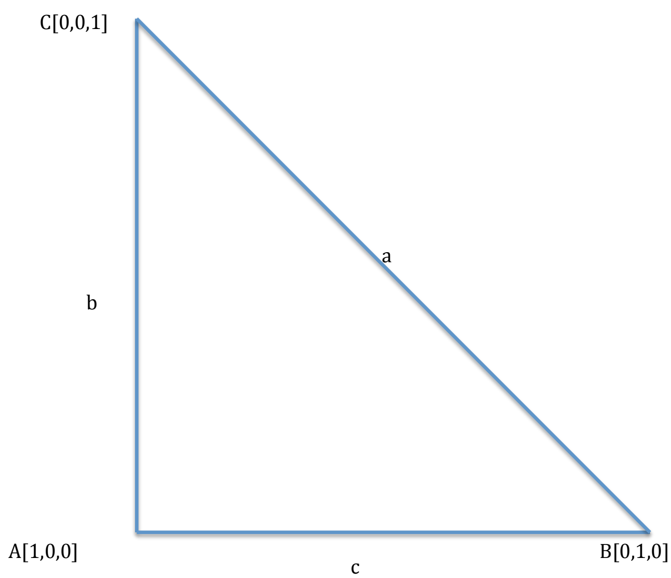


Figure 12.1:

Chapter 13

Product of quasi–projective varieties and tensors

13.1 Products

In Chapter 2, Section 2.5, we have seen how the product $\mathbb{P}^1 \times \mathbb{P}^1$ can be interpreted as a projective variety, and precisely a quadric of maximal rank, by means of the Segre map. Now we want to give a structure of algebraic variety to all products of algebraic varieties. We will seen that this can be done by generalizing the definition of the Segre map to any product of projective spaces $\mathbb{P}^n \times \mathbb{P}^m$.

Let $\mathbb{P}^n$, $\mathbb{P}^m$ be projective spaces over the same field K. The cartesian product $\mathbb{P}^n \times \mathbb{P}^m$ is simply a set: we want to define an injective map from $\mathbb{P}^n \times \mathbb{P}^m$ to a suitable projective space, so that the image is a projective variety, which will be identified with our product.

Let $N = (n+1)(m+1) - 1$ and define $\sigma : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ in the following way: $\sigma([x_0, \dots, x_n], [y_0, \dots, y_m]) = [x_0y_0, x_0y_1, \dots, x_iy_j, \dots, x_ny_m]$. Using coordinates w_{ij}, $i = 0, \dots, n$, $j = 0, \dots, m$, in $\mathbb{P}^N$, σ is defined by

$$\{w_{ij} = x_iy_j, \; i = 0, \dots, n, j = 0, \dots, m.$$

It is easy to observe that σ is a well–defined map.

Let $\Sigma_{n,m}$ (or simply Σ) denote the image $\sigma(\mathbb{P}^n \times \mathbb{P}^m)$.

Proposition 13.1.1. *σ is injective and $\Sigma_{n,m}$ is a closed subset of $\mathbb{P}^N$.*

Proof. If $\sigma([x], [y]) = \sigma([x'], [y'])$, then there exists $\lambda \neq 0$ such that $x'_iy'_j = \lambda x_iy_j$ for all i, j. In particular, if $x_h \neq 0$, $y_k \neq 0$, then also $x'_h \neq 0$, $y'_k \neq 0$, and for all i $x'_i = \lambda \frac{y_k}{y'_k} x_i$, so

$[x_0, \dots, x_n] = [x'_0, \dots, x'_n]$. Similarly for the second point.
To prove the second assertion, I claim: $\Sigma_{n,m}$ is the closed set of equations:

$$\{w_{ij}w_{hk} = w_{ik}w_{hj}, \ i, h = 0, \dots, n; j, k = 0 \dots, m. \tag{13.1}$$

It is clear that if $[w_{ij}] \in \Sigma$, then it satisfies (13.1).

Conversely, assume that $[w_{ij}]$ satisfies (13.1) and that $w_{\alpha\beta} \neq 0$. Then

$$\begin{aligned}[][w_{00}, \dots, w_{ij}, \dots, w_{nm}] &= [w_{00}w_{\alpha\beta}, \dots, w_{ij}w_{\alpha\beta}, \dots, w_{nm}w_{\alpha\beta}] = \\ &= [w_{0\beta}w_{\alpha 0}, \dots, w_{i\beta}w_{\alpha j}, \dots, w_{n\beta}w_{\alpha m}] = \\ &= \sigma([w_{0\beta}, \dots, w_{n\beta}], [w_{\alpha 0}, \dots, w_{\alpha m}]).\end{aligned}$$

□

σ is called the Segre map and $\Sigma_{n,m}$ the Segre variety or biprojective space. Note that Σ is covered by the affine open subsets $\Sigma^{ij} = \Sigma \cap W_{ij}$, where $W_{ij} = \mathbb{P}^N \setminus V_P(w_{ij})$. Moreover $\Sigma^{ij} = \sigma(U_i \times V_j)$, where $U_i \times V_j$ is naturally identified with $\mathbb{A}^{n+m}$.

Proposition 13.1.2. *$\sigma|_{U_i \times V_j} : U_i \times V_j = \mathbb{A}^{n+m} \to \Sigma^{ij}$ is an isomorphism of varieties.*

Proof. Assume by simplicity $i = j = 0$. Choose non–homogeneous coordinates on U_0: $u_i = x_i/x_0$ and on V_0: $v_j = y_j/y_0$. So $u_1, \dots u_n, v_1, \dots, v_m$ are coordinates on $U_0 \times V_0$. Take non–homogeneous coordinates also on W_{00}: $z_{ij} = w_{ij}/w_{00}$.

Using these coordinates we have:

$$\begin{aligned}\sigma|_{U_i \times V_j} :&(u_1, \dots u_n, v_1, \dots, v_m) \to (v_1, \dots, v_m, u_1, u_1v_1, \dots, u_1v_m, \dots, u_nv_m)\\ &\qquad || \\ &([1, u_1, \dots, u_n], [1, v_1, \dots, v_m])\end{aligned}$$

i.e. $\sigma(u_1, \dots, v_m) = (z_{01}, \dots, z_{nm})$, where

$$\begin{cases} z_{i0} = u_i, & \text{if } i = 1, \dots, n; \\ z_{0j} = v_j, & \text{if } j = 1, \dots, m; \\ z_{ij} = u_iv_j = z_{i0}z_{0j} & \text{otherwise.} \end{cases}$$

Hence $\sigma|_{U_0 \times V_0}$ is regular. Its inverse maps $(z_{01}, \dots, z_{nm})$ to $(z_{10}, \dots, z_{n0}, z_{01}, \dots, z_{0m})$, so it is also regular. □

Corollary 13.1.3. *$\mathbb{P}^n \times \mathbb{P}^m$ is irreducible and birational to $\mathbb{P}^{n+m}$.*

Proof. The first assertion follows from Exercise 5, Chapter 6, considering the covering of Σ by the open subsets Σ^{ij}. Indeed, $\Sigma^{ij} \cap \Sigma^{hk} = \sigma((U_i \times V_j) \cap (U_h \times V_k)) = \sigma((U_i \cap U_h) \times (V_j \cap V_k))$, and $U_i \cap U_h \neq \emptyset \neq V_j \cap V_k$.

For the second assertion, by Theorem 12.2.3, it is enough to note that $\Sigma_{n,m}$ and $\mathbb{P}^{n+m}$ contain isomorphic open subsets, i.e. Σ^{ij} and $\mathbb{A}^{n+m}$. □

From now on, we shall identify $\mathbb{P}^n \times \mathbb{P}^m$ with $\Sigma_{n,m}$. If $X \subset \mathbb{P}^n$, $Y \subset \mathbb{P}^m$ are any quasi–projective varieties, then $X \times Y$ will be automatically identified with $\sigma(X \times Y) \subset \Sigma$.

Proposition 13.1.4. *If X and Y are projective varieties (resp. quasi–projective varieties), then $X \times Y$ is projective (resp. quasi–projective).*

Proof.

$$\begin{aligned}\sigma(X \times Y) &= \bigcup_{i,j} (\sigma(X \times Y) \cap \Sigma^{ij}) = \\ &= \bigcup_{i,j} (\sigma(X \times Y) \cap (U_i \times V_j)) = \\ &= \bigcup_{i,j} (\sigma((X \cap U_i) \times (Y \cap V_j))).\end{aligned}$$

If X and Y are projective varieties, then $X \cap U_i$ is closed in U_i and $Y \cap V_j$ is closed in V_j, so their product is closed in $U_i \times V_j$; since $\sigma|_{U_i \times V_j}$ is an isomorphism, also $\sigma(X \times Y) \cap \Sigma^{ij}$ is closed in Σ^{ij}, so $\sigma(X \times Y)$ is closed in Σ, by Lemma 9.1.3.

If X, Y are quasi–projective, the proof is similar: $X \cap U_i$ is locally closed in U_i and $Y \cap V_j$ is locally closed in V_j, so $X \cap U_i = Z \setminus Z'$, $Y \cap V_j = W \setminus W'$, with Z, Z', W, W' closed. Therefore $(Z \setminus Z') \times (W \setminus W') = Z \times W \setminus ((Z' \times W) \cup (Z \times W'))$, which is locally closed.

As for the irreducibility, see Exercise 1. □

Example 13.1.5. $\mathbb{P}^1 \times \mathbb{P}^1$

The example of $\mathbb{P}^1 \times \mathbb{P}^1$, the Segre quadric, has already been studied in Section 2.5.

We recall that $\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ is given by the parametric equations $\{w_{ij} = x_i y_j,\ i = 0, 1,$ $j = 0, 1$. Σ has only one non–trivial equation: $w_{00}w_{11} - w_{01}w_{10}$, hence Σ is a quadric. The equation of Σ can be written as

$$\begin{vmatrix} w_{00} & w_{01} \\ w_{10} & w_{11} \end{vmatrix} = 0. \tag{13.2}$$

Σ contains two families of special closed subsets parametrised by $\mathbb{P}^1$, i.e.

$$\{\sigma(\{P\} \times \mathbb{P}^1)\}_{P \in \mathbb{P}^1} \quad \text{and} \quad \{\sigma(\mathbb{P}^1 \times \{Q\})\}_{Q \in \mathbb{P}^1}.$$

If $P = [a_0, a_1]$, then $\sigma(\{P\} \times \mathbb{P}^1)$ is given by the equations:

$$\begin{cases} w_{00} = a_0 y_0 \\ w_{01} = a_0 y_1 \\ w_{10} = a_1 y_0 \\ w_{11} = a_1 y_1 \end{cases}$$

hence it is a line. Cartesian equations of $\sigma(\{P\} \times \mathbb{P}^1)$ are:

$$\begin{cases} a_1 w_{00} - a_0 w_{10} = 0 \\ a_1 w_{01} - a_0 w_{11} = 0; \end{cases}$$

they express the proportionality of the rows of the matrix (13.2) with coefficients $[a_1, -a_0]$. Similarly, $\sigma(\mathbb{P}^1 \times \{Q\})$ is the line of equations

$$\begin{cases} a_1 w_{00} - a_0 w_{01} = 0 \\ a_1 w_{10} - a_0 w_{11} = 0. \end{cases}$$

Hence Σ contains two families of lines, called the rulings of Σ: two lines of the same ruling are clearly disjoint, while two lines of different rulings intersect at one point ($\sigma(P, Q)$). Conversely, through any point of Σ there pass two lines, one for each ruling.

Note that Σ is exactly the quadric surface of Section 12.3 e) and that the projection π_P of centre $P[1, 0, 0, 0]$ realizes an explicit birational map between $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$. The two lines contained in Σ passing through P have equations $w_{10} = w_{11} = 0$ and $w_{01} = w_{11} = 0$ respectively; they are contracted to the points $E_0[1, 0, 0]$, $E_1[0, 1, 0]$ of $\mathbb{P}^2$ respectively. Conversely, the line $x_2 = 0$ in $\mathbb{P}^2$ passing through E_0, E_1 is contracted to P by π_p^{-1}.

13.2 Tensors

The product of projective spaces has a coordinate-free description in terms of tensors. Precisely, let $\mathbb{P}^n = \mathbb{P}(V)$ and $\mathbb{P}^m = \mathbb{P}(W)$. The tensor product $V \otimes W$ of the vector spaces V, W is constructed as follows: let $K(V \times W)$ be the K-vector space with basis $V \times W$ obtained as the set of formal finite linear combinations of type $\Sigma_i a_i (v_i, w_i)$ with $a_i \in K$. Let U be the vector subspace generated by all elements of the form:

$(v, w) + (v', w) - (v + v', w)$,

$(v, w) + (v, w') - (v, w + w')$,

$(\lambda v, w) - \lambda(v, w)$,
$(v, \lambda w) - \lambda(v, w)$,
with $v, v' \in V$, $w, w' \in W$, $\lambda \in K$. The tensor product is by definition the quotient $V \otimes W := K(V \times W)/U$. The class of a pair (v, w) is denoted by $v \otimes w$, and called a decomposable tensor. So $V \otimes W$ is generated by the decomposable tensors; more precisely, a general element $\omega \in V \otimes W$ is of the form $\Sigma_{i=1}^k v_i \otimes w_i$, with $v_i \in V$, $w_i \in W$. The minimum k such that an expression of this type exists is called the tensor rank of ω.

There is a natural bilinear map $\otimes : V \times W \to V \otimes W$, such that $(v, w) \to v \otimes w$. It enjoys the following universal property: for any K-vector space Z with a bilinear map $f : V \times W \to Z$, there exists a unique linear map $\bar{f} : V \otimes W \to Z$ such that f factorizes in the form $f = \bar{f} \circ \otimes$.

If $\dim V = n+1$, $\dim W = m+1$, and bases $\mathcal{B} = (e_0, \dots, e_n), \mathcal{B}' = (e'_0, \dots, e'_m)$ are given, then $(e_0 \otimes e'_0, \dots, e_i \otimes e'_j, \dots e_n \otimes e'_m)$ is a basis of $V \otimes W$: therefore $\dim V \otimes W = (n+1)(m+1)$.

If $v = x_0 e_0 + \dots + x_n e_n$, $w = y_0 e'_0 + \dots + y_m e'_m$, then $v \otimes w = \sum_{i,j} x_i y_j e_i \otimes e'_j$. So, passing to the projective spaces, the map $\otimes$ defines precisely the Segre map

$$\sigma : \mathbb{P}(V) \times \mathbb{P}(W) \to \mathbb{P}(V \otimes W), \quad ([v], [w]) \to [v \otimes w].$$

Indeed in coordinates we have $([x_0, \dots, x_n], [y_0, \dots, y_m]) \to [w_{00}, \dots, w_{nm}]$, with $w_{ij} = x_i y_j$. The image of $\otimes$ is the set of decomposable tensors, or rank one tensors.

The tensor product $V \otimes W$ has the same dimension, and is therefore isomorphic to the vector space of $(n+1) \times (m+1)$ matrices. The coordinates w_{ij} can be interpreted as the entries of such a $(n+1) \times (m+1)$ matrix. The equations of the Segre variety $\Sigma_{n,m}$ are the 2×2 minors of the matrix, therefore $\Sigma_{n,m}$ can be interpreted as the set of matrices of rank one.

The construction of the tensor product can be iterated, to construct $V_1 \otimes V_2 \otimes \dots \otimes V_r$. The following properties can easily be proved:

1. $V_1 \otimes (V_2 \otimes V_3) \simeq (V_1 \otimes V_2) \otimes V_3$;
2. $V \otimes W \simeq W \otimes V$;
3. $V^* \otimes W \simeq Hom(V, W)$, where $f \otimes w \to (V \to W : v \to f(v)w)$.

Also the Veronese morphism has a coordinate free description, in terms of symmetric tensors. Given a vector space V, for any $d \geq 0$ the d-th symmetric power of V, $S^d V$ or $Sym^d V$, is constructed as follows. We consider the tensor product of d copies of V: $V \otimes \dots \otimes V = V^{\otimes d}$, and we consider its subvector space U generated by all tensors of the form $v_1 \otimes \dots v_d - v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(d)}$, where $v_1, \dots, v_d$ vary in V and σ varies in the symmetric group on d elements $\mathcal{S}_d$. Then by definition $S^dV := V^{\otimes d}/U$. The equivalence class $[v_1 \otimes \cdots \otimes v_d]$ is denoted as a product $v_1 \dots v_d$. The elements of S^dV are called symmetric tensors.

There is a natural multilinear and symmetric map $V \times \cdots \times V = V^d \to S^dV$, such that $(v_1, \dots, v_d) \to v_1 \dots v_d$, which enjoys the universal property. S^dV is generated by the products $v_1 \dots v_d$.

In characteristic 0, S^dV can also be interpreted as a subspace of $V^{\otimes d}$, by considering the following map, that is an isomorphism to the image:

$$S^dV \to V^{\otimes d}, \quad v_1 \dots v_d \to \Sigma_{\sigma \in \mathcal{S}_d} \frac{1}{d!} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}.$$

For instance, in S^2V the product v_1v_2 can be identified with $\frac{1}{2}(v_1 \otimes v_2 + v_2 \otimes v_1)$.

If $\mathcal{B} = (e_0, \dots, e_n)$ is a basis of V, then it is easy to check that a basis of S^dV is formed by the monomials of degree d in $e_0, \dots, e_n$; therefore $\dim S^dV = \binom{n+d}{d}$.

The symmetric algebra of V is $SV := \oplus_{d \geq 0} S^dV = K \oplus V \oplus S^2V \oplus \dots$. An inner product can be naturally defined to give it the structure of a K-algebra, which results to be isomorphic to the polynomial ring in $n+1$ variables, where $n+1 = \dim V$.

If $char K = 0$ the Veronese morphism can be interpreted in the following way (up to projectivity):

$$v_{n,d} : \mathbb{P}(V) \to \mathbb{P}(S^dV), \ [v] = [x_0e_0 + \dots x_ne_n] \to [v^d] = [(x_0e_0 + \cdots + x_ne_n)^d].$$

Moreover S^2V can be interpreted as the space of symmetric $(n+1) \times (n+1)$ matrices, and the Veronese variety $V_{n,2}$ as the subset of the symmetric matrices of rank one, because its equations express precisely the vanishing of the minors of order 2 (see Section 10.6).

Exercises 13.2.1. 1. Using Exercise 5 of Chapter 6, prove that, if $X \subset \mathbb{P}^n$, $Y \subset \mathbb{P}^m$ are irreducible projective varieties, then $X \times Y$ is irreducible.

2. Let L, M, N be the following lines in $\mathbb{P}^3$:

$$L : x_0 = x_1 = 0, M : x_2 = x_3 = 0, N : x_0 - x_2 = x_1 - x_3 = 0.$$

Let X be the union of lines meeting L, M and N: write equations for X and describe it: is it a projective variety? If yes, of what dimension and degree?

3. Let X, Y be quasi–projective varieties, identify $X \times Y$ with its image via the Segre map. Check that the two projection maps $X \times Y \xrightarrow{p_1} X$, $X \times Y \xrightarrow{p_2} Y$ are regular. (Hint: use the open covering of the Segre variety by the Σ^{ij}'s.)

Chapter 14

The dimension of an intersection

Our aim in this chapter is to investigate the dimension of the intersection of two algebraic varieties.

14.1 The theorem of the intersection

Theorem 14.1.1. *Let K be an algebraically closed field. Let $X, Y \subset \mathbb{P}^n$ be quasi–projective varieties. Assume that $X \cap Y \neq \emptyset$. Then if Z is any irreducible component of $X \cap Y$, then* $\dim Z \geq \dim X + \dim Y - n$.

To prove Theorem 14.1.1, the main ingredient will be the following theorem, known as "Krull's principal ideal theorem".

Theorem 14.1.2. *Let R be a noetherian ring, let $a \in R$ be a non-invertible element. Then, for any prime ideal $\mathcal{P} \subset R$, minimal over the ideal (a) generated by a, the height of $\mathcal{P}$ is at most 1, i.e. $ht\mathcal{P} \leq 1$. If moreover a is a non-zero divisor, then $ht\mathcal{P} = 1$.*

We postpone the proof of Theorem 14.1.2 to the end of this chapter and proceed to the proof of Theorem 14.1.1. It will be divided in three steps. Note first that, possibly passing to the closure, we can assume that X, Y are projective varieties. Then we can assume that $X \cap Y$ intersects $U_0 \simeq \mathbb{A}^n$, so, possibly after restricting X and Y to $\mathbb{A}^n$, we may work with irreducible closed subsets of the affine space. Put $r = \dim X$, $s = \dim Y$.

Step 1. Assume that $X = V(F)$ is an irreducible hypersurface, with F irreducible polynomial of $K[x_1, \dots, x_n]$. The irreducible components of $X \cap Y$ correspond, by the Nullstellensatz, to the minimal prime ideals containing $I(X \cap Y)$ in $K[x_1, \dots, x_n]$. We recall (Corollary 3.2.9) that $I(X \cap Y) = \sqrt{I(X) + I(Y)} = \sqrt{\langle I(Y), F \rangle}$. So those prime ideals are the minimal prime ideals over $\langle I(Y), F\rangle$. They correspond bijectively to the minimal prime ideals containing $\langle f\rangle$ in $\mathcal{O}(Y)$, where f is the regular function on Y defined by F. We distinguish two cases:

(i) if $Y \subset X = V(F)$, then $f = 0$ and $Y \cap X = Y$; since $s = \dim Y > r + s - n = (n-1) + s - n$, the theorem is easily true in this case;

(ii) if $Y \not\subset X$, then $f \neq 0$, moreover f is not invertible, otherwise $X \cap Y = \emptyset$: hence the minimal prime ideals over $\langle f\rangle$ in $\mathcal{O}(Y)$, which is an integral domain, have all height one by Theorem 14.1.2. So for all Z, irreducible component of $X \cap Y$, $\dim Z = \dim Y - 1 = r + s - n$ (Theorem 7.2.4).

Step 2. Assume that $I(X)$ is generated by $n - r$ polynomials (where $n - r$ is the codimension of X): $I(X) = \langle F_1, \dots, F_{n-r}\rangle$. Then we can argue by induction on $n - r$: we first intersect Y with $V(F_1)$, whose irreducible components are all hypersurfaces, and apply Step 1: all irreducible components of $Y \cap V(F_1)$ have dimension either s or $s - 1$. Then we intersect each of these components with $V(F_2)$, and so on. We conclude that every irreducible component Z has $\dim Z \geq \dim Y - (n - r) = r + s - n$.

Step 3. We use the isomorphism $\psi : X \cap Y \simeq (X \times Y) \cap \Delta_{\mathbb{A}^n}$ (see Exercise 1, Chapter 10). Note that $X \times Y$ is irreducible by Proposition 6.4.2. ψ preserves the irreducible components and their dimensions, so we consider instead of X and Y, the algebraic sets $X \times Y$ and $\Delta_{\mathbb{A}^n}$, contained in $\mathbb{A}^{2n}$. We have $\dim X \times Y = r + s$ (Proposition 7.2.7). $\Delta_{\mathbb{A}^n}$ is a linear subspace of $\mathbb{A}^{2n}$, so it satisfies the assumption of Step 2; indeed it has dimension n in $\mathbb{A}^{2n}$ and is defined by n linear equations. Hence, for all Z we have: $\dim Z \geq (r + s) + n - 2n = r + s - n$. □

The above theorem can be seen as a generalization of the Grassmann relation for linear subspaces. However, it is not an existence theorem, because it says nothing about $X \cap Y$ being non–empty. But for projective varieties, the following more precise version of the theorem holds:

Theorem 14.1.3. *Let $X, Y \subset \mathbb{P}^n$ be projective varieties of dimensions r, s. If $r + s - n \geq 0$, then $X \cap Y \neq \emptyset$.*

Proof. Let $C(X)$, $C(Y)$ be the affine cones associated to X and Y. Then $C(X) \cap C(Y)$ is certainly non–empty, because it contains the origin $O(0, 0, \dots, 0)$. Assume we know that $C(X)$ has dimension $r + 1$ and $C(Y)$ has dimension $s + 1$: then by Theorem 14.1.1 all the irreducible components Z of $C(X) \cap C(Y)$ have dimension $\geq (r + 1) + (s + 1) - (n + 1) = r + s - n + 1 \geq 1$, hence Z contains points different from O. These points give rise to points of $\mathbb{P}^n$ belonging to $X \cap Y$. The conclusion of the proof will follow from next proposition. □

Proposition 14.1.4. *Let $Y \subset \mathbb{P}^n$ be a projective variety.*

Then $\dim Y = \dim C(Y) - 1$*. If $S(Y)$ denotes the homogeneous coordinate ring, hence also* $\dim Y = \dim S(Y) - 1$.

Proof. Let $p : \mathbb{A}^{n+1} \setminus \{O\} \to \mathbb{P}^n$ be the canonical morphism. Let us recall that $C(Y) = p^{-1}(Y) \cup \{O\}$. Assume that $Y_0 := Y \cap U_0 \neq \emptyset$ and consider also $C(Y_0) = p^{-1}(Y_0) \cup \{O\}$. Then we have:

$$C(Y_0) = \{(\lambda, \lambda a_1, \dots, \lambda a_n) \mid \lambda \in K, (a_1, \dots, a_n) \in Y_0\}.$$

So we can define a birational map between $C(Y_0)$ and $Y_0 \times \mathbb{A}^1$ as follows:

$$(y_0, y_1, \dots, y_n) \in C(Y_0) \to ((y_1/y_0, \dots, y_n/y_0), y_0) \in Y_0 \times \mathbb{A}^1,$$

$$((a_1, \dots, a_n), \lambda) \in Y_0 \times \mathbb{A}^1 \to (\lambda, \lambda a_1, \dots, \lambda a_n) \in C(Y_0).$$

Therefore $\dim C(Y_0) = \dim(Y_0 \times \mathbb{A}^1) = \dim Y_0 + 1$. To conclude, it is enough to remark that $\dim Y = \dim Y_0$ and $\dim C(Y) = \dim C(Y_0) = \dim S(Y)$. □

We observe that also $C(Y)$ and $Y \times \mathbb{P}^1$ are birationally equivalent.

Corollary 14.1.5. *1. If $X, Y \subset \mathbb{P}^2$ are projective curves over an algebraically closed field, then $X \cap Y \neq \emptyset$.*
2. $\mathbb{P}^1 \times \mathbb{P}^1$ is not isomorphic to $\mathbb{P}^2$.

Proof. 1. is a straightforward application of Theorem 14.1.3.

To prove 2., assume by contradiction that $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ is an isomorphism. Let L, L' be two skew lines in $\mathbb{P}^1 \times \mathbb{P}^1$; since φ is an isomorphism, then $\varphi(L)$, $\varphi(L')$ are rational disjoint curves in $\mathbb{P}^2$, but this contradicts 1. □

If $X, Y \subset \mathbb{P}^n$ are varieties of dimensions r, s, then $r+s-n$ is called the *expected dimension* of $X \cap Y$. If all irreducible components Z of $X \cap Y$ have the expected dimension, then we say that the intersection $X \cap Y$ is *proper* or that *X and Y intersect properly.*

For example, two plane projective curves X, Y intersect properly if they don't have any common irreducible component. In this case, it is possible to predict the number of points of intersections. Precisely, it is possible to associate to every point $P \in X \cap Y$ a number $i(P; X, Y)$, called the *multiplicity of intersection of X and Y at P*, in such a way that

$$\sum_{P \in X \cap Y} i(P; X, Y) = dd',$$

where d is the degree of X and d' is the degree of Y. This result is the Theorem of Bézout, and is the first result of the branch of algebraic geometry called Intersection Theory. For a proof of the Theorem of Bézout, see for instance the classical [W], or [F].

14.2 Complete intersections

Let X be a closed subvariety of $\mathbb{P}^n$ (resp. of $\mathbb{A}^n$) of codimension r. X is called a *complete intersection* if $I_h(X)$ (resp. $I(X)$) is generated by r polynomials, the minimum possible number.

Hence, if X is a complete intersection of codimension r, then X is certainly the intersection of r hypersurfaces. Conversely, if X is intersection of r hypersurfaces, then, by Theorem 14.1.1, using induction, we deduce that $\dim X \geq n - r$; even assuming equality, we cannot conclude that X is a complete intersection, but simply that $I(X)$ is the radical of an ideal generated by r polynomials.

Example 14.2.1. *The skew cubic (again).*

Let $X \subset \mathbb{P}^3$ be the skew cubic. The homogeneous ideal of X is generated by the three polynomials F_1, F_2, F_3, the 2×2–minors of the matrix

$$M = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix},$$

which are linearly independent polynomials of degree 2. Note that $I_h(X)$ does not contain any linear polynomial, because X is not contained in any hyperplane, and that the homogeneous component of minimal degree 2 of $I_h(X)$ is a vector space of dimension 3. Hence $I_h(X)$ cannot be generated by two polynomials, i.e. X is not a complete intersection.

Nevertheless, X is the intersection of the surfaces $V_P(F)$, $V_P(G)$, where

$$F = F_1 = \begin{vmatrix} x_0 & x_1 \\ x_1 & x_2 \end{vmatrix} \text{ and } G = \begin{vmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_0 \end{vmatrix}.$$

Indeed, clearly $F, G \in I_h(X)$ so $X \subset V_P(F) \cap V_P(G)$. Conversely, observe that

$$G = x_0F - x_3(x_0x_3 - x_1x_2) + x_2(x_1x_3 - x_2^2) = x_0F_1 - x_3F_2 + x_2F_3.$$

If $P[x_0, \ldots, x_3] \in V_P(F) \cap V_P(G)$, then P is a zero also of $G - x_0F = x_0x_3^2 - 2x_1x_2x_3 + x_2^3$, and therefore also of

$$x_2(x_0x_3^2 - 2x_1x_2x_3 + x_2^3) = x_1^2x_3^2 - 2x_1x_2^2x_3 + x_2^4 = (x_1x_3 - x_2^2)^2 = F_3^2,$$

because $x_0x_2 = x_1^2$. Hence P is a zero also of $F_3 = x_1x_3 - x_2^2$. So P annihilates $G - x_0F - x_2F_3 = x_3(x_0x_3 - x_1x_2) = x_3F_2$. If P satisfies the equation $x_3 = 0$, then it satisfies also $x_2 = 0$ and $x_1 = 0$, therefore $P = [1, 0, 0, 0] \in X$. If $x_3 \neq 0$, then $P \in V_P(F_1, F_2, F_3) = X$.

The geometric description of this phenomenon is that the skew cubic X is the set-theoretic intersection of a quadric and a cubic, which are tangent along X, so their intersection is X "counted with multiplicity 2".

This example motivates the following definition: X is a *set–theoretic complete intersection* if codim$X = r$ and the ideal of X is the radical of an ideal generated by r polynomials. It is an open problem if all irreducible curves of $\mathbb{P}^3$ are set–theoretic complete intersections. For more details, see [K].

14.3 Krull's principal ideal theorem

We conclude this chapter with the proof of Krull's principal ideal Theorem 14.1.2.

Proof. Let $\mathcal{P}$ be a prime ideal, minimal among those containing (a), let $R_{\mathcal{P}}$ be the localization. Then $ht\mathcal{P} = \dim R_{\mathcal{P}}$, because of the bijection between prime ideals of $R_{\mathcal{P}}$ and prime ideals of R contained in $\mathcal{P}$. Moreover $\mathcal{P}R_{\mathcal{P}}$ is a minimal prime ideal over $aR_{\mathcal{P}}$, the ideal generated by a in $R_{\mathcal{P}}$. So, we can replace the ring R with its localization $R_{\mathcal{P}}$, or, in other words, we can assume that R is a local ring and that its maximal ideal $\mathcal{M}$ is minimal over (a).

It is enough to prove that, for any prime ideal $\mathcal{Q}$ of R, with $\mathcal{Q} \neq \mathcal{M}$, we have $ht\mathcal{Q} = 0$. Indeed this will imply $ht\mathcal{M} \leq 1$. Let $j : R \to R_{\mathcal{Q}}$ be the natural homomorphism. For any integer i, $i \geq 1$, we consider $\mathcal{Q}^i$, and its saturation with $\mathcal{Q}$: $\mathcal{Q}^{(i)} := j^{-1}(\mathcal{Q}^i R_{\mathcal{Q}})$, called the i-th symbolic power of $\mathcal{Q}$. It is $\mathcal{Q}$-primary. We have $\mathcal{Q}^i \subset \mathcal{Q}^{(i)}$ and

$$\mathcal{Q} = \mathcal{Q}^{(1)} \supseteq \mathcal{Q}^{(2)} \supseteq \cdots \supseteq \mathcal{Q}^{(i)} \supseteq \ldots.$$

We also have

$$(a) + \mathcal{Q} \supseteq (a) + \mathcal{Q}^{(2)} \supseteq \cdots \supseteq (a) + \mathcal{Q}^{(i)} \supseteq \ldots. \tag{14.1}$$

We observe that in $R/(a)$ there is only one prime ideal, $\mathcal{M}/(a)$, because R is local and $\mathcal{M}$ is minimal over (a), therefore $R/(a)$ has dimension 0; since it is noetherian of dimension 0, $R/(a)$ is artinian, and we can conclude that the chain of ideals (14.1) is stationary, so there exists an integer n such that $(a) + \mathcal{Q}^{(n)} = (a) + \mathcal{Q}^{(n+1)}$.

Let $q \in \mathcal{Q}^{(n)}$: so $q \in (a) + \mathcal{Q}^{(n+1)}$, and it can be written in the form $q = ra + q'$, where $r \in R$, $q' \in \mathcal{Q}^{(n+1)} \subset \mathcal{Q}^{(n)}$. Therefore $ra = q - q' \in \mathcal{Q}^{(n)}$; but $a \notin \mathcal{Q}$ (because $\mathcal{M}$ is minimal over (a)), and $\mathcal{Q}^{(n)}$ is $\mathcal{Q}$-primary, so $r \in \mathcal{Q}^{(n)}$. We conclude that $\mathcal{Q}^{(n)} = a\mathcal{Q}^{(n)} + \mathcal{Q}^{(n+1)}$.

We can apply now Nakayama's lemma (Theorem 14.3.1 below), and get $\mathcal{Q}^{(n)} = \mathcal{Q}^{(n+1)}$. Therefore $\mathcal{Q}^n R_{\mathcal{Q}} = \mathcal{Q}^{n+1} R_{\mathcal{Q}}$. We apply Nakayama's lemma again, and we conclude that

$\mathcal{Q}^n R_{\mathcal{Q}} = (0)$. So every element of the maximal ideal $\mathcal{Q}R_{\mathcal{Q}}$ of $R_{\mathcal{Q}}$ is nilpotent, which implies that $ht\mathcal{Q}R_{\mathcal{Q}} = 0$.

We recall here the statement of Nakayama's lemma.

Theorem 14.3.1. *Let $I \subset R$ be an ideal contained in the Jacobson radical of R (the intersection of the maximal ideals). Let M be a finitely generated R-module, let $N \subset M$ be a submodule.*

If $M = N + IM$, then $M = N$.

We have applied Nakayama's lemma the first time in the situation where R is a local ring and $I = (a) \subset \mathcal{M}$, which is the Jacobson radical of R. The R-module M is $\mathcal{Q}^{(n)}$ and its submodule N is $\mathcal{Q}^{(n+1)}$. The second time, we are instead in the situation where the ring is $R_{\mathcal{Q}}$, $I = \mathcal{Q}R_{\mathcal{Q}}$, the module M is $\mathcal{Q}^n R_{\mathcal{Q}}$ and N is (0).

To conclude the proof of the theorem, we observe that the second assertion follows from the first one, because if $\mathcal{P}$ is a prime ideal of height zero, all its elements are zero-divisors. Indeed, let $r \in \mathcal{P}$, $r \neq 0$; we can find an element $t \notin \mathcal{P}$ belonging to the intersection $\cap_i \mathcal{P}_i$ of the prime ideals of height zero different from $\mathcal{P}$ (there is a finite number of such ideals because R is noetherian). Otherwise $\mathcal{P} \subset \cap_i \mathcal{P}_i$, but this would imply $\mathcal{P} \subset \mathcal{P}_i$ for some i. Now observe that rt belongs to the intersection of all minimal prime ideals of R, so rt is nilpotent: there exists $\alpha \geq 0$ such that $(rt)^\alpha = 0$. Since $t \notin \mathcal{P}$, it is not nilpotent, so $t^\alpha \neq 0$. Hence there is a minimum $\beta \geq 0$ such that $r^\beta t^\alpha \neq 0$ but $r^{\beta+1} t^\alpha = r(r^\beta t^\alpha) = 0$. This proves that r is a zero-divisor. □

Exercises 14.3.2. 1. Let $X \subset \mathbb{P}^2$ be the union of three points not lying on a line. Prove that the homogeneous ideal of X cannot be generated by two polynomials.

Chapter 15

Complete varieties

We work over an algebraically closed field K.

In this chapter, we will prove that the algebra of regular functions $\mathcal{O}(X)$ of an irreducible projective variety X is the base field K, i.e. that the only regular functions on X are the constants. We will obtain this theorem as a consequence of the theorem of completeness of projective varieties. The property of a variety to be complete can be seen as an analogue of compactness in the context of algebraic geometry.

15.1 Complete varieties

Definition 15.1.1. Let X be a quasi–projective variety. X is *complete* if, for any quasi–projective variety Y, the natural projection on the second factor $p_2 : X \times Y \to Y$ is a closed map.

Note that both projections p_1, p_2 are morphisms: see Exercise 3, Chapter 14.

We recall that a topological space X is compact if and only if the above projection map is closed with respect to the product topology. Here the product variety $X \times Y$ does not carry the product topology but the Zariski topology, that is in general strictly finer (Proposition 2.4.1).

Example 15.1.2. *The affine line $\mathbb{A}^1$ is not complete: let $X = Y = \mathbb{A}^1$, $p_2 : \mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2 \to \mathbb{A}^1$ is the map such that $(x_1, x_2) \to x_2$. Then $Z := V(x_1x_2 - 1)$ is closed in $\mathbb{A}^2$ but $p_2(Z) = \mathbb{A}^1 \setminus \{O\}$ is not closed.*

Proposition 15.1.3. *(i) If $f : X \to Y$ is a regular map and X is complete, then $f(X)$ is a closed complete subvariety of Y.*

(ii) If X is complete, then all closed subvarieties of X are complete.

Proof. (i) Let $\Gamma_f \subset X \times Y$ be the graph of f: $\Gamma_f = \{(x, f(x)) \mid x \in X\}$. It is clear that $f(X) = p_2(\Gamma_f)$, so to prove that $f(X)$ is closed it is enough to check that Γ_f is closed in $X \times Y$. Let us consider the diagonal of Y: $\Delta_Y = \{(y, y) \mid y \in Y\} \subset Y \times Y$. If $Y \subset \mathbb{P}^n$, then $\Delta_Y = \Delta_{\mathbb{P}^n} \cap (Y \times Y)$, so it is closed in $Y \times Y$, because $\Delta_{\mathbb{P}^n}$ is the closed subset defined in $\Sigma_{n,n}$ by the equations $w_{ij} - w_{ji} = 0$, $i, j = 0, \dots, n$. There is a natural map $f \times 1_Y : X \times Y \to Y \times Y$, $(x, y) \to (f(x), y)$, such that $(f \times 1_Y)^{-1}(\Delta_Y) = \Gamma_f$. It is easy to see that $f \times 1_Y$ is regular, so Γ_f is closed, so also $f(X)$ is closed.

Let now Z be any variety and consider $p_2 : f(X) \times Z \to Z$ and the regular map $f \times 1_Z : X \times Z \to f(X) \times Z$. There is a commutative diagram:

$$\begin{array}{ccc} X \times Z & \xrightarrow{p_2'} & Z \\ \downarrow{\scriptstyle f \times 1_Z} & \nearrow & {\scriptstyle p_2} \\ f(X) \times Z & & \end{array}$$

If $T \subset f(X) \times Z$, then $(f \times 1_Z)^{-1}(T)$ is closed and $p_2(T) = p_2'((f \times 1_Z)^{-1}(T))$ is closed because X is complete. We conclude that $f(X)$ is complete.

(ii) Let $T \subset X$ be a closed subvariety and Y be any variety. We have to prove that $p_2 : T \times Y \to Y$ is closed. If $Z \subset T \times Y$ is closed, then Z is closed also in $X \times Y$, hence $p_2(Z)$ is closed because X is complete. □

Corollary 15.1.4. *1. If X is a complete variety, then $\mathcal{O}(X) \simeq K$.*

2. If X is an affine complete irreducible variety, then X is a point.

Proof. 1. If $f \in \mathcal{O}(X)$, f can be interpreted as a regular map $f : X \to \mathbb{A}^1$. By Proposition 15.1.3, (i), $f(X)$ is a closed complete subvariety of $\mathbb{A}^1$, which is not complete. Hence $f(X)$ has dimension < 1 and is irreducible, hence it is a point, so $f \in K$.

2. By part 1., $\mathcal{O}(X) \simeq K$. But $\mathcal{O}(X) \simeq K[x_1, \dots, x_n]/I(X)$, hence $I(X)$ is maximal. By the Nullstellensatz, X is a point. □

15.2 Completeness of projective varieties

Before stating Theorem 15.2.2 of completeness of projective varieties, we give a characterization of the closed subsets of a biprojective space $\mathbb{P}^n \times \mathbb{P}^m$, that will be needed in its proof. It is expressed in terms of equations in two series of variables, corresponding to the homogeneous coordinates $[x_0, \dots, x_n]$ on $\mathbb{P}^n$ and $[y_0, \dots, y_m]$ on $\mathbb{P}^m$.

Let $\sigma : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ be the Segre map. A closed subvariety X in $\mathbb{P}^N$ is defined by finitely many equations $F_k(w_{00}, \dots, w_{nm})$, where the F_k are homogeneous polynomials in the w_{ij}. On the subvariety $X \cap \Sigma$, where Σ is the Segre variety, we have $w_{ij} = x_i y_j$, so we can make this substitution in F_k and get equations $G_k(x_0, \dots, x_n; y_0, \dots, y_m) = 0$, where $G_k = F_k(x_0y_0, \dots, x_ny_m)$: they are equations characterizing the subset $\sigma^{-1}(X)$. Note that each G_k is homogeneous in each set of variables x_i and y_j, and of the same degree in both.

Conversely, it is easy to see that a polynomial with this property of bihomogeneity can always be written as a polynomial in the products x_iy_j, and the possible ambiguity depending on the choice disappears in view of the equations of the Segre variety. So it describes a subset of $\mathbb{P}^n \times \mathbb{P}^m$ whose image in σ is closed. However, equations that are bihomogeneous in x_i and y_j always define an algebraic closed subvariety of $\mathbb{P}^n \times \mathbb{P}^m$ even if the degrees of homogeneity in the two sets of variables are different. Indeed if $G(x_0, \dots, x_n; y_0, \dots, y_m)$ has degree r in x_i and s in y_j, and for instance $r > s$, then the equation $G = 0$ is equivalent to the system of equations $y_i^{r-s}G = 0$, $i = 0, \dots, m$, and these define an algebraic variety.

We will need the answer to the analogous question also for the product $\mathbb{P}^n \times \mathbb{A}^m$. Let us assume that $\mathbb{A}^m = U_0 \subset \mathbb{P}^m$, defined by $y_0 \neq 0$. If we have a closed subset of $\mathbb{P}^n \times \mathbb{P}^m$ defined by equations $G_k(x_0, \dots, x_n; y_0, \dots, y_m) = 0$, with G_k homogeneous of degree r_k in y_j, dividing by $y_0^{r_k}$ and setting $v_j = y_j/y_0$, we get equations $g_k(x_0, \dots, x_n; v_1, \dots, v_m) = 0$ that are homogeneous in the x_i and in general non-homogeneous in the v_j.

These observations can be collected in the following result.

Theorem 15.2.1. *A subset $X \subset \mathbb{P}^n \times \mathbb{P}^m$ is a closed algebraic subvariety if and only if it is defined by a system of equations $G_k(x_0, \dots, x_n; y_0 \dots, y_m) = 0$, homogeneous separately in each set of variables. Every closed algebraic subvariety of $\mathbb{P}^n \times \mathbb{A}^m$ is defined by a system of equations $g_k(x_0, \dots, x_n; v_1, \dots, v_m) = 0$ that are homogeneous in $x_0, \dots, x_n$.*

Theorem 15.2.2. *Let $X \subset \mathbb{P}^n$ be a projective irreducible variety over an algebraically closed field K. Then X is complete.*

Proof. (see [S], Theorem 3, Ch.1, §5)

1. It is enough to prove that $p_2 : \mathbb{P}^n \times \mathbb{A}^m \to \mathbb{A}^m$ is closed, for any positive n, m. This can be observed by using the local character of closedness and the existence of an affine open covering of any quasi–projective varieties.

Indeed, let us assume first that $p_2 : \mathbb{P}^n \times Y \to Y$ is a closed map for any quasi-projective variety Y. We observe that $X \times Y$ is closed in $\mathbb{P}^n \times Y$, because X is closed in $\mathbb{P}^n$. So, if $Z \subset X \times Y$ is closed, it is also closed in $\mathbb{P}^n \times Y$, which implies that $p_2(Z)$ is closed in Y. So we can replace X with $\mathbb{P}^n$.

Secondly, since being closed is a local property, it is enough to cover Y by affine open subsets U_i, and prove the theorem for each of them. Hence we can assume that Y is an affine variety. Finally, if $Y \subset \mathbb{A}^m$ is closed, then $\mathbb{P}^n \times Y$ is closed in $\mathbb{P}^n \times \mathbb{A}^m$, so it is enough to prove the theorem in the particular case $X = \mathbb{P}^n$ and $Y = \mathbb{A}^m$.

2. If $x_0, \dots, x_n$ are homogeneous coordinates on $\mathbb{P}^n$ and $y_1, \dots, y_m$ are non-homogeneous coordinates on $\mathbb{A}^m$, then any closed subvariety of $\mathbb{P}^n \times \mathbb{A}^m$ can be characterised as the set of common zeros of a set of polynomials in the variables $x_0, \dots, x_n, y_1, \dots, y_m$, homogeneous in the first group of variables $x_0, \dots, x_n$ (Theorem 15.2.1).

3. Let $Z \subset \mathbb{P}^n \times \mathbb{A}^m$ be closed. Then Z is the set of solutions of a system of equations

$$\{G_i(x_0, \dots, x_n; y_1, \dots, y_m) = 0, i = 1, \dots, t,$$

where G_i is homogeneous in the x's. A point $P(\overline{y}_1, \dots, \overline{y}_m)$ is in $p_2(Z)$ if and only if the system

$$\{G_i(x_0, \dots, x_n; \overline{y}_1, \dots, \overline{y}_m) = 0, i = 1, \dots, t,$$

has a solution in $\mathbb{P}^n$, i.e. if the ideal of $K[x_0, \dots, x_n]$ generated by $G_1(x; \overline{y})$,..., $G_t(x; \overline{y})$ has at least one zero in $\mathbb{P}^n$. Hence

$$\begin{aligned} p_2(Z) &= \{(\overline{y}_1, \dots, \overline{y}_m) \in \mathbb{A}^m | \ \forall\ d \geq 1\ \langle G_1(x; \overline{y}), \dots, G_t(x; \overline{y})\rangle \not\supset K[x_0, \dots, x_n]_d\} \\ &= \bigcap_{d\geq 1} \{(\overline{y}_1, \dots, \overline{y}_m) | \ \langle G_1(x; \overline{y}), \dots, G_t(x; \overline{y})\rangle \not\supset K[x_0, \dots, x_n]_d\} = \cap_{d \geq 1} T_d, \quad (15.1) \end{aligned}$$

where $T_d = \{(\overline{y}_1, \dots, \overline{y}_m) | \ \langle G_1(x; \overline{y}), \dots, G_t(x; \overline{y})\rangle \not\supset K[x_0, \dots, x_n]_d\}$. To conclude the proof of the theorem it is enough to prove that T_d is closed in $\mathbb{A}^m$ for any $d \geq 1$.

Let $\{M_\alpha\}_{\alpha = 1, \dots, \binom{n+d}{d}}$ be the set of the monomials of degree d in $K[x_0, \dots, x_n]$; let $d_i =$ deg $G_i(x; \overline{y})$, let $\{N_i^\beta\}_\beta$ be the set of the monomials of degree $d - d_i$.

Note that $P(\overline{y}_1, \dots, \overline{y}_m) \notin T_d$ if and only if $M_\alpha = \sum_i G_i(x; \overline{y}) F_{i,\alpha}(x_0, \dots, x_n)$, for all α and for suitable polynomials $F_{i,\alpha}$ homogeneous of degree $d - d_i$. So $P \notin T_d$ if and only if, for all index α, M_α is a linear combination of the polynomials $\{G_i(x; \overline{y}) N_i^\beta\}$, i.e. the matrix A of the coordinates of the polynomials $G_i(x; \overline{y}) N_i^\beta$ with respect to the basis $\{M_\alpha\}$ has maximal rank $\binom{n+d}{d}$. So T_d is the set of zeros of the minors of a fixed order of the matrix A, hence it is closed. □

Corollary 15.2.3. *Let X be a projective variety. Then $\mathcal{O}(X) \simeq K$.*

Corollary 15.2.4. *Let X be a projective variety, let $\varphi : X \to Y \subset \mathbb{P}^n$ be any regular map. Then $\varphi(X)$ is a projective variety. In particular, if $X \simeq Y$, then Y is projective.*

Corollary 15.2.4 says that the notion of projective variety, differently from that of affine variety, is invariant by isomorphism, i.e. quasi-projective varieties that are isomorphic to projective varieties are already projective.

In algebraic terms, Theorem 15.2.2 can be seen as a result in Elimination Theory. Indeed it can be reformulated by saying that, given a system of algebraic equations in two sets of variables, $x_0, \dots, x_n$ and $y_1, \dots, y_m$, homogeneous in the first ones, it is possible to find another system of algebraic equations only in $y_1, \dots, y_m$, such that $\bar{y}_1, \dots, \bar{y}_m$ is a solution of the second system if and only if there exist $\bar{x}_0, \dots, \bar{x}_n$, that, together with $\bar{y}_1, \dots, \bar{y}_m$, are a solution of the first system. In other words, it is possible to eliminate a set of homogeneous variables from any system of algebraic equations.

Example 15.2.5. Let $S = K[x_0, \dots, x_n]$. Let $d \geq 1$ be an integer number and consider S_d, the vector space of homogeneous polynomials of degree d. As an application of Theorem 15.2.2, we shall prove that the set of (proportionality classes of) reducible polynomials is a projective algebraic set in $\mathbb{P}(S_d)$.

We denote by $X \subset \mathbb{P}(S_d)$ the set of reducible polynomials. For any integer k, $0 < k < d$, let $X_k \subseteq X$ be the set of polynomials of the form F_1F_2 with $\deg F_1 = k, \deg F_2 = d - k$. Then $X = \bigcup_{k=1}^{d-1} X_k$. Let $f_k : \mathbb{P}(S_k) \times \mathbb{P}(S_{d-k}) \to \mathbb{P}(S_d)$ be the multiplication of polynomials, i.e. $f_k([F_1], [F_2]) = [F_1F_2]$. f_k is clearly a regular map, and its image is $X_k = X_{d-k}$. Since the domain is a projective variety, and precisely a Segre variety, it follows from Theorem 15.2.2 that also X_k is projective.

In the special case $d = 2$, the quadratic polynomials, the equations of $X = X_1$ are the minors of order 3 of the matrix associated to the quadric.

Chapter 16

The tangent space and the notion of smoothness

We will always assume K algebraically closed. In this chapter we follow the approach of Šafarevič [S]. We define the tangent space $T_{X,P}$ at a point P of an *affine* variety $X \subset \mathbb{A}^n$ as the union of the lines passing through P and "touching" X at P. It results to be an affine subspace of $\mathbb{A}^n$. Then we will find a "local" characterization of $T_{X,P}$, this time interpreted as a vector space, the direction of $T_{X,P}$, only depending on the local ring $\mathcal{O}_{X,P}$: this will allow to define the tangent space at a point of any quasi–projective variety.

16.1 Tangent space to an affine variety

Assume first that $X \subset \mathbb{A}^n$ is closed and $P = O = (0, \dots, 0)$. Let L be a line through P: if $A(a_1, \dots, a_n)$ is another point of L, then a general point of L has coordinates $(ta_1, \dots, ta_n)$, $t \in K$. If $I(X) = (F_1, \dots, F_m)$, then the intersection $X \cap L$ is determined by the following system of equations in the indeterminate t:

$$F_1(ta_1, \dots, ta_n) = \cdots = F_m(ta_1, \dots, ta_n) = 0.$$

The solutions of this system of equations are the roots of the greatest common divisor $G(t)$ of the polynomials $F_1(ta_1, \dots, ta_n), \dots, F_m(ta_1, \dots, ta_n)$ in $K[t]$, i.e. the generator of the ideal they generate. We may factorize $G(t)$ as $G(t) = ct^e(t - \alpha_1)^{e_1} \dots (t - \alpha_s)^{e_s}$, where $c \in K$, $\alpha_1, \dots, \alpha_s \neq 0$, $e, e_1, \dots, e_s$ are non-negative, and $e > 0$ if and only if $P \in X \cap L$. The number e is by definition the **intersection multiplicity at** P **of** X **and** L. If $G(t)$ is identically zero, then $L \subset X$ and the intersection multiplicity is, by definition, $+\infty$.

Note that the polynomial $G(t)$ doesn't depend on the choice of the generators $F_1, \dots, F_m$ of $I(X)$, but only on the ideal $I(X)$ and on L.

Definition 16.1.1. The line L is **tangent to the variety** X **at** P if the intersection multiplicity of L and X at P is at least 2 (in particular, if $L \subset X$). The **tangent space to** X **at** P is the union of the lines that are tangent to X at P; it is denoted $T_{P,X}$.

We will see now that $T_{P,X}$ is an affine subspace of $\mathbb{A}^n$. Assume that $P \in X$: then the polynomials F_i may be written in the form $F_i = L_i + G_i$, where L_i is a homogeneous linear polynomial (possibly zero) and G_i contains only terms of degree ≥ 2. Then

$$F_i(ta_1, \dots, ta_n) = tL_i(a_1, \dots, a_n) + G_i(ta_1, \dots, ta_n),$$

where the last term is divisible by t^2. Let L be the line $\overline{OA}$, with $A = (a_1, \dots, a_n)$. We note that the intersection multiplicity of X and L at P is the maximal power of t dividing the greatest common divisor, so L is tangent to X at P if and only if $L_i(a_1, \dots, a_n) = 0$ for all $i = 1, \dots, m$.

Therefore the point A belongs to $T_{P,X}$ if and only if

$$L_1(a_1, \dots, a_n) = \dots = L_m(a_1, \dots, a_n) = 0.$$

This shows that $T_{P,X}$ is a linear subspace of $\mathbb{A}^n$, whose equations are the linear components of the equations defining X.

Example 16.1.2. (i) $T_{O,\mathbb{A}^n} = \mathbb{A}^n$, because $I(\mathbb{A}^n) = (0)$.

(ii) If X is a hypersurface, with $I(X) = (F)$, we write as above $F = L + G$; then $T_{O,X} = V(L)$: so $T_{O,X}$ is either a hyperplane if $L \neq 0$, or the whole space $\mathbb{A}^n$ if $L = 0$. For instance, if X is the affine plane cuspidal cubic $V(x^3 - y^2) \subset \mathbb{A}^2$, $T_{O,X} = \mathbb{A}^2$.

Assume now that $P \in X$ has coordinates $(y_1, \dots, y_n)$. With an affine transformation we may translate P to the origin $(0, \dots, 0)$, taking as new coordinates functions on $\mathbb{A}^n$ $x_1 - y_1, \dots, x_n - y_n$. This corresponds to considering the K-isomorphism $K[x_1, \dots, x_n] \longrightarrow K[x_1 - y_1, \dots, x_n - y_n]$, which takes a polynomial $F(x_1, \dots, x_n)$ to its Taylor expansion

$$G(x_1 - y_1, \dots, x_n - y_n) = F(y_1, \dots, y_n) + d_P F + d_P^{(2)} F + \dots,$$

where $d_P^{(i)} F$ denotes the i^{th} differential of F at P: it is a homogeneous polynomial of degree i in the variables $x_1 - y_1, \dots, x_n - y_n$. In particular the linear term is

$$d_P F = \frac{\partial F}{\partial x_1}(P)(x_1 - y_1) + \dots + \frac{\partial F}{\partial x_n}(P)(x_n - y_n).$$

We get that, if $I(X) = (F_1, \ldots, F_m)$, then $T_{P,X}$ is the affine subspace of $\mathbb{A}^n$ defined by the equations

$$d_P F_1 = \cdots = d_P F_m = 0.$$

The affine space $\mathbb{A}^n$, which may identified with K^n, can be given a natural structure of K-vector space with origin P, so in a natural way $T_{P,X}$ is a vector subspace (with origin P). The functions $x_1 - y_1, \ldots, x_n - y_n$ form a basis of the dual space $(K^n)^*$ and their restrictions generate $T^*_{P,X}$. Note moreover that $\dim T_{P,X} = \dim T^*_{P,X} = k$ if and only if $n - k$ is the maximal number of polynomials linearly independent among $d_P F_1, \ldots, d_P F_m$. If $d_P F_1, \ldots, d_P F_{n-k}$ are these polynomials, then they form a basis of the orthogonal $T^{\perp}_{P,X}$ of the vector space $T_{P,X}$ in $(K^n)^*$, because they vanish on $T_{P,X}$.

16.2 Zariski tangent space

Let us define now the *differential of a regular function.* Let $f \in \mathcal{O}(X)$ be a regular function on X. We want to define the differential of f at P. Since X is closed in $\mathbb{A}^n$, f is induced by a polynomial $F \in K[x_1, \ldots, x_n]$ as well as by all polynomials of the form $F + G$ with $G \in I(X)$. Fix $P \in X$: then $d_P(F + G) = d_P F + d_P G$ so the differentials of two polynomials inducing the same function f on X differ by the term $d_P G$ with $G \in I(X)$. By definition, $d_P G$ is zero along $T_{P,X}$, so we may define $d_p f$ as a regular function on $T_{P,X}$, the differential of f at P: it is the function on $T_{P,X}$ induced by $d_P F$. Since $d_P F$ is a linear combination of $x_1 - y_1, \ldots, x_n - y_n$, $d_p f$ can also be seen as an element of $T^*_{P,X}$, the dual vector space.

There is a natural map $d_p : \mathcal{O}(X) \to T^*_{P,X}$, which sends f to $d_p f$. Because of the rules of derivation, it is clear that $d_P(f + g) = d_P f + d_P g$ and $d_P(fg) = f(P) d_P g + g(P) d_P f$. In particular, if $c \in K$, $d_p(cf) = c d_P f$. So d_p is a linear map of K-vector spaces. We denote again by d_P the restriction of d_P to $I_X(P)$, the maximal ideal of the regular functions on X which are zero at P. Since clearly $f = f(P) + (f - f(P))$ then $d_P f = d_P(f - f(P))$, so this restriction doesn't modify the image of the map.

Proposition 16.2.1. *The map $d_P : I_X(P) \to T^*_{P,X}$ is surjective and its kernel is $I_X(P)^2$. Therefore $T^*_{P,X} \simeq I_X(P)/I_X(P)^2$ as K-vector spaces.*

Proof. Let $\varphi \in T^*_{P,X}$ be a linear form on $T_{P,X}$. φ is the restriction of a linear form on K^n: $\lambda_1(x_1 - y_1) + \ldots + \lambda_n(x_n - y_n)$, with $\lambda_1, \ldots, \lambda_n \in K$. Let G be the polynomial of degree 1 $\lambda_1(x_1 - y_1) + \ldots + \lambda_n(x_n - y_n)$: the function g induced by G on X is zero at P and coincides with its own differential, so $\varphi = d_P g$ and d_P is surjective.

Let now $g \in I_X(P)$ such that $d_p g = 0$, g induced by a polynomial G. Note that $d_P G$ may be interpreted as a linear form on K^n which vanishes on $T_{P,X}$, hence as an element of $T_{P,X}^{\perp}$. So $d_P G = c_1 d_p F_1 + \ldots + c_m d_p F_m$ ($c_1, \ldots, c_m$ suitable elements of K). Let us consider the polynomial $G - c_1 F_1 - \ldots - c_m F_m$: since its differential at P is zero, it doesn't have any term of degree 0 or 1 in $x_1 - y_1, \ldots, x_n - y_n$, so it belongs to $I(P)^2$. Since $G - c_1 F_1 - \ldots - c_m F_m$ defines the function g on X, we conclude that $g \in I_X(P)^2$. $\square$

Corollary 16.2.2. *The tangent space $T_{P,X}$ is isomorphic to $(I_X(P)/I_X(P)^2)^*$ as an abstract K-vector space.*

Corollary 16.2.3. *Let $\varphi : X \to Y$ be an isomorphism of affine varieties and $P \in X$, $Q = \varphi(P)$. Then the tangent spaces $T_{P,X}$ and $T_{Q,Y}$ are isomorphic.*

Proof. φ induces the comorphism $\varphi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$, which results to be an isomorphism such that $\varphi^* I_Y(Q) = I_X(P)$ and $\varphi^* I_Y(Q)^2 = I_X(P)^2$. So there is an induced homomorphism

$$I_Y(Q)/I_Y(Q)^2 \to I_X(P)/I_X(P)^2.$$

which is an isomorphism of K-vector spaces. By dualizing we get the claim. $\square$

The above map from $T_{P,X}$ to $T_{Q,Y}$ is called the *differential of φ at P* and is denoted by $d_P\varphi$.

Now we would like to find a "more local" characterization of $T_{P,X}$. To this end we consider the local ring of P in X: $\mathcal{O}_{P,X}$. We recall the natural map $\mathcal{O}(X) \to \mathcal{O}_{P,X} = \mathcal{O}(X)_{I_X(P)}$, the last one being the localization. It is natural to extend the map $d_P : \mathcal{O}(X) \to T^*_{P,X}$ to $\mathcal{O}_{P,X}$ setting

$$d_P\Big(\frac{f}{g}\Big) = \frac{g(P)d_P f - f(P)d_P g}{g(P)^2}.$$

As in the proof of Proposition 16.2.1 one proves that the map $d_P : \mathcal{O}_{P,X} \to T^*_{P,X}$ induces an isomorphism $\mathcal{M}_{P,X}/\mathcal{M}^2_{P,X} \to T^*_{P,X}$, where $\mathcal{M}_{P,X}$ is the maximal ideal of $\mathcal{O}_{P,X}$. So by duality we have: $T_{P,X} \simeq (\mathcal{M}_{P,X}/\mathcal{M}^2_{P,X})^*$. This proves that the tangent space $T_{P,X}$ is a *local invariant* of P in X.

Definition 16.2.4. Let X be any quasi-projective variety, $P \in X$. The *Zariski tangent space* of X at P is the vector space $(\mathcal{M}_{P,X}/\mathcal{M}^2_{P,X})^*$.

It is an abstract vector space, but if $X \subset \mathbb{A}^n$ is closed, taking the dual of the comorphism associated to the inclusion morphism $X \hookrightarrow \mathbb{A}^n$, we have an embedding of $T_{P,X}$ into $T_{P,\mathbb{A}^n} = \mathbb{A}^n$. If $X \subset \mathbb{P}^n$ and $P \in U_i = \mathbb{A}^n$, then $T_{P,X} \subset U_i$: its projective closure $\mathbb{T}_{P,X}$ is called the *embedded tangent space* to X at P.

16.3 Smoothness

As we have seen the tangent space $T_{P,X}$ is invariant by isomorphism. In particular its dimension is invariant. If $X \subset \mathbb{A}^n$ is closed, $I(X) = (F_1, \dots, F_m)$, then $\dim T_{P,X} = n - r$, where r is the dimension of the K-vector space generated by $\{d_P F_1, \dots, d_p F_m\}$.

Since $d_P F_i = \frac{\partial F_i}{\partial x_1}(P)(x_1 - y_1) + \dots + \frac{\partial F_i}{\partial x_n}(P)(x_n - y_n)$, r is the rank of the following $m \times n$ matrix, the *Jacobian matrix of X at P*:

$$J(P) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(P) & \dots & \frac{\partial F_1}{\partial x_n}(P) \\ \dots & \dots & \dots \\ \frac{\partial F_m}{\partial x_1}(P) & \dots & \frac{\partial F_m}{\partial x_n}(P) \end{pmatrix}.$$

The *generic Jacobian matrix of X* is instead the following matrix with entries in $\mathcal{O}(X)$ (the entries are the functions on X induced by the partial derivatives of the polynomials F_i):

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \dots & \dots & \dots \\ \frac{\partial f_m}{\partial x_1} & \dots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}.$$

The rank of J is ρ when all minors of order $\rho + 1$ are functions identically zero on X, while at least one minor of order ρ is different from zero at some point. Hence, for all $P \in X$ rk $J(P) \leq \rho$, and rk $J(P) < \rho$ if and only if all minors of order ρ of J vanish at P. It is then clear that there is a non-empty open subset of X where $\dim T_{P,X}$ is minimal, equal to $n - \rho$, and a proper (possibly empty) closed subset formed by the points P such that $\dim T_{P,X} > n - \rho$.

Definition 16.3.1. The points of an irreducible variety X for which $\dim T_{P,X} = n - \rho$ (the minimal) are called *smooth* or *non-singular* (or *simple) points* of X. The remaining points are called *singular* (or multiple). X is a *smooth* variety if all its points are smooth.

If X is quasi-projective, the same argument may be repeated for any affine open subset.

Example 16.3.2. Let $X \subset \mathbb{A}^n$ be the irreducible hypersurface $V(F)$, with F irreducible generator of $I(X)$. Then $J = (\frac{\partial F}{\partial x_1} \dots \frac{\partial F}{\partial x_n})$ is a row matrix. So rk $J = 0$ or 1. If rk $J = 0$, then $\frac{\partial F}{\partial x_i} = 0$ in $\mathcal{O}(X)$ for all i. So $\frac{\partial F}{\partial x_i} \in I(X) = (F)$. Since the degree of $\frac{\partial F}{\partial x_i}$ is $\leq \deg F - 1$, it follows that $\frac{\partial F}{\partial x_i} = 0$ in the polynomial ring. If the characteristic of K is zero this means that F is constant: a contradiction. If char $K = p$, then $F \in K[x_1^p, \dots, x_n^p]$; since K is algebraically closed, then all coefficients of F are p-th powers, so $F = G^p$ for a suitable polynomial G; but again this is impossible because F is irreducible. So always rk

$J = 1 = \rho$. Hence for P general in X, i.e. for P varying in a suitable non-empty open subset of X, $\dim T_{P,X} = n-1$. For some particular points, the singular points of X, we can have $\dim T_{P,X} = n$, i.e. $T_{P,X} = \mathbb{A}^n$.

So in the case of a hypersurface $\dim T_{P,X} \geq \dim X$ for every point P in X, and equality holds in the points of the smooth locus of X. The general case can be reduced to the case of hypersurfaces in view of the following theorem.

Theorem 16.3.3. *Every quasi-projective irreducible variety X is birational to a hypersurface in some affine space.*

Proof. We observe that we can reduce the proof to the case in which X is affine, closed in $\mathbb{A}^n$. Let $m = \dim X$. We have to prove that the field of rational functions $K(X)$ is isomorphic to a field of the form $K(t_1, \dots, t_{m+1})$, where $t_1, \dots, t_{m+1}$ satisfy only one non-trivial relation $F(t_1, \dots, t_{m+1}) = 0$, where F is an irreducible polynomial with coefficients in K. This will follow from the "Abel's primitive element Theorem" 16.3.5 concerning extensions of fields. To state it, we need some preliminaries.

Let $K \subset L$ be an extension of fields. Let $a \in L$ be algebraic over K, and let $f_a \in K[x]$ be its minimal polynomial: it is irreducible and monic. Let E be the splitting field of f_a.

Definition 16.3.4. An element a, algebraic over K, is *separable* if f_a does not have any multiple root in E, i.e. if f_a and its derivative f'_a don't have any common factor of positive degree. Otherwise a is inseparable. If $K \subset L$ is an algebraic extension of fields, it is called separable if any element of L is separable.

In view of the fact that f_a is irreducible in $K[x]$, and that the GCD of two polynomials is independent of the field where one considers the coefficients, if a is inseparable, then f'_a is the zero polynomial. If char $K = 0$, this implies that f_a is constant, which is a contradiction. So in characteristic 0, any algebraic extension is separable. If char $K = p > 0$, then $f_a \in K[x^p]$, and f_a is called an inseparable polynomial. In particular algebraic inseparable elements can exist only in positive characteristic.

Theorem 16.3.5 (Abel's primitive element Theorem.)**.** *Let $K \subseteq L = K(y_1, \dots, y_m)$ be an algebraic finite extension. If L is a separable extension, then there exists $\alpha \in L$, called a primitive element of L, such that $L = K(\alpha)$ is a simple extension.*

For a proof, see for instance [L], or any textbook on Galois theory.

To prove Theorem 16.3.3 we need also a second ingredient, that I state here without proof.

Theorem 16.3.6 (Existence of separating transcendence bases)**.** *Let K be an algebraically closed field and $E \supset K$ a finitely generated field extension of K with $tr.d.E/K = m$. Then any set of generators of E over K contains a transcendence basis $\{x_1, \dots, x_m\}$ such that E is a separable algebraic extension of $K(x_1, \dots, x_m)$.*

Proof. See for instance [ZS]. □

Proof of Theorem 16.3.3. The field of rational functions of X is of the form $K(X) = Q(K[X]) = K(t_1, \dots, t_n)$, where $t_1, \dots, t_n$ are the coordinate functions on X and $tr.d.K(X)/K = m$. By Theorem 16.3.6, possibly after renumbering them, we can assume that the first m coordinate functions $t_1, \dots, t_m$ are algebraically independent over K, and $K(X)$ is a separable algebraic extension of $L := K(t_1, \dots, t_m)$. So in our situation we can apply Theorem 16.3.5: there exists a primitive element α such that $K(X) = L(\alpha) = K(t_1 \dots, t_m, \alpha)$. Therefore there exists an irreducible polynomial $f \in L[x]$ such that $K(X) = L[x]/(f)$. Multiplying f by a suitable element of $K[t_1, \dots, t_m]$, invertible in L, we can eliminate the denominator of f and replace f by a polynomial $g \in K[t_1, \dots, t_m, x] \subset L[x]$. Now $K[t_1, \dots, t_m, x]/(g)$ is contained in $L[x]/(g) = K(X)$, and its quotient field is again $K(X)$. But $K[t_1, \dots, t_m, x]/(g)$ is the coordinate ring of the hypersurface $Y \subset \mathbb{A}^{m+1}$ of equation $g = 0$. It is clear that X and Y are birationally equivalent, because they have the same field of rational functions. This concludes the proof. □

One can show that the coordinate functions on Y, $t_1, \dots, t_{m+1}$, can be chosen to be linear combinations of the original coordinate functions on X: this means that Y is obtained as a suitable birational projection of X.

Theorem 16.3.7. *The dimension of the tangent space at a non-singular point of an irreducible variety X is equal to* $\dim X$.

Proof. It is enough to prove the claim under the assumption that X is affine. Let Y be an affine hypersurface birational to X (which exists by the previous theorem) and $\varphi : X \dashrightarrow Y$ be a birational map. There exist open non-empty subsets $U \subset X$ and $V \subset Y$ such that $\varphi : U \to V$ is an isomorphism. The set of smooth points of Y is an open subset W of Y such that $W \cap V$ is non-empty and $\dim T_{P,Y} = \dim Y = \dim X$ for all $P \in W \cap V$. But $\varphi^{-1}(W \cap V) \subset U$ is open non-empty and $\dim T_{Q,X} = \dim X$ for all $Q \in \varphi^{-1}(W \cap V)$. This proves the theorem. □

We will denote by X_{sing} the closed set, possibly empty, of singular points of X, and by X_{sm} the smooth locus of X, i.e. the open non empty subset of its smooth points.

Corollary 16.3.8. *The singular points of an affine variety X closed in $\mathbb{A}^n$ with $\dim X = m$, are the points P of X where the Jacobian matrix $J(P)$ has rank strictly less than $n-m$.*

To find the singular points of a projective variety, it is useful to remember the following Euler relation for homogeneous polynomials.

Proposition 16.3.9 (Euler's formula)**.** *Let $F(x_0,\dots,x_n)$ be a homogeneous polynomial of degree d. Then $dF = x_0F_{x_0} + \dots + x_nF_{x_n}$, where, for every $i = 0,\dots,n$, F_{x_i} denotes the (formal) partial derivative of F with respect to x_i.*

Proof. Since $d = degF$, we have $F(tx_0,\dots,tx_n) = t^dF(x_0,\dots,x_n)$. To get the desired formula it is enough to derive with respect to t and then put $t = 1$. □

Let now $X \subset \mathbb{P}^n$ be a hypersurface with $I_h(X) = \langle F(x_0,\dots,x_n)\rangle$, $degF = d$.

Proposition 16.3.10. *Let K be a field of characteristic p; assume that $p = 0$ or d does not divide p. Then the singular points of X are the common zeros of the partial derivatives of F, i.e. $X_{sing} = V_P(F_{x_0},\dots,F_{x_n})$.*

Proof. We denote by $f(x_1,\dots,x_n)$ the dehomogenized ${}^aF = F(1,x_1,\dots,x_n)$ of F with respect to x_0. We observe that, for $i = 1,\dots,n$, ${}^a(F_{x_i}) = f_{x_i}$, and that ${}^aF_{x_0} = df - x_1f_{x_1} - \dots - x_nf_{x_n}$, in view of Proposition 16.3.9. So, if $P \in U_0$, $f(P) = f_{x_1}(P) = \dots = f_{x_n}(P) = 0$ if and only if $F_{x_0}(P) = \dots = F_{x_n}(P) = 0$. □

Therefore, to look for the singular points of an affine hypersurface X, one has to consider the system of equations defined by the equation of X and its partial derivatives, whereas in the projective case it is enough to consider the system of the partial derivatives, because Euler's relation garantees that by consequence also the equation of the hypersurface is satisfied.

For an affine variety X of higher codimension $n-m$, one has to impose the vanishing of the equations of X and of the minors of order $n-m$ of the Jacobian matrix. In the projective case, using again Euler's relation, one can check that the singular points are those that annihilate the homogeneous polynomials $F_1,\dots,F_r$ generating $I_h(X)$ and also the minors of order $n-m$ of the homogeneous $r \times (n+1)$ Jacobian matrix $(\partial F_i/\partial x_j)_{ij}$.

Euler formula is useful also to write the equations of the embedded tangent space $\mathbb{T}_{P,X}$ to a projective variety X at a point P. Assume first that $X \subset \mathbb{P}^n$ is a hypersurface $V_P(F)$, $F \in K[x_0,\dots,x_n]$. Assume that $P \in U_0$, and use non-homogeneous coordinates $u_i = x_i/x_0$ on U_0, so that $X \cap U_0$ is the zero locus of ${}^aF = F(1,u_1,\dots,u_n) =: f(u_1,\dots,u_n)$. If P

has non-homogeneous coordinates $a_1, \dots, a_n$, the affine tangent space $T_{P,X\cap U_0}$ has equation $\sum_{i=1}^n \frac{\partial f}{\partial u_i}(P)(u_i - a_i) = 0$. By definition $\mathbb{T}_{P,X}$ is its projective closure, so it is

$$\mathbb{T}_{P,X} = \{[x_0 \dots, x_n] \mid \sum_{i=1}^n \frac{\partial F}{\partial x_i}(1, a_1, \dots, a_n)(x_i - a_i x_0) = 0\}.$$

From Euler formula, using that $F(1, a_1, \dots, a_n) = 0$, we get that

$$\sum_{i=1}^n \frac{\partial F}{\partial x_i}(1, a_1, \dots, a_n)(-a_i x_0) = \frac{\partial F}{\partial x_0}(1, a_1, \dots, a_n)x_0.$$

We conclude that $\mathbb{T}_{P,X}$ is defined by the equation $\sum_{i=0}^n \frac{\partial F}{\partial x_i}(P)x_i = 0$.

If X is the projective variety with ideal $I_h(X) = (F_1, \dots, F_r)$, then, repeating the previous argument, we get that its tangent space is defined by the linear polynomials $\sum_{i=0}^n \frac{\partial F_k}{\partial x_i}(P)x_i$, for $k = 1, \dots, r$.

We note that the affine tangent space, when X is affine, or the embedded tangent space, when X is projective, to X at P is the intersection of the tangent spaces to the hypersurfaces containing X.

Now we would like to study a variety X in a neighbourhood of a smooth point. We have seen that P is smooth for X if and only if $\dim T_{P,X} = \dim X$. Assume X affine: in this case the local ring of P in X is $\mathcal{O}_{P,X} \simeq \mathcal{O}(X)_{I_X(P)}$. But by Theorem 7.2.4, we have: $\dim \mathcal{O}_{P,X} = \mathrm{ht}\mathcal{M}_{P,X} = \mathrm{ht}I_X(P) = \dim \mathcal{O}(X) = \dim X$ and $\dim T_{P,X} = \dim_K \mathcal{M}_{P,X}/\mathcal{M}^2_{P,X}$. Therefore P is smooth if and only if

$$\dim_K \mathcal{M}_{P,X}/\mathcal{M}^2_{P,X} = \dim \mathcal{O}_{P,X}$$

(the first one is a dimension as K-vector space, the second one is a Krull dimension). By Nakayama's Lemma (Theorem 14.3.1) a basis of $\mathcal{M}_{P,X}/\mathcal{M}^2_{P,X}$ corresponds bijectively to a minimal system of generators of the ideal $\mathcal{M}_{P,X}$. Indeed, since the residue field of $\mathcal{O}_{P,X}$ is isomorphic to K, we can interpret any scalar in K as an element $[a]_{\mathcal{M}_{P,X}} \in \mathcal{O}_{P,X}/\mathcal{M}_{P,X} \simeq K$, and the product giving the structure of K-vector space to $\mathcal{M}_{P,X}/\mathcal{M}^2_{P,X}$ operates as follows: $[a]_{\mathcal{M}_{P,X}}[m]_{\mathcal{M}^2_{P,X}} = [am]_{\mathcal{M}^2_{P,X}}$ (the definition is well posed). Now, given elements $f_1, \dots, f_r \in \mathcal{M}_{P,X}$, we call $\alpha = \langle f_1, \dots, f_r\rangle$ the ideal they generate. We apply Nakayama's Lemma with notations as in Theorem 14.3.1, where the module M is the maximal ideal $\mathcal{M}_{P,X}$, its submodule N is the ideal α, and the ideal I is again $\mathcal{M}_{P,X}$. We get $\mathcal{M}_{P,X} = \alpha$.

Therefore P is smooth for X if and only if $\mathcal{M}_{P,X}$ is minimally generated by r elements, where $r = \dim \mathcal{O}_{P,X}$, in other words if and only if $\mathcal{O}_{P,X}$ is a *regular local ring.*

For example, if X is a curve, P is smooth if and only if $T_{P,X}$ has dimension 1, i.e. $\mathcal{M}_{P,X}$ is principal: $\mathcal{M}_{P,X} = (t)$.

Observe that the set of common zeros of the functions in $\mathcal{M}_{P,X}$ is precisely the point P. The fact that $\mathcal{M}_{P,X}$ is principal generated by t means that P is defined in X by the only equation $t = 0$ in a suitable neighborhood of P. This is called a local equation of P. If P is a singular point, then the minimal number of generators of $\mathcal{M}_{P,X}$ is bigger than one, equal to the dimension of the tangent space $T_{P,X}$. So to define P we need more than one local equation.

Let P be a smooth point of X and $\dim X = n$. Functions $u_1, \ldots, u_n \in \mathcal{O}_{P,X}$ are called *local parameters* at P if $u_1, \ldots, u_n \in \mathcal{M}_{P,X}$ and their residues $\bar{u}_1, \ldots, \bar{u}_n$ in $\mathcal{M}_{P,X}/\mathcal{M}^2_{P,X}$ ($= T^*_{P,X}$) form a basis, or equivalently if $u_1, \ldots, u_n$ is a minimal set of generators of $\mathcal{M}_{P,X}$. Recalling the isomorphism

$$d_P : \mathcal{M}_{P,X}/\mathcal{M}^2_{P,X} \to T^*_{P,X}$$

we deduce that $u_1, \ldots, u_n$ are local parameters if and only if $d_P\bar{u}_1, \ldots, d_P\bar{u}_n$ are linearly independent linear forms on $T_{P,X}$ (which is a vector space of dimension n), if and only if the system of linear equations on $T_{P,X}$

$$d_P\bar{u}_1 = \ldots = d_P\bar{u}_n = 0$$

has only the trivial solution P (which is the origin of the vector space $T_{P,X}$).

Let $u_1, \ldots, u_n$ be local parameters at P. There exists an open affine neighborhood of P on which $u_1, \ldots, u_n$ are all regular. We replace X by this neighborhood, so we assume that X is affine and that $u_1, \ldots, u_n$ are polynomial functions on X. Let X_i be the closed subset $V(u_i)$ of X: it has codimension 1 in X, because u_i is not identically zero on X ($u_1, \ldots, u_n$ is a minimal set of generators of $\mathcal{M}_{P,X}$).

Proposition 16.3.11. *In this notation, P is a smooth point of X_i, for all $i = 1, \ldots, n$, and $\bigcap_i T_{P,X_i} = \{P\}$.*

Proof. Assume that U_i is a polynomial inducing u_i, then $X_i = V(U_i) \cap X = V(I(X) + (U_i))$. So $I(X_i) \supset I(X) + (U_i)$. By considering the linear parts of the polynomials of the previous ideal, we get: $T_{P,X_i} \subset T_{P,X} \cap V(d_P U_i)$. By the assumption on the u_i, it follows that $T_{P,X} \cap V(d_P U_1) \cap \cdots \cap V(d_P U_n) = \{P\}$. Since $\dim T_{P,X} = n$, we can deduce that $T_{P,X} \cap V(d_P U_i)$ is strictly contained in $T_{P,X}$, and $\dim T_{P,X} \cap V(d_P U_i) = n - 1$. So $\dim T_{P,X_i} \leq n - 1 = \dim X_i$, hence P is a smooth point on X_i, equality holds and $T_{P,X_i} = T_{P,X} \cap V(d_P U_i)$. Moreover $\bigcap T_{P,X_i} = \{P\}$. □

Note that $\bigcap_i X_i$ has no positive-dimensional component Y passing through P: otherwise the tangent space to Y at P would be contained in T_{P,X_i} for all i, against the fact that $\bigcap T_{P,X_i} = \{P\}$.

Definition 16.3.12. Let X be a smooth variety. Subvarieties $Y_1, \dots, Y_r$ of X are called *transversal at* P, with $P \in \bigcap Y_i$, if the intersection of the tangent spaces T_{P,Y_i} has dimension as small as possible, i.e. if $\operatorname{codim}_{T_{P,X}}(\bigcap T_{P,Y_i}) = \sum \operatorname{codim}_X Y_i$.

Taking $T_{P,X}$ as ambient variety, one gets the relation:

$$\dim \bigcap T_{P,Y_i} \geq \sum \dim T_{P,Y_i} - (r-1) \dim T_{P,X};$$

hence

$$\operatorname{codim}_{T_{P,X}}(\bigcap T_{P,Y_i}) = \dim T_{P,X} - \dim \bigcap T_{P,Y_i} \leq \sum (\dim T_{P,X} - \dim T_{P,Y_i}) =$$

$$= \sum \operatorname{codim}_{T_{P,X}}(T_{P,Y_i}) \leq \sum \operatorname{codim}_X Y_i.$$

If equality holds, P is a smooth point for Y_i for all i, moreover we get that P is a smooth point for the set $\bigcap Y_i$.

For example, if X is a surface and $P \in X$ is smooth, there is a neighbourhood U of P such that P is the transversal intersection of two curves in U, corresponding to local parameters u_1, u_2. If P is singular we need three functions u_1, u_2, u_3 to generate the maximal ideal $\mathcal{M}_{P,X}$.

16.4 Tangent cone

To conclude this chapter I want to mention the tangent cone to a variety X at a point P. To introduce it we consider first the case where X is a closed affine variety $X \subset \mathbb{A}^n$ and $P = O(0, \dots, 0)$. The tangent cone to X at O, $TC_{O,X}$, is the union of the lines through O which are "limit positions" of secant lines to X. To formalize this idea, we consider in $\mathbb{A}^{n+1} = \mathbb{A}^n \times \mathbb{A}^1$ the closed set $\tilde{X}$ of pairs (a, t), with $a = (a_1, \dots, a_n) \in \mathbb{A}^n$ and $t \in \mathbb{A}^1$, such that $at \in X$. Let $\varphi : \tilde{X} \to \mathbb{A}^1$, $\psi : \tilde{X} \to \mathbb{A}^n$ be the projections. If $X \neq \mathbb{A}^n$, $\tilde{X}$ results to be reducible: $\tilde{X} = \tilde{X}_1 \cup \tilde{X}_2$, where $\tilde{X}_2 = \{(a, 0) \mid a \in \mathbb{A}^n\} \simeq \mathbb{A}^n$, $\tilde{X}_1 = \overline{\varphi^{-1}(\mathbb{A}^1 \setminus 0)}$. We consider the restrictions φ_1, ψ_1 of the projections to $\tilde{X}_1$. $\psi_1(\tilde{X}_1)$ results to be the closure of the union of the secant lines of X through O. The tangent cone $TC_{O,X}$ is by definition $\psi_1(\varphi_1^{-1}(0))$.

Let us write the equations of $TC_{O,X}$. We note first that the equations of $\tilde{X}$ are of the form $F(a_1 t, \dots, a_n t) = 0$ where $F \in I(X)$. Write F as sum of its homogeneous components

$F = F_k + \cdots + F_d$, where F_k is the non-zero component of minimal degree, and $k \geq 1$ because $O \in X$. Then $F(at) = t^k F_k(a) + \cdots + t^d F_d(a)$. The equation of the component $\tilde{X}_2$ inside $\tilde{X}$ is $t = 0$. The equations of the tangent cone are $F_k = 0$ for all $F \in I(X)$, they are given by the initial forms of the polynomials of $I(X)$, i.e. the non-zero homogeneous components of minimal degree. Since all equations are homogeneous, it is clear that we get a cone. Moreover $TC_{O,X} \subseteq T_{O,X}$, and equality holds if and only if O is a smooth point of X.

As in the case of the tangent space, we can extend the definition to any point, by translation, and then find a characterization that allows to prove that the tangent cone is invariant by isomorphism.

In the particular case $n = 2$, with X a curve defined by the equation $F(x, y) = 0$, the tangent cone at O is defined by the vanishing of the initial form $F_k(x, y)$. Being a homogeneous polynomial in two variables, it factorizes as a product of k linear forms (counting multiplicities), defining k lines: the tangent lines to X at O.

For instance, in the case of the cuspidal cubic $V(x^3 - y^2)$ the tangent cone at the origin has equation $y^2 = 0$: it is the line $y = 0$ "counted with multiplicity 2. If X is the cubic of equation $x^2 - y^2 + x^3 = 0$, the tangent cone consists in the two distinct lines $x - y = 0$ and $x + y = 0$: the cubic is nodal.

The tangent cone allows to define the multiplicity of a point on X and to start an analysis of the singularities.

Exercises 16.4.1. 1. Assume char $K \neq 2$. Find the singular points of the following surfaces in $\mathbb{A}^3$:

1. $xy^2 = z^3$;

2. $x^2 + y^2 = z^2$;

3. $xy + x^3 + y^3 = 0$.

2. Suppose that char $K \neq 3$. Determine the singular locus of the projective variety in $\mathbb{P}^5$ given by the equations:

$$\sum_{i=0}^{5} x_i = 0, \quad \sum_{i=0}^{5} x_i^3 = 0.$$

Chapter 17

Finite morphisms and blow–ups

In this section we will see the notion of finite morphism, and a fundamental example of a morphism which is not finite: the blow-up of a variety at a point, or, more in general, along a subvariety. The blow-up is the main ingredient in the resolution of singularities of an algebraic variety. As usual we will assume that K is algebraically closed.

17.1 Finite morphisms

First of all we will give an interpretation in geometric terms of the notions of integral elements and integral extensions introduced and studied in Chapters 4 and 8.

Let $f : X \to Y$ be a dominant morphism of affine varieties, i.e. we assume that $f(X)$ is dense in Y. Then the comorphism $f^* : K[Y] \to K[X]$ is injective (by Exercise 4, Chapter 12): we will often identify $K[Y]$ with its image $f^*K[Y] \subset K[X]$.

Definition 17.1.1. f is a finite morphism if $K[X]$ is an integral extension of $K[Y]$.

This means that, for any regular function φ on X, there is a relation of integral dependence

$$\varphi^r + f^*(g_1)\varphi^{r-1} + \cdots + f^*(g_r) = 0 \tag{17.1}$$

with $g_1, \ldots, g_r \in K[Y]$. Finite morphisms enjoy the following properties.

Proposition 17.1.2. *1. The composition of finite morphisms is a finite morphism.*

2. Let $f : X \to Y$ be a finite morphism of affine varieties. Then, for any $y \in Y$, $f^{-1}(y)$ is a finite set.

3. Finite morphisms are surjective, i.e. $f^{-1}(y)$ is non-empty for any $y \in Y$.

4. *Finite morphisms are closed maps.*

Proof. 1. It follows from the transitivity of integral dependence, Corollary 4.0.3.

2. Let X be a closed subset of $\mathbb{A}^n$, so $K[X]$ is generated by the coordinate functions $t_1, \ldots, t_n$. Let $y \in Y$. We want to prove that any coordinate function t_i takes only a finite number of values on the set $f^{-1}(y)$. For the function t_i there is a relation of integral dependence of type (17.1): $t_i^r + f^*(g_1)t_i^{r-1} + \cdots + f^*(g_r) = 0 \in K[X]$ with $g_1, \ldots, g_r \in K[Y]$. We apply this relation to $x \in f^{-1}(y)$ and we get $t_i^r(x) + g_1(y)t_i^{r-1}(x) + \cdots + g_r(y) = 0$. This means that the i-th coordinate of any point in $f^{-1}(y)$ has to satisfy a (monic) equation of degree r, so there are only finitely many possibilities for this coordinate. This proves what we want.

3. This is a consequence of the property of Lying over - LO (Section 8.1). Let $y = (y_1, \ldots, y_m) \in Y \subset \mathbb{A}^m$, let $u_1, \ldots, u_m$ be the coordinate functions on Y. A point $x \in X$ belongs to $f^{-1}(y)$ if and only if $u_i(f(x)) = f^*(u_i)(x) = y_i$ for any i, or equivalently if and only if the function $f^*(u_i) - y_i$ vanishes on x, i.e. it belongs to the ideal $I_X(x)$. In view of the relative version of the Nullstellensatz (Proposition 9.1.5), the condition $f^{-1}(y) = \emptyset$ is therefore equivalent to the fact that the ideal generated by $f^*(u_1) - y_1, \ldots, f^*(u_m) - y_m$ in $K[X]$ is the entire ring $K[X]$, in particular it is not contained in any maximal ideal. Consider now the maximal ideal $I_Y(y)$ of regular functions on Y vanishing in y, it is generated by $u_1 - y_1, \ldots, u_m - y_m$. But, from the Lying over applied to the integral extension $f^*K[Y] \subset K[X]$, it follows that there is a prime ideal $\mathcal{P}$ of $K[X]$ over $f^*(I_Y(y))$, which is generated by $f^*(u_1) - y_1, \ldots, f^*(u_m) - y_m$. This implies that $f^{-1}(y) \neq \emptyset$.

4. Let $f : X \to Y$ be a finite morphism and $Z \subset X$ an irreducible closed subset. We consider the restriction of f to Z, i.e. $\bar{f} : Z \to \overline{f(Z)}$. We observe that, via the comorphism $\bar{f}^* : K[\overline{f(Z)}] \to K[Z]$, $K[Z] \simeq K[X]/I_X(Z)$ is an integral extension of $K[\overline{f(Z)}]$, because it is enough to reduce modulo $I_X(Z)$ the integral equations of the elements of $K[X]$. So, applying (3) to the finite morphism $\bar{f}$, we conclude that $\bar{f}$ is surjective, i.e. $f(Z) = \overline{f(Z)}$.

□

An example of non-finite morphism is the projection $V(xy - 1) \to \mathbb{A}^1$. Instead the projection $p_2 : V(y - x^2) \to \mathbb{A}^1$ is finite.

Theorem 17.1.3 (Geometric interpretation of the Normalization Lemma)**.** *Let $X \subset \mathbb{A}^n$ be an affine irreducible variety of dimension d. Then there exists a finite morphism $X \to \mathbb{A}^d$. Moreover the morphism can be taken to be a projection.*

Proof. The coordinate ring of X is an integral K-algebra, finitely generated by the coordinate functions, whose quotient field has transcendence degree d over K. The Normalization Lemma (Theorem 4.0.4) then asserts that there exist elements $z_1, \dots, z_d$ algebraically independent over K, such that $K[X]$ is an integral extension of the K-algebra $B = K[z_1, \dots, z_d]$. But B is the coordinate ring of $\mathbb{A}^d$ and the inclusion $B \hookrightarrow K[X]$ can be seen as the comorphism of a finite morphism $f : X \to \mathbb{A}^d$. The proof of Normalization Lemma shows that $z_1, \dots, z_d$ can be chosen linear combinations of the generators of $K[X]$. In this case, f results to be a projection. □

One can prove that being a finite morphism is a local property, in the following sense: let $f : X \to Y$ be a morphism of affine varieties. Then f is finite if and only if any $y \in Y$ has an affine open neighbourhood V, such that $U := f^{-1}(V)$ is affine, and the restriction $f \mid: U \to V$ is a finite morphism. This property allows to give the definition of finite morphism between arbitrary varieties, as a morphism which is finite when restricted to the open subsets of an affine open covering. See [S] for more details and consequences.

For instance one can obtain the following non-trivial facts, that I quote here only for information.

Example 17.1.4. *1. Let $X \subset \mathbb{P}^n$ be a closed algebraic set, let $\Lambda \subset \mathbb{P}^n$ be a linear subspace of dimension d such that $X \cap \Lambda = \emptyset$. Then the restriction of the projection $\pi_\Lambda : X \to \mathbb{P}^{n-d-1}$ defines a finite morphism from X to $\pi_\Lambda(X)$.*

2. Let $X \subset \mathbb{P}^n$ be a closed algebraic set and $F_0, \dots, F_r$ be homogeneous polynomials of the same degree d without any common zero on X. Then $\varphi : X \to \mathbb{P}^r$ defined by the polynomials $F_0, \dots, F_r$ is a finite morphism to the image.

For a proof of the first property, see [S]. To prove the second one, we observe that φ is the composition of the Veronese morphism $v_{n,d}$ with a projection. The conclusion follows from part 1., remembering that $v_{n,d}$ is an isomorphism (Section 10.6). The upshot is that, if φ is defined by the same homogeneous polynomials on the whole X, then it is a finite morphism; in particular all the fibres are finite.

17.2 Blow-up

We will define now the blow-up (or blowing-up) of an affine space at the origin $O(0,\dots,0)$. It is a variety X with a morphism $\sigma : X \to \mathbb{A}^n$ which results to be birational and not finite. The idea is that X is obtained from $\mathbb{A}^n$ by replacing the point O with a $\mathbb{P}^{n-1}$, which can be interpreted as $\mathbb{P}(T_{O,\mathbb{A}^n})$, the set of the tangent directions to $\mathbb{A}^n$ at O.

To construct X we first consider the product $\mathbb{A}^n \times \mathbb{P}^{n-1}$, which is a quasi-projective variety via the Segre map. Let $x_1,\dots,x_n$ be the coordinates of $\mathbb{A}^n$, and $y_1,\dots,y_n$ the homogeneous coordinates of $\mathbb{P}^{n-1}$. We recall that the closed subsets of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ are zeros of polynomials in the two series of variables, which are homogeneous in $y_1,\dots,y_n$.

Definition 17.2.1. Let X be the closed subset of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ defined by the system of equations

$$\begin{cases} x_i y_j = x_j y_i, i, j = 1,\dots,n. \end{cases} \tag{17.2}$$

The blow-up of $\mathbb{A}^n$ at O is the variety X together with the map $\sigma : X \to \mathbb{A}^n$ defined by restricting the first projection of $\mathbb{A}^n \times \mathbb{P}^{n-1}$. O is also called the centre of the blow-up.

The equations (17.2) express that $y_1,\dots,y_n$ are proportional to $x_1,\dots,x_n$. Let us see what this means. Let $P \in \mathbb{A}^n$ be a point, we consider $\sigma^{-1}(P)$. We distinguish two cases:

1) If $P \neq O$, then $\sigma^{-1}(P)$ consists of a single point and precisely, if $P = (a_1,\dots,a_n)$, $\sigma^{-1}(P)$ is the pair $((a_1,\dots,a_n),[a_1,\dots,a_n])$.

2) If $P = O$, then $\sigma^{-1}(O) = \{O\} \times \mathbb{P}^{n-1} \simeq \mathbb{P}^{n-1}$, because if $x_1 = \cdots = x_n = 0$ there are no restrictions on $y_1 \dots, y_n$. It is a standard notation to denote $\sigma^{-1}(O)$ by E. It is called the *exceptional divisor* of the blow-up.

It is easy to check that σ gives an isomorphism between $X \setminus \sigma^{-1}(O)$ and $\mathbb{A}^n \setminus \{O\}$. Indeed both σ and σ^{-1} so restricted are regular.

The points of $\sigma^{-1}(O)$ are in bijection with the set of lines through O in $\mathbb{A}^n$. Indeed if L is a line through O, it can be parametrized by $\{x_i = a_i t,\ t \in K$, with $(a_1,\dots,a_n) \neq (0,\dots,0)$. Then $\sigma^{-1}(L \setminus O)$ is parametrized by

$$\begin{cases} x_i = a_i t \\ y_i = a_i t, t \neq 0, \end{cases} \tag{17.3}$$

or, which is the same, by

$$\begin{cases} x_i = a_i t \\ y_i = a_i, t \neq 0. \end{cases} \tag{17.4}$$

If we add also $t = 0$, we find the closure $L' = \overline{\sigma^{-1}(L \setminus O)}$, it is a line meeting $\sigma^{-1}(O)$ at the point $O \times [a_1, \dots, a_n]$: L' can be interpreted as the line L "lifted at the level $[a_1, \dots, a_n]$". So we have a bijection associating to the line L passing through O the point $\overline{\sigma^{-1}(L \setminus O)} \cap \sigma^{-1}(O) = L' \cap E$.

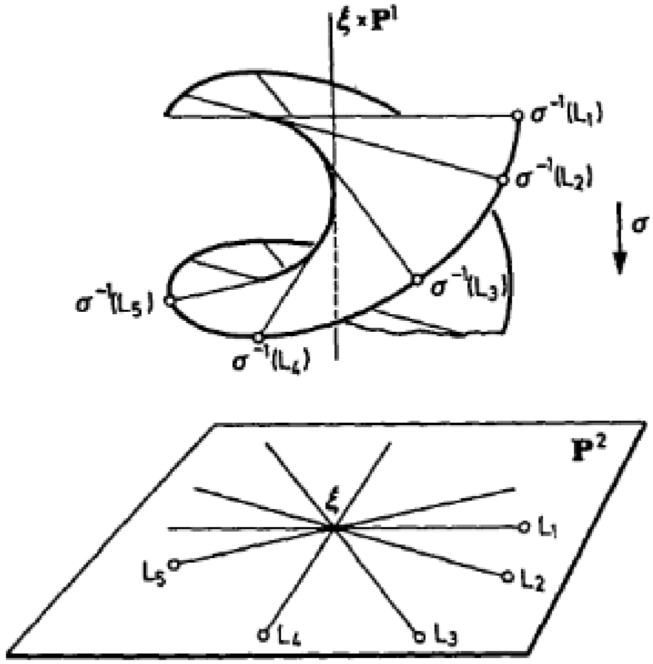


Figure 17.1: Blow-up of the plane

Finally we note that X is irreducible: indeed $X = (X \setminus E) \cup E$; $X \setminus E$ is isomorphic to $\mathbb{A}^n \setminus O$, so it is irreducible; moreover every point of E belongs to a line L', the closure of $\sigma^{-1}(L \setminus O) \subset X \setminus E$. Hence $X \setminus E$ is dense in X, which implies that X is irreducible.

Therefore X is birational to $\mathbb{A}^n$: they are both irreducible and contain the isomorphic open subsets $X \setminus \sigma^{-1}(O)$ and $\mathbb{A}^n \setminus O$. In particular $\dim X = n$, and $\sigma^{-1}(O) = E \simeq \mathbb{P}^{n-1}$ has codimension 1 in X. The tangent space $T_{O,\mathbb{A}^n}$ coincides with $\mathbb{A}^n = K^n$, and the set of the lines through O can be interpreted as the projective space $\mathbb{P}(T_{O,\mathbb{A}^n})$. So there is a bijection between the exceptional divisor E and $\mathbb{P}(T_{O,\mathbb{A}^n})$.

Figure 17.2, taken from the book [S], illustrates the case of the plane.

If we consider the second projection $p_2 : X \to \mathbb{P}^{n-1}$, for any $[a] = [a_1, \dots, a_n] \in \mathbb{P}^{n-1}$, $p_2^{-1}[a]$ is the line L' of (17.4). X with the map p_2 is an example of non-trivial line bundle, called the universal bundle over $\mathbb{P}^{n-1}$.

If Y is a closed subvariety of $\mathbb{A}^n$ passing through O, it is clear that $\sigma^{-1}(Y)$ contains the exceptional divisor $E = \sigma^{-1}(O)$. It is called the total trasform of Y in the blow-up. We define the *strict or proper transform of* Y in the blow-up of $\mathbb{A}^n$ as the closure $\widetilde{Y} := \overline{\sigma^{-1}(Y \setminus O)}$. It is interesting to consider the intersection $\widetilde{Y} \cap E$, it depends on the behaviour of Y in a neighborhood of O, and allows to analyse its singularities at O.

Example 17.2.2.

1. Let $Y \subset \mathbb{A}^2$ be the plane cubic curve of equation $y^2 - x^2 = x^3$. The origin is a singular point of Y, with multiplicity 2, and the tangent cone $TC_{O,Y}$ is the union of the two lines of equations $x - y = 0$, $x + y = 0$, respectively. We consider the blow-up $X \subset \mathbb{A}^2 \times \mathbb{P}^1$ of $\mathbb{A}^2$ with centre O. Using coordinates t_0, t_1 in $\mathbb{P}^1$, X is defined by the unique equation $xt_1 = t_0 y$. Then $\sigma^{-1}(Y)$ is defined by the system

$$\begin{cases} y^2 - x^2 = x^3 \\ xt_1 = t_0 y \end{cases}$$

As usual $\mathbb{P}^1$ is covered by the two open subsets $U_0 : t_0 \neq 0$ and $U_1 : t_1 \neq 0$, so $\mathbb{A}^2 \times \mathbb{P}^1 = (\mathbb{A}^2 \times U_0) \cup (\mathbb{A}^2 \times U_1)$, the union of two copies of $\mathbb{A}^3$, and we can study X considering its intersection X_0, X_1 with each of them. If $t_0 \neq 0$, we use $t = t_1/t_0$ as affine coordinate; if $t_1 \neq 0$ we use $u = t_0/t_1$. X_0 has equation $y = tx$ and X_1 has equation $x = uy$. For $\sigma^{-1}(Y) \cap X_0$ we get the equations $y^2 - x^2 - x^3 = 0$ and $y = tx$ in $\mathbb{A}^3$ with coordinates x, y, t. Substituting we get $t^2x^2 - x^2 - x^3 = x^2(t^2 - 1 - x) = 0$. So there are two components: one is defined by $x = y = 0$, which is $E \cap X_0$; the other is defined by $\begin{cases} x = t^2 - 1 \\ y = t(t^2 - 1) \end{cases}$, it is $\widetilde{Y} \cap X_0$. Note that it meets E at the two points $P(0, 0, 1), Q(0, 0, -1)$. They correspond on E to the two tangent lines to Y at O: $y - x = 0$ and $x + y = 0$.

If we work on the other open set $\mathbb{A}^2 \times U_1$, $\sigma^{-1}(Y)$ is defined by $x = uy$ and $y^2 - u^2y^2 - u^3y^3 = y^2(1 - u^2 - u^3y) = 0$. So $\widetilde{Y} \cap X_1$ is defined by $\begin{cases} x = uy \\ 1 - u^2 - u^3y = 0 \end{cases}$. We find the same two points of intersection with E: $(0, 0, 1), (0, 0, -1)$.

The restriction of the projection $\sigma : \widetilde{Y} \to Y$ is an isomorphism outside the points P, Q on $\widetilde{Y}$ and O on Y. The result is that the two branches of the singularity O have been separated, and the singularity has been resolved.

2. Let $Y \subset \mathbb{A}^2$ be the cuspidal cubic curve of equation $y^2 - x^3 = 0$. The total transform

is defined by

$$\begin{cases} y^2 - x^3 = 0 \\ xt_1 = t_0 y. \end{cases}$$

On the first open subset it becomes $y^2 - x^3 = 0$ together with $y = tx$; replacing and simplifying t, which corresponds to E, we get the equations for $\widetilde{Y}$:

$$\begin{cases} x = t^2 \\ y = t^3 \end{cases}.$$

This is the affine skew cubic, that meets E at the unique point $(0,0,0)$, corresponding to the tangent line to Y at O: $y = 0$. By the way, we can check that E is the tangent line to $\widetilde{Y}$ at $(0,0,0)$. On the second open subset, we have the equations $y^2 - x^3 = 0$ together with $x = uy$; the strict transform is defined by $1 - u^3 y = 0$ and $x = uy$. There is no point of intersection with E in this affine chart. The map $\sigma : \widetilde{Y} \to Y$ is therefore regular, birational, bijective, but not biregular; Y and $\widetilde{Y}$ cannot be isomorphic, because one is smooth and the other is not smooth.

3. Let $Y = V(x^2 - x^4 - y^4) \subset \mathbb{A}^2$. O is a singular point of multiplicity 2 with tangent cone the line $x = 0$ counted twice. Let $\widetilde{Y}$ be the strict transform of Y in the blow-up of the plane in the origin. Proceeding as in the previous example we find that $\widetilde{Y}$ meets the exceptional divisor $E = O \times \mathbb{P}^1$ at the point $O' = ((0,0),[0,1])$, which belongs only to the second open subset $\mathbb{A}^2 \times U_1$. In coordinates $x, y, u = t_0/t_1$, $\widetilde{Y}$ is defined by the equations

$$\begin{cases} x = uy \\ u^2 - u^4 y^2 - y^2 = 0 \end{cases},$$

and $O' = (0,0,0)$. We compute the equation of the tangent space $T_{O',\widetilde{Y}}$, it is $x = 0$: it is a 2-plane in $\mathbb{A}^3$, so $\widetilde{Y}$ is singular at O'. The tangent cone $TC_{O',\widetilde{Y}}$ is $x = 0, u^2 - y^2 = 0$, the union of two lines in the tangent plane.

Let us consider a second blow-up σ', of $\mathbb{A}^3$ in O'. It is contained in $\mathbb{A}^3 \times \mathbb{P}^2$; using coordinates z_0, z_1, z_2 in $\mathbb{P}^2$, it is defined by

$$rk \begin{pmatrix} x & y & u \\ z_0 & z_1 & z_2 \end{pmatrix} < 2.$$

We first work on the open subset $\mathbb{A}^3 \times U_0 \simeq \mathbb{A}^5$; we put $z_0 = 1$ and we work with affine coordinates x, y, u, z_1, z_2; the exceptional divisor E' is defined by $x = y = u = 0$, and the total transform $\sigma'^{-1}(\widetilde{Y})$ of $\widetilde{Y}$ by

$$\begin{cases} x = uy \\ y = z_1 x \\ u = z_2 x \\ x^2(z_2^2 - z_1^2 - u^4 z_1^2) = 0 \end{cases}.$$

Replacing $x = uy$ in the second and third equation we get the equivalent system

$$\begin{cases} x = uy \\ y(1 - z_1 u) = 0 \\ u(1 - z_2 y) = 0 \\ x^2(z_2^2 - z_1^2 - u^4 z_1^2) = 0 \end{cases}.$$

Combining the factors of the four equations in all possible ways, we find that, on $\mathbb{A}^3 \times U_0$, $\sigma'^{-1}(\widetilde{Y})$ is union of E' and of the strict transform $\widetilde{Y}'$ defined by

$$\begin{cases} x = uy \\ 1 - z_1 u = 0 \\ 1 - z_2 y = 0 \\ z_2^2 - z_1^2 - u^4 z_1^2 = 0 \end{cases}.$$

The intersection $\widetilde{Y}' \cap E' \cap (\mathbb{A}^3 \times U_0)$ results to be empty.

We then work on the open subset $\mathbb{A}^3 \times U_1 \simeq \mathbb{A}^5$; we put $z_1 = 1$ and we work with affine coordinates x, y, u, z_0, z_2. Proceeding as in the first case, we find the equations of the total transform

$$\begin{cases} x = uy \\ y(z_0 - u) = 0 \\ u = z_2 y \\ y^2(z_2^2 - 1 - z_2^4 y^4) = 0 \end{cases}.$$

The strict transform results to be defined by

$$\begin{cases} x = uy \\ z_0 - u = 0 \\ u = z_2 y \\ z_2^2 - 1 - z_2^4 y^4 = 0 \end{cases},$$

and its intersection with the exceptional divisor $x = y = u = 0$ is the union of the two points P, Q of coordinates $((0,0,0),[0,1,\pm 1]) \in \mathbb{A}^3 \times \mathbb{P}^2$. Considering the third open subset $\mathbb{A}^3 \times U_2 \simeq \mathbb{A}^5$ one finds the same two points.

In conclusion, we consider the composition of the two blow-ups $\widetilde{Y}' \xrightarrow{\sigma'} \widetilde{Y} \xrightarrow{\sigma} Y$, which is birational. In the first blow-up σ, we pass from Y, with a singularity at the blown-up point O with one tangent line, to $\widetilde{Y}$ with a node in O', its point of intersection with E. In the second blow-up σ', O' is replaced by two points on the second exceptional divisor E'. To verify if $\widetilde{Y}'$ is smooth, it is enough to check if P, Q are smooth, and this can be checked easily (taking the differentials at P and Q of the equations of $\widetilde{Y}'$).

The singularity of Y is called a *tacnode*. We have just checked that to resolve it two blow-ups are needed. What allows to distinguish the singularity of the curve of Example 2 from the present example, is the multiplicity of intersection at the point O of the tangent line at the singular point O with the curve: it is 3 in Example 2 and 4 in Example 3.

The general problem of the *resolution of singularities* is, given a variety Y, to find a birational morphism $f : Y' \to Y$ with Y' non-singular. It is possible to prove that, if Y is a curve, the problem can be solved with a finite sequence of blow-ups. If $\dim Y > 1$, the problem is much more difficult, and is presently completely solved only in characteristic 0 (see for instance [rH], Ch. V, 3).

To conclude this chapter, we will see a different way to introduce the blow-up of $\mathbb{A}^n$ at O. Let $p : \mathbb{A}^n \setminus O \to \mathbb{P}^{n-1}$ be the natural projection $(a_1, \ldots, a_n) \to [a_1, \ldots, a_n]$. Let Γ be the graph of p, $\Gamma \subset (\mathbb{A}^n \setminus O) \times \mathbb{P}^{n-1} \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$. We immediately have that the closure of Γ in $\mathbb{A}^n \times \mathbb{P}^{n-1}$ is precisely the blow-up X of $\mathbb{A}^n$ at O. This interpretation suggests how to extend Definition 17.2.1 and define the blow up of a variety X along a subvariety Y.

Suppose that X is an affine variety and $I = I_X(Y) \subset K[X]$ is the ideal of a subvariety Y of X. Suppose that $I = (f_0, \ldots, f_r)$. Let λ be the rational map $X \dashrightarrow \mathbb{P}^r$ defined by $\lambda = [f_0, \ldots, f_r]$. The blow-up of X along Y is the closure of the graph of λ, together with the projection map to X. Similarly one can define the blow-up of a projective variety along a subvariety, provided that its ideal is generated by homogeneous polynomials all of the same degree. For details, see for instance [C].

Exercises 17.2.3. Let $Y \subset \mathbb{P}^2$ be a smooth plane projective curve of degree $d > 1$, defined by the equation $f(x, y, z) = 0$. Let $C(Y) \subset \mathbb{A}^3$ be the affine variety defined by the same polynomial f: $C(Y)$ is the affine cone of Y. Let $O(0,0,0) \in \mathbb{A}^3$ be the origin, vertex of $C(Y)$. Let $\sigma : X \to \mathbb{A}^3$ be the blow-up in O.

1. Show that $C(Y)$ has only one singular point, the vertex O;

2. show that $\widetilde{C(Y)}$, the strict transform of $C(Y)$, is nonsingular (cover it with open affine subsets);

3. let E be the exceptional divisor; show that $\widetilde{C(Y)} \cap E$ is isomorphic to Y.

Chapter 18

Grassmannians

In this chapter we will see how the antisymmetric tensors play an important role in algebraic geometry, providing an ambient space in which naturally embeds the Grassmannian of subspaces of fixed dimension of a vector space, or, equivalently, of a projective space.

18.1 Exterior powers of a vector space

To define the exterior powers of the vector space V, one proceeds in a way which is similar to the one used to define its symmetric powers. We define the d-th exterior power $\wedge^d V$ as the quotient $V^{\otimes d}/\Lambda$, where Λ is generated by the tensors of the form $v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_j \otimes \cdots \otimes v_d$, with $v_i = v_j$ for some $i \neq j$. The following notation is used: $[v_1 \otimes \cdots \otimes v_d] = v_1 \wedge \cdots \wedge v_d$.

There is a natural multilinear alternating map $V \times \cdots \times V = V^d \to \wedge^d V$, that enjoys the universal property. Given a basis $\mathcal{B} = (e_1, \ldots, e_n)$ of V, a basis of $\wedge^d V$ is formed by the tensors $e_{i_1} \wedge \ldots \wedge e_{i_d}$, with $1 \leq i_1 < \ldots < i_d \leq n$. Therefore $\dim \wedge^d V = \binom{n}{d}$. The exterior algebra of V is the following direct sum: $\wedge V = \oplus_{d \geq 0} \wedge^d V = K \oplus V \oplus \wedge^2 V \oplus \ldots$. To define an inner product that gives it the structure of algebra we can proceed as follows.

Step 1. Fixed $v_1, \ldots, v_p \in V$, for any d we define $f : V^d \to \wedge^{d+p} V$ posing $f(x_1, \ldots, x_d) = x_1 \wedge \ldots \wedge x_d \wedge v_1 \wedge \ldots \wedge v_p$. Since f results to be multilinear and alternating, by the universal property we get a factorization of f through $\wedge^d V$, which gives a linear map $\bar{f} : \wedge^d V \to \wedge^{d+p} V$, extending f. For any $\omega \in \wedge^d V$, we denote $\bar{f}(\omega)$ by $\omega \wedge v_1 \wedge \ldots \wedge v_p$.

Step 2. Fixed $\omega \in \wedge^d V$, consider the map $g : V^p \to \wedge^{d+p} V$ such that $g(y_1, \ldots, y_p) = \omega \wedge y_1 \wedge \ldots \wedge y_p$: it is multilinear and alternating, therefore it factorizes through $\wedge^p V$ and we get a linear map $\bar{g} : \wedge^p V \to \wedge^{d+p} V$, extending g. We denote $\bar{g}(\sigma) := \omega \wedge \sigma$.

Step 3. For any $d, p \geq 0$ we have got a map $\wedge : \wedge^d V \times \wedge^p V \to \wedge^{d+p} V$, that results to be bilinear, and extends to an inner product $\wedge : (\wedge V) \times (\wedge V) \to \wedge V$, which gives $\wedge V$ the required structure of algebra. It is a graded algebra, the non-zero homogeneous components are those of degree from 0 to $n = \dim V$.

Proposition 18.1.1. *Let V be a vector space of dimension n.*

(i) Vectors $v_1, \dots, v_p \in V$ are linearly dependent if and only if $v_1 \wedge \dots \wedge v_p = 0$.

(ii) Let $v \in V$ be a non-zero vector, and $\omega \in \wedge^p V$. Then $\omega \wedge v = 0$ if and only if there exists $\Phi \in \wedge^{p-1} V$ such that $\omega = \Phi \wedge v$. In this case we say that v divides ω.

Proof. The proof of (i) is standard. If $\omega = \Phi \wedge v$, then $\omega \wedge v = (\Phi \wedge v) \wedge v = \Phi \wedge (v \wedge v) = 0$. Conversely, if $\omega \wedge v = 0$, $v \neq 0$, we choose a basis of V, $\mathcal{B} = (e_1, \dots, e_n)$ with $e_1 = v$. Write $\omega = \Sigma_{i_1 < \dots < i_p} a_{i_1 \dots i_p} e_{i_1} \wedge \dots \wedge e_{i_p}$. Then $0 = \omega \wedge e_1 = \Sigma_{i_1 < \dots < i_p} (\pm) a_{i_1 \dots i_p} e_1 \wedge e_{i_1} \wedge \dots \wedge e_{i_p}$. If $i_1 = 1$, the corresponding summand does not appear in this sum, so it remains a linear combination of linearly independent tensors, which implies $a_{i_1 \dots i_p} = 0$ every time $i_1 > 1$. Therefore $\omega = e_1 \wedge \Phi$ for a suitable Φ. □

Proposition 18.1.2. *Let $\omega \neq 0$ be an element of $\wedge^p V$. Then ω is totally decomposable if and only if the subspace of V: $W = \{v \in V \mid v \text{ divides } \omega\}$ has dimension p.*

Proof. If $\omega = x_1 \wedge \dots \wedge x_p \neq 0$, then $x_1, \dots, x_p$ are linearly independent and belong to W. So we can extend them to a basis of V adding vectors $x_{p+1}, \dots, x_n$. If $v \in W$, $v = \alpha_1 x_1 + \dots + \alpha_n x_n$, and v divides ω, then $\omega \wedge v = 0$, i.e. $x_1 \wedge \dots \wedge x_p \wedge (\alpha_1 x_1 + \dots + \alpha_n x_n) = 0$. This implies $\alpha_{p+1} x_1 \wedge \dots \wedge x_p \wedge x_{p+1} + \dots + \alpha_n x_1 \wedge \dots \wedge x_p \wedge x_n = 0$, therefore $\alpha_{p+1} = \dots = \alpha_n = 0$, so $v \in \langle x_1, \dots, x_p \rangle$.

Conversely, if $(x_1, \dots, x_p)$ is a basis of W, we can complete it to a basis of V and write $\omega = \Sigma a_{i_1 \dots i_p} x_{i_1} \wedge \dots \wedge x_{i_p}$. But x_1 divides ω, so $\omega \wedge x_1 = 0$. Replacing ω with its explicit expression, we obtain that $a_{i_1 \dots i_p} = 0$ if $1 \notin \{i_1, \dots, i_p\}$. Repeating this argument for $x_2, \dots, x_p$, it remains $\omega = a_{1 \dots p} x_1 \wedge \dots \wedge x_p$. □

With explicit computations, one can prove the following proposition.

Proposition 18.1.3. *Let V be a vector space with $\dim V = n$. Let $\mathcal{B} = (e_1, \dots, e_n)$ be a basis of V and $v_1, \dots, v_n$ be any vectors. Then $v_1 \wedge \dots \wedge v_n = \det(A) e_1 \wedge \dots \wedge e_n$, where A is the matrix of the coordinates of the vectors $v_1, \dots, v_n$ with respect to $\mathcal{B}$.*

Corollary 18.1.4. *Let $v_1, \dots, v_p \in V$, with $v_i = \Sigma a_{ij} e_j$, $i = 1, \dots, p$. Then $v_1 \wedge \dots \wedge v_p = \Sigma_{i_1 < \dots < i_p} a_{i_1 \dots i_p} e_{i_1} \wedge \dots \wedge e_{i_p}$, with $a_{i_1 \dots i_p} = \det(A_{i_1 \dots i_p})$, the determinant of the $p \times p$ submatrix of A containing the columns of indices $i_1, \dots, i_p$.*

18.2 The Plücker embedding

We are now ready to introduce the Grassmannian and to give it an interpretation as projective variety via the Plücker map. Let V be a vector space of dimension n, and r be a positive integer, $1 \leq r \leq n$. The Grassmannian $G(r, V)$ is the set whose elements are the subspaces of V of dimension r. It is usual also to denote it by $G(r, n)$.

There is a natural bijection between $G(r, V)$ and the set of the projective subspaces of $\mathbb{P}(V)$ of dimension $r-1$, denoted by $\mathbb{G}(r-1, \mathbb{P}(V))$ or $\mathbb{G}(r-1, n-1)$. Let $W \in G(r, V)$; if $(w_1, \dots, w_r)$ and $(x_1, \dots, x_r)$ are two bases of W, then $w_1 \wedge \cdots \wedge w_r = \lambda x_1 \wedge \cdots \wedge x_r$, where $\lambda \in K$ is the determinant of the matrix of the change of basis. Therefore W uniquely determines an element of $\wedge^r V$ up to proportionality. This allows to define a map, called the Plücker map, $\psi : G(r, V) \to \mathbb{P}(\wedge^r V)$, such that $\psi(W) = [w_1 \wedge \cdots \wedge w_r]$.

Proposition 18.2.1. *The Plücker map is injective.*

Proof. Assume $\psi(W) = \psi(W')$, where W, W' are subspaces of V of dimension r with bases $(x_1, \dots, x_r)$ and $(y_1, \dots, y_r)$. So there exists $\lambda \neq 0$ in K such that $x_1 \wedge \cdots \wedge x_r = \lambda y_1 \wedge \cdots \wedge y_r$. This implies $x_1 \wedge \cdots \wedge x_r \wedge y_i = 0$ for any i, so y_i is linearly dependent from $x_1, \dots, x_r$, so $y_i \in W$. Therefore $W' \subset W$. The reverse inclusion is similar. $\square$

In coordinates with respect to the basis of $\wedge^r V$ $\{e_{i_1} \wedge \ldots \wedge e_{i_r}, 1 \leq i_1 < \ldots < i_r \leq n\}$, $\psi(W)$ is given by the minors of maximal order r of the matrix of the coordinates of the vectors of a basis of W, with respect to $e_1, \dots, e_n$.

Example 18.2.2.

(i) $r = n-1$: $\wedge^{n-1} V$ has dimension n. It results to be isomorphic to the dual vector space V^*, and an explicit isomorphism is obtained associating to $e_1 \wedge \cdots \wedge \hat{e}_k \wedge \cdots \wedge e_n$ the linear form e_k^* of the dual basis. In this case the Plücker map is surjective, so $\psi(G(n-1, n)) \simeq \mathbb{P}(V^*)$.

(ii) $n = 4, r = 2$: $G(2, 4)$ or $\mathbb{G}(1, 3)$, the Grassmannian of lines in $\mathbb{P}^3$. In this case $\psi : \mathbb{G}(1, 3) \to \mathbb{P}(\wedge^2 V) \simeq \mathbb{P}^5$. Let (e_0, e_1, e_2, e_3) be a basis of V. Let $\ell = \mathbb{P}(L)$ be the line of $\mathbb{P}^3$ obtained by projectivisation of the vector subspace $L \subset V$ of dimension 2, let $L = \langle x, y \rangle$; then $\psi(\ell) = [x \wedge y]$. Its Plücker coordinates are traditionally denoted by $p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23}$, with $p_{ij} = x_i y_j - x_j y_i$, the 2×2 minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{pmatrix}.$$

This time ψ is not surjective; its image is the subset of $\wedge^2 V$ of the totally decomposable tensors. Assume $char(K) \neq 2$. They satisfy the equation of degree 2: $p_{01}p_{23} - p_{02}p_{13} +$

$p_{03}p_{12} = 0$, which represents a quadric of maximal rank in $\mathbb{P}^5$, called the Klein quadric. The fact that this equation is satisfied can be seen by considering the 4×4 matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{pmatrix}:$$

its determinant is precisely the above equation (consider the development of the determinant according to the first two rows).

For instance the line of equations $x_2 = x_3 = 0$, obtained projectivising the subspace $\langle e_0, e_1 \rangle$, has Plücker coordinates $[1, 0, 0, 0, 0, 0]$.

In general we can prove the following theorem.

Theorem 18.2.3. *The image of the Plücker map is a closed subset in $\mathbb{P}(\wedge^r V)$.*

Proof. The image of the Plücker map is the set of the proportionality classes of totally decomposable tensors. By Proposition 18.1.2, a tensor $\omega \in \wedge^r V$ is totally decomposable if and only if the subspace $W = \{v \in V \mid v \text{ divides } \omega\}$ has dimension r. We consider the linear map $\Phi : V \to \wedge^{r+1} V$, such that $\Phi(v) = \omega \wedge v$. The kernel of Φ is equal to W. So ω is totally decomposable if and only if the rank of Φ is $n - r$. Fixed a basis $\mathcal{B} = (e_1, \ldots, e_n)$ of V, we write $\omega = \Sigma_{i_1 < \cdots < i_r} a_{i_1 \ldots i_r} e_{i_1} \wedge \ldots \wedge e_{i_r}$. We then consider the basis of $\wedge^{r+1} V$ associated to $\mathcal{B}$ and we construct the matrix A of Φ with respect to these bases: its minors of order $n - r + 1$ are equations of the image of ψ, and they are polynomials in the coordinates $a_{i_1 \ldots i_r}$ of ω. $\square$

From now on we shall identify the Grassmannian with the projective algebraic set that is its image in the Plücker map. The equations obtained in Theorem 18.2.3 are nevertheless not generators for the ideal of the Grassmannian. For instance, in the case $n = 4, r = 2$, let $\omega = p_{01} e_0 \wedge e_1 + p_{02} e_0 \wedge e_2 + \ldots$. Then:

$\Phi(e_0) = \omega \wedge e_0 = p_{12} e_0 \wedge e_1 \wedge e_2 + p_{13} e_0 \wedge e_1 \wedge e_3 + p_{23} e_0 \wedge e_2 \wedge e_3$;
$\Phi(e_1) = \omega \wedge e_1 = -p_{02} e_0 \wedge e_1 \wedge e_2 - p_{03} e_0 \wedge e_1 \wedge e_3 + p_{23} e_1 \wedge e_2 \wedge e_3$;
$\Phi(e_2) = \omega \wedge e_2 = p_{01} e_0 \wedge e_1 \wedge e_2 - p_{03} e_0 \wedge e_2 \wedge e_3 + p_{13} e_1 \wedge e_2 \wedge e_3$;
$\Phi(e_3) = \omega \wedge e_3 = p_{01} e_0 \wedge e_1 \wedge e_3 + p_{02} e_0 \wedge e_2 \wedge e_3 + p_{12} e_1 \wedge e_2 \wedge e_3$.

So the matrix is

$$\begin{pmatrix} p_{12} & -p_{02} & p_{01} & 0 \\ p_{13} & -p_{03} & 0 & p_{01} \\ p_{23} & 0 & -p_{03} & p_{02} \\ 0 & p_{23} & p_{13} & p_{12} \end{pmatrix}.$$

Its 3×3 minors are equations defining $\mathbb{G}(1,3)$, but the radical of the ideal generated by these minors is in fact $(p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12})$.

To find equations for the Grassmannian and to prove that it is irreducible, it is convenient to give an explicit open covering with affine open subsets. In $\mathbb{P}(\wedge^r V)$, let $U_{i_1 \dots i_r}$ be the affine open subset where the Plücker coordinate $p_{i_1 \dots i_r} \neq 0$. To simplify notation we assume $i_1 = 1, i_2 = 2, \dots, i_r = r$, and we put $U = U_{1 \dots r}$. If $W \in G(r,n) \cap U$, and $w_1, \dots, w_r$ is a basis of W, then the first minor of the matrix M of the coordinates of $w_1, \dots, w_r$ is non-degenerate. So we can choose a new basis of W such that M is of the form

$$M = \begin{pmatrix} 1 & 0 & \dots & 0 & \alpha_{1,r+1} & \dots & \alpha_{1,n} \\ 0 & 1 & \dots & 0 & \alpha_{2,r+1} & \dots & \alpha_{2,n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & \alpha_{r,r+1} & \dots & \alpha_{r,n} \end{pmatrix}.$$

Conversely, any matrix of this form defines a subspace $W \in G(r,n) \cap U$. So there is a bijection between $G(r,n) \cap U$ and $K^{r(n-r)}$, i.e. the affine space of dimension $r(n-r)$. The coordinates of W result to be equal to 1 and all minors of all orders of the submatrix of the last $n-r$ columns of M. Therefore they are expressed as polynomials in the $r(n-r)$ elements of the last $n-r$ columns of M. This shows that $G(r,n) \cap U$ is an affine subvariety of U isomorphic to $\mathbb{A}^{r(n-r)}$. By homogenising the equations obtained in this way, one gets equations for $G(r,n)$.

For instance, in the case $n = 4, r = 2$, the matrix M becomes

$$M = \begin{pmatrix} 1 & 0 & \alpha_{13} & \alpha_{14} \\ 0 & 1 & \alpha_{23} & \alpha_{24} \end{pmatrix}.$$

One gets $1 = p_{01}, \alpha_{23} = p_{02}, \alpha_{24} = p_{03}, -\alpha_{13} = p_{12}, -\alpha_{14} = p_{13}, \alpha_{13}\alpha_{24} - \alpha_{23}\alpha_{14} = p_{23}$. If we make the substitutions and homogenise the last equation with respect to p_{01}, we find the equation of the Klein quadric $p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0$.

Theorem 18.2.4. *$G(r,n)$ is an irreducible projective variety of dimension $r(n-r)$, and it is rational.*

Proof. We remark that $G(r,n) \cap U_{i_1 \dots i_r}$ is the set of the subspaces W which are complementar to the subspace of equations $x_{i_1} = \dots = x_{i_r} = 0$. It is clear that they have two by two non-empty intersection. Therefore, the projective algebraic set $G(r,n)$ has an affine open covering with irreducible varieties isomorphic to $\mathbb{A}^{r(n-r)}$. Using Exercise 5 of Chapter 6, we conclude that $G(r,n)$ is irreducible. Its dimension is equal to the dimension of any open subset of the open covering, $r(n-r)$. Since it is irreducible and contains open subsets isomorphic to the affine space, it is rational. □

Assume $char(K) \neq 2$. In the special case $r=2$ with $n \geq 4$, using the Plücker coordinates $[\dots, p_{ij}, \dots]$, the equations of the Grassmannian $G(2,n)$ are of the form $p_{ij}p_{hk} - p_{ih}p_{jk} + p_{ik}p_{jh} = 0$, for any $i < j < h < k$.

Also in the case of $G(2,n)$, as for $\mathbb{P}^n \times \mathbb{P}^m$ and $V_{n,2}$, there is an interpretation in terms of matrices, that I expose here without entering in all the details. Given a tensor in $\wedge^2 V$ with coordinates $[p_{ij}]$, we can consider the skew-symmetric $n \times n$ matrix whose term of position i,j is p_{ij}, with the conditions $p_{ii} = 0$ and $p_{ji} = -p_{ij}$. In this way we can construct an isomorphism between $\wedge^2 V$ and the vector space of skew-symmetric matrices of order n.

From ${}^tA = -A$, it follows $\det(A) = (-1)^n \det(A)$. If n is odd, this implies $\det(A) = 0$. If n is even, one can prove that $\det(A)$ is a square. For instance if $n = 2$, and $A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$, then $\det(A) = a^2$.

If $n = 4$, and $P = \begin{pmatrix} 0 & p_{12} & p_{13} & p_{14} \\ -p_{12} & 0 & p_{23} & p_{24} \\ -p_{13} & -p_{23} & 0 & p_{34} \\ -p_{14} & -p_{24} & -p_{34} & 0 \end{pmatrix}$, then $\det(P) = (p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23})^2$.

In general, for a skew-symmetric matrix A of even order $2n$, one defines the **pfaffian of** A, $pf(A)$, in one of the following equivalent ways:

(i) by recursion: if $n = 1$, $pf \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = a$; if $n > 1$, one defines

$$pf(A) = \Sigma_{i=2}^{2n} (-1)^i a_{1i} Pf(A_{1i}),$$

where A_{1i} is the matrix obtained from A by removing the rows and the columns of indices 1 and i. Then one verifies that $pf(A)^2 = \det(A)$;

(ii) (in characteristic 0) given the matrix A, one considers the tensor $\omega = \Sigma_{i,j=1}^{2n} a_{ij} e_i \wedge e_j \in \wedge^2 K^{2n}$. Then one defines the pfaffian of A as the unique constant such that $pf(A) e_1 \wedge \dots \wedge e_{2n} = \frac{1}{n!} \omega \wedge \dots \wedge \omega$.

For a skew-symmetric matrix of odd order, one defines the pfaffian to be 0.

Proposition 18.2.5. *A 2-tensor $\omega \in \wedge^2 V$ is totally decomposable if and only if $\omega \wedge \omega = 0$.*

Proof. If ω is decomposable, the conclusion easily follows. Conversely, if $\omega = \Sigma_{i,j=1}^{2n} a_{ij} e_i \wedge e_j$ and $\omega \wedge \omega = 0$, then the pfaffians of the principal minors of order 4 of the matrix A corresponding to ω are all 0, therefore from definition (ii) it follows that the pfaffians of the principal minors of all orders are 0, and also $\det(A) = 0$. In conclusion A has rank 2. Then one checks that ω is the $\wedge$ product of two vectors corresponding to two linearly independent rows of A. For instance, if $a_{12} \neq 0$, then $\omega = (a_{12}e_2 + \ldots + a_{1n}e_n) \wedge (-a_{12}e_1 + a_{23}e_3 + \ldots + a_{2n}e_n)$. □

The equations of $G(2, n)$ are the pfaffians of the principal minors of order 4 of the matrix P. They are all zero if and only if the rank of P is 2. Therefore the points of the Grassmannian $G(2, n)$, for any n, can be interpreted as (proportionality classes of) skew-symmetric matrices of order n and rank 2.

The subvarieties of the Grassmannian $\mathbb{G}(r, n)$ correspond to subvarieties of $\mathbb{P}^n$ covered by linear spaces of dimension r. Conversely, any subvariety of $\mathbb{P}^n$ covered by linear spaces of dimension r gives rise to a subvariety of the Grassmannian.

Example 18.2.6.

1. **Pencils of lines.** A pencil of lines in $\mathbb{P}^n$ is the set of lines passing through a fixed point O and contained in a 2-plane π such that $O \in \pi$. Assume that O has coordinates $[y_0, \ldots, y_n]$, and fix two points $A, B \in \pi$, different from O. Let $A = [a_0, \ldots, a_n]$, $B[b_0, \ldots, b_n]$. Then a general line of the pencil is generated by O and by a point of coordinates $[\ldots, \lambda a_i + \mu b_i, \ldots]$. Therefore the Plücker coordinates of a general line of the pencil are $p_{ij} = y_i(\lambda a_j + \mu b_j) - y_j(\lambda a_i + \mu b_i) = \lambda q_{ij} + \mu q'_{ij}$, where q_{ij}, q'_{ij} are the Plücker coordinates of the lines OA and OB respectively. So the lines of the pencil are represented in the Grassmannian by the points of a line. Conversely one can check that any line contained in a Grassmannian of lines represents the lines of a pencil.

2. **Lines in a smooth quadric surface.** Let $\Sigma : x_0x_3 - x_1x_2 = \det \begin{pmatrix} x_0 & x_1 \\ x_2 & x_3 \end{pmatrix} = 0$ be the Segre quadric in $\mathbb{P}^3$. A line of the first ruling of Σ is characterised by a constant ratio of the rows of the matrix $\begin{pmatrix} x_0 & x_1 \\ x_2 & x_3 \end{pmatrix}$. Therefore it can be generated by two points with coordinates $[x_0, x_1, 0, 0]$, $[0, 0, x_0, x_1]$. The Plücker coordinates of such a line are $[x_0^2, 0, x_0x_2, -x_0x_2, 0, x_2^2]$. This parametrizes a conic contained in $\mathbb{G}(1, 3)$. Similarly, the lines of the second ruling describe the points of another conic, indeed the coordinates are $[0, x_0^2, x_0x_1, x_0x_1, x_1^2, 0]$. These two conics are disjoint and contained in disjoint planes.

3. **Planes in** $\mathbb{G}(1,3)$. One can prove that $\mathbb{G}(1,3)$ contains two families of planes, and no linear space of dimension > 2. The planes of one family correspond to stars of lines in $\mathbb{P}^3$ (lines in $\mathbb{P}^3$ through a fixed point), while the planes of the second family correspond to the lines contained in the planes of $\mathbb{P}^3$. The geometry of the lines in $\mathbb{P}^3$ translates to give a decription of the geometry of the planes contained in $\mathbb{G}(1,3)$. Since on an algebraically closed field of characteristic $\neq 2$ two quadric hypersurfaces are projectively equivalent if and only if they have the same rank, one obtains a description of the geometry of all quadrics of maximal rank in $\mathbb{P}^5$.

Exercises 18.2.7. 1. Let ℓ, ℓ' two distinct lines in $\mathbb{P}^3$. Let $[p_{ij}]$ be the Plücker coordinates of ℓ and $[q_{ij}]$ those of ℓ', $0 \leq i < j \leq 3$. Prove that $\ell \cap \ell' \neq \emptyset$ if and only if

$$p_{01}q_{23} - p_{02}q_{13} + p_{03}q_{12} + p_{12}q_{03} - p_{13}q_{02} + p_{23}q_{01} = 0.$$

(Hint: fix points on the two lines to get the Plücker coordinates.)

Chapter 19

Fibres of a morphism and lines on hypersurfaces

In this last chapter we will state the Theorem on the dimension of the fibres of a morphism, and we will see an application, involving Grassmannians, about the existence of lines on a hypersurface of given degree in a projective space.

19.1 Fibres of a morphism

Let us recall that the *fibres of a morphism* are the inverse images of the points of the codomain. More precisely, if $f : X \to Y$ is a morphism, for any $y \in Y$, the fibre of f over y is $f^{-1}(y)$. Since in the Zariski topology every point is closed, the fibre $f^{-1}(y)$ is closed in X, and we want to study the dimensions of its irreducible components. We have seen in Chapter 17 that finite morphisms have the property that all the fibres are finite and non-empty, so all irreducible components have dimension 0.

The following theorem gives informations about the behaviour of the fibres of general morphisms.

Theorem 19.1.1 (Theorem on the dimension of the fibres.)**.** *Let $f : X \to Y$ be a dominant morphism of algebraic sets. Then:*

1. $\dim(X) \geq \dim(Y)$*;*

2. for any $y \in Y$, and for any irreducible component F of $f^{-1}(y)$, $\dim F \geq \dim(X) - \dim(Y)$*;*

3. there exists a non-empty open subset $U \subset Y$, such that $\dim f^{-1}(y) = \dim(X) - \dim(Y)$ *for any $y \in U$;*

4. the sets $Y_k = \{y \in Y \mid \dim f^{-1}(y) \geq k\}$ are closed in Y (upper semicontinuity of the dimension of the fibres).

Before giving a sketch of the proof, let us see an example.

Example 19.1.2. Let V be an affine variety and consider $W \subset V \times \mathbb{A}^r$ defined by s linear equations with coefficients in $K[V]$:

$$\left\{ \sum_{j=1}^r a_{ij}x_j = 0, \ a_{ij} \in K[V], \ i = 1, \dots, s, \right.$$

where $x_1, \dots, x_r$ are coordinates on $\mathbb{A}^r$. Let $\varphi : W \to V$ be the projection. For $P \in V$, $\varphi^{-1}(P)$ is the set of solutions of the system of linear equations with constant coefficients

$$\sum_{j=1}^r a_{ij}(P)x_j, \ a_{ij}(P) \in K, \ i = 1, \dots, s,$$

so its dimension is $r - rk(a_{ij}(P))$. For any $k \in \mathbb{N}$ the set $\{P \in V \mid rk(a_{ij}(P)) \leq k\}$ is closed in V, defined by the vanishing of the minors of order $k+1$, and it is precisely V_{r-k}, the subset of V where the dimension of the fibre is $\geq r-k$.

The meaning of this example is that we have a family of subspaces of $\mathbb{A}^r$ defined by a system of linear equations with coefficients parametrized by V. A "general" space of the family has minimal dimension $r - rkA$, where $A = (a_ij)$ is the matrix of the coefficients of the system. General spaces correspond to the points of an open non-empty subset of V. There are closed subsets in V corresponding to spaces of higher dimension, where the rank of A decreases.

Proof of Theorem 19.1.1. 1. Since f is dominant, there is the K-homomorphism $f^* : K(Y) \hookrightarrow K(X)$, and $tr.d.K(Y)/K \leq tr.d.K(X)/K$, because algebraically independent elements of $K(Y)$ remain algebraically independent in $K(X)$. So $\dim(Y) \leq \dim(X)$.

2. Fix $y \in Y$. We observe that we can replace Y with an affine open neighborhood U of y and X with $f^{-1}(U)$. So we can assume that Y is closed in an affine space $\mathbb{A}^N$. Let $n = \dim(X), m = \dim(Y)$. We observe that we can find a polynomial G in N variables which does not vanish identically on any irreducible component of Y. For instance, we can fix a point on any irreducible component and choose a hyperplane not passing through any of these points. Then all irreducible components of $Y^{(1)} := Y \cap V(G)$ have dimension $m-1$. Repeating this argument, we can find a chain of subvarieties of Y of the form $Y \supset Y^{(1)} \supset \dots \supset Y^{(m)} \supset Y^{(m+1)}$, where all irreducible components of $Y^{(i)}$ have dimension $m-i$. In particular the irreducible components of $Y^{(m)}$ are points, among which there is y, and $Y^{(m)}$ is defined by m equations of the form $g_1 = \dots = g_m = 0$, with $g_1, \dots, g_m \in K[Y]$.

Possibly restricting the open set U, we can assume that $Y^{(m)} \cap U = \{y\}$. Hence, the fibre $f^{-1}(y)$ is defined by the system of m equations $f^*(g_1) = \cdots = f^*(g_m) = 0$. The conclusion follows from the Theorem of the intersection 14.1.1.

3. See [S].

4. By induction on the dimension of Y. It is obviously true if $\dim Y = 0$. We know from 3. that there is an open subset U of Y such that $\dim f^{-1}(y) = n - m$ if and only if $y \in U$. Let Z be the complement of U in Y; thus $Z = Y_{n-m+1}$. Let $Z_1, \dots, Z_r$ be the irreducible components of Z. We can now apply the induction to the restrictions of f, $f^{-1}(Z_j) \to Z_j$ for each j , and we obtain the result. □

As a consequence of Theorem 19.1.1, we are able to prove the following very useful proposition.

Proposition 19.1.3. *Let $f : X \to Y$ be a surjective morphism of projective algebraic sets. Assume that Y is irreducible and that all fibres of f are irreducible and of the same dimension r, then X is irreducible of dimension* $\dim(Y) + r$.

Proof. Note first of all that $r = \dim(X) - \dim(Y)$. Let Z be an irreducible closed subset of X, and consider the restriction $f|_Z : Z \to Y$; its fibres are $f|_Z^{-1}(y) = f^{-1}(y) \cap Z$. There are three possibilities:

(a) $f(Z) \neq Y$. Then $f(Z)$ is a proper closed subset of Y;

(b) $f(Z) = Y$ and $\dim(Z) < r + \dim(Y)$. Then 2. of Theorem 19.1.1 shows that there is a nonempty open subset U of Y such that for $y \in U$, $\dim(f^{-1}(y) \cap Z) = \dim(Z) - \dim(Y) < r = \dim(X) - \dim(Y)$. Thus, for $y \in U$, the fibre is not contained in Z.

(c) $f(Z) = Y$ and $\dim(Z) \geq r + \dim(Y)$. Then again 2. of Theorem 19.1.1 shows that $\dim(f^{-1}(y) \cap Z) \geq \dim(Z) - \dim(Y) \geq r$ for all y; thus $f^{-1}(y) \subset Z$ for all $y \in Y$, so $Z = X$.

Now let $Z_1, \dots, Z_r$ be the irreducible components of X . We claim that (c) holds for at least one of the Z_i. Otherwise, there will be an open subset U in Y, such that for $y \in U$, $f^{-1}(y)$ is contained in none of the Z_i; but $f^{-1}(y)$ is irreducible and $f^{-1}(y) = \bigcup_i (f^{-1}(y) \cap Z_i)$ so this is impossible. We conclude that X is irreducible. □

19.2 Lines on hypersurfaces

As an important application, we will study the existence of lines on hypersurfaces of fixed degree. Let $S = K[x_0, \dots, x_n]$, let $d \geq 1$ be an integer number, then $\mathbb{P}(S_d)$ is a projective space of dimension $N = \binom{n+d}{d} - 1$, parametrizing the hypersurfaces of degree d in $\mathbb{P}^n$. Among them there are reducible and even non-reduced hypersurfaces (i.e. those corresponding to non square-free polynomials). Let us introduce the *incidence correspondence* line-hypersurface as follows. Let $\mathbb{G}(1,n)$ be the Grassmannian parametrising the lines in $\mathbb{P}^n$. We consider the product variety $\mathbb{G}(1,n) \times \mathbb{P}(S_d)$, whose points are the pairs $(\ell, [F])$, where ℓ is a line in $\mathbb{P}^n$ and $F \in S_d$, that we can identify with the hypersurface $V_P(F)$. By definition the incidence variety (or correspondence) is $\Gamma_d := \{(\ell, [F]) \mid \ell \subset V_P(F)\} \subset \mathbb{G}(1,n) \times \mathbb{P}(S_d)$.

Proposition 19.2.1. *Γ_d is a projective algebraic set, i.e. it is the set of zeros of a set of bihomogeneous polynomials in two series of variables: the Plücker coordinates p_{ij} on the Grasmannian and the coefficients $a_{i_0 \dots i_n}$ of F.*

Proof. Let $P = (p_{ij})$ be the skew-symmetric matrix, whose elements are the coordinates of a line ℓ: it has rank two and from Proposition 18.2.5, it follows that each non-zero row of P contains the coordinates of a point of ℓ. So the rows of P are a system of generators of a vector subspace W of dimension 2, such that $\ell = \mathbb{P}(W)$. Hence the coordinates of any point of ℓ are linear combinations of the rows of P, of the form $(x_0 = \Sigma_i \lambda_i p_{0i}, \dots, x_n = \Sigma_i \lambda_i p_{ni})$. A line ℓ is contained in $V_P(F)$ if and only if the equation $F(\Sigma_i \lambda_i p_{0i}, \dots, \Sigma_i \lambda_i p_{ni}) = 0$ is an identity in $\lambda_0, \dots, \lambda_n$. Therefore, Γ_d is the set of common zeros of the coefficients of the monomials of degree d in $\lambda_0, \dots, \lambda_n$: they are homogeneous of degree 1 in the coefficients of F and of degree d in the p_{ij}'s. □

Example 19.2.2.

Let $n = d = 3$, $F = x_0^3 - x_1 x_2 x_3 \in S_3$. We put

$$\begin{cases} x_0 = \lambda_1 p_{01} + \lambda_2 p_{02} + \lambda_3 p_{03} \\ x_1 = -\lambda_0 p_{01} + \lambda_2 p_{12} + \lambda_3 p_{13} \\ x_2 = -\lambda_0 p_{02} - \lambda_1 p_{12} + \lambda_3 p_{23} \\ x_3 = -\lambda_0 p_{03} - \lambda_1 p_{13} - \lambda_2 p_{23} \end{cases}$$

then we replace in F, and we get the identity $(\lambda_1 p_{01} + \lambda_2 p_{02} + \lambda_3 p_{03})^3 - (-\lambda_0 p_{01} + \lambda_2 p_{12} + \lambda_3 p_{13})(-\lambda_0 p_{02} - \lambda_1 p_{12} + \lambda_3 p_{23})(-\lambda_0 p_{03} - \lambda_1 p_{13} - \lambda_2 p_{23}) = 0$. By equating to zero the coefficients of the 20 monomials of degree 3 in $\lambda_0, \dots, \lambda_3$ we get the equations representing the lines contained in $V_P(F)$.

As a matter of fact, for this particular surface finding the lines contained in it is particularly simple. Indeed, we can distinguish the lines contained in the hyperplane "at infinity" from the lines which are projective closure of a line in $\mathbb{A}^3$. The first ones are contained in $x_0 = 0$, and it is clear that there are only three of them: $x_0 = x_1 = 0, x_0 = x_2 = 0, x_0 = x_3 = 0$. To find the others we dehomogenize F and get the equation $x_1 x_2 x_3 - 1 = 0$, and consider the parametrization of a general line in $\mathbb{A}^3$: $x_i = a_i t + b_i$, $i = 1, 2, 3$. By substituting, we immediately see that there are no solutions. We conclude that the surface contains only three lines.

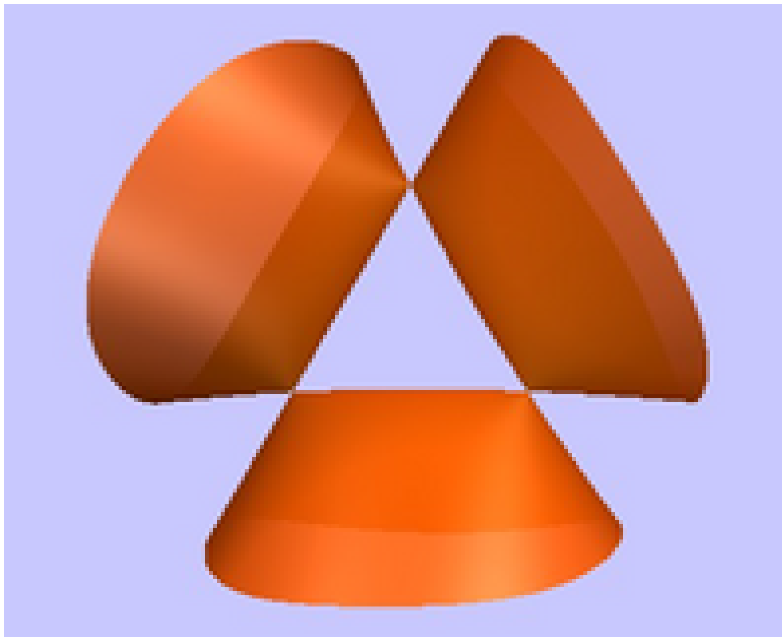


Figure 19.1: The cubic surface of Example 19.2.2

We consider now the restrictions to Γ_d of the two projections, and we get $\varphi_1 : \Gamma_d \to \mathbb{G}(1, n)$, $\varphi_2 : \Gamma_d \to \mathbb{P}(S_d)$. We will see now that the fibres of φ_1 are all irreducible and of the same dimension; this will allow to compute the dimension of Γ_d and get informations on the fibres of φ_2.

1. $\varphi_1(\Gamma_d) = \mathbb{G}(1, n)$, because any line ℓ is contained in some hypersurface of degree d. Indeed, up to a change of coordinates, we can assume that $\ell : x_0 = x_1 = \cdots = x_{n-2} = 0$. So $\ell \subset V_P(F)$ if and only if $F(0, \ldots, 0, x_{n-1}, x_n) \equiv 0$, if and only if the coefficients of the monomials containing only x_{n-1}, x_n vanish, i.e. F is of the form $x_0 G_0 + \cdots + x_{n-2} G_{n-2}$. So $\varphi_1^{-1}(\ell)$ is a linear subspace of dimension $N - (d + 1)$, because the $d + 1$ monomials $x_{n-1}^d, x_{n-1}^{d-1} x_n, \ldots, x_n^d$ don't appear in F. In particular we have that the fibres of φ_1 are all irreducible and of the same dimension. By applying Proposition 19.1.3, we obtain that Γ_d is irreducible of dimension $\dim \mathbb{G}(1, n) + \dim \varphi_1^{-1}(\ell) = 2(n - 1) + N - (d + 1)$.

2. Consider now $\varphi_2 : \Gamma_d \to \mathbb{P}(S_d) = \mathbb{P}^N$. If $\dim \Gamma_d < N$, then φ_2 cannot be surjective.

This happens if

$$\dim(\Gamma_d) = 2(n-1) + N - (d+1) < N \text{ if and only if } d > 2n-3.$$

We have proved the following theorem.

Theorem 19.2.3. *If $d > 2n-3$, there is an open non-empty subset $U \subset \mathbb{P}(S_d)$, such that if $[F] \in U$ then the hypersurface $V_P(F)$ does not contain any line; shortly, a "general" hypersurface of degree $d > 2n-3$ in $\mathbb{P}^n$ does not contain any line. The hypersurfaces containing a line form a proper closed subset in $\mathbb{P}(S_d)$.*

Example 19.2.4. Let $n = 3$, the case of surfaces in $\mathbb{P}^3$. Theorem 19.2.3 says that a general surface of degree ≥ 4 does not contain any line. Let us analyse the cases $d = 1, 2, 3$.

- $d = 1$: the surface is a plane, the lines contained in a plane form a $\mathbb{P}^2$.
- $d = 2$: the surface is a quadric, any quadric contains lines, and precisely, if its rank is 4, it contains two families of dimension 1 parametrised by two conics in $\mathbb{G}(1,3)$; if the rank is 3, the quadric is a cone, and it contains a family of dimension 1 of lines, parametrised by a conic in $\mathbb{G}(1,3)$. In both cases of rank $3, 4$ the fibres of φ_2 have dimension 1. If the rank is 2 or 1, the quadric is a pair of distinct planes or one plane with multiplicity 2, and the fibres of φ_2 have dimension 2.
- $d = 3$: in this case $N = 19 = \dim \Gamma_d$. Two cases can occur: either φ_2 is surjective, and a general fibre has dimension 0, or it is not surjective. In the second case, $\varphi_2(\Gamma_3)$, the variety of the cubic surfaces containing at least one line, has dimension < 19, so the fibres of $\Gamma_3 \to \varphi_2(\Gamma_3)$ have all dimension > 0. Hence, if a cubic surface contains a line, it contains by consequence infinitely many lines. But in Example 19.2.2 we have seen an explicit example of a cubic surface containing finitely many lines; this shows that the first possibility occurs, i.e. a "general" cubic surface contains finitely many lines. Theorem 19.1.1 explains the meaning of the adjective "general": it means that the property holds true in an open dense subset of $\mathbb{P}^{19}$.

It is a classical fact that any smooth cubic surface contains exactly 27 lines, whose configuration is completely described (see for instance [rH]). Figure 19.2 shows the Clebsch cubic surface, the only one having 27 real lines. In particular, among these 27 lines there are many pairs of skew lines.

It is a nice application of the theory we have developed so far to prove that such a cubic surface is rational.

Theorem 19.2.5. *Let $S \subset \mathbb{P}^3$ be a cubic surface containing two skew lines. Then S is rational.*

Proof. Let ℓ, ℓ' be two skew lines contained in S. For any point $P \in \mathbb{P}^3$, $P \notin \ell \cup \ell'$, there is exactly one line r_P passing through P and meeting both ℓ and ℓ': r_P is the intersection of the two planes passing through P and containing ℓ and ℓ' respectively. So we can consider the rational map $f : \mathbb{P}^3 \dashrightarrow \ell \times \ell' \simeq \mathbb{P}^1 \times \mathbb{P}^1$, such that $f(P) = (r_P \cap \ell, r_p \cap \ell')$, the pair of points of intersection of r_P with ℓ and ℓ'. We consider now the restriction $\bar{f}$ of f to S, and we get a birational map. Indeed, for any pair of points $x \in \ell$ and $x' \in \ell'$, the line joining x and x', if not contained in S, meets S in a third point. Since not all lines meeting ℓ and ℓ' can be contained in S, this defines the rational inverse of $\bar{f}$. Therefore S is birational to $\mathbb{P}^1 \times \mathbb{P}^1$, that is birational to $\mathbb{P}^2$. By transitivity we conclude that S is rational. $\square$

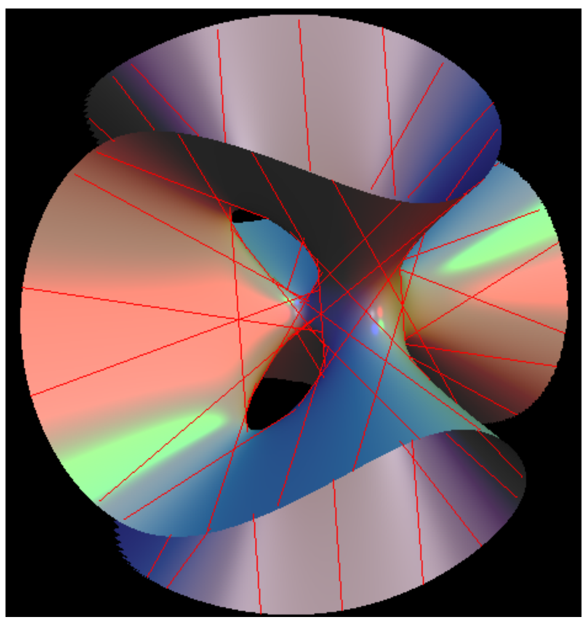


Figure 19.2: The Clebsch cubic surface

Possible equations for the Clebsch cubic surface, for different choices of coordinates, are

$$x^2y + y^2z + z^2w + w^2x = 0$$

or

$$x_0 + x_1 + x_2 + x_3 + x_4 = x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0.$$

The following equation represents the Cayley cubic surface with 4 singular points of multiplicity 2, containing 9 lines

$$xyz + yzw + zwx + wxy = 0.$$

Figure 19.2 is the image of such a surface.

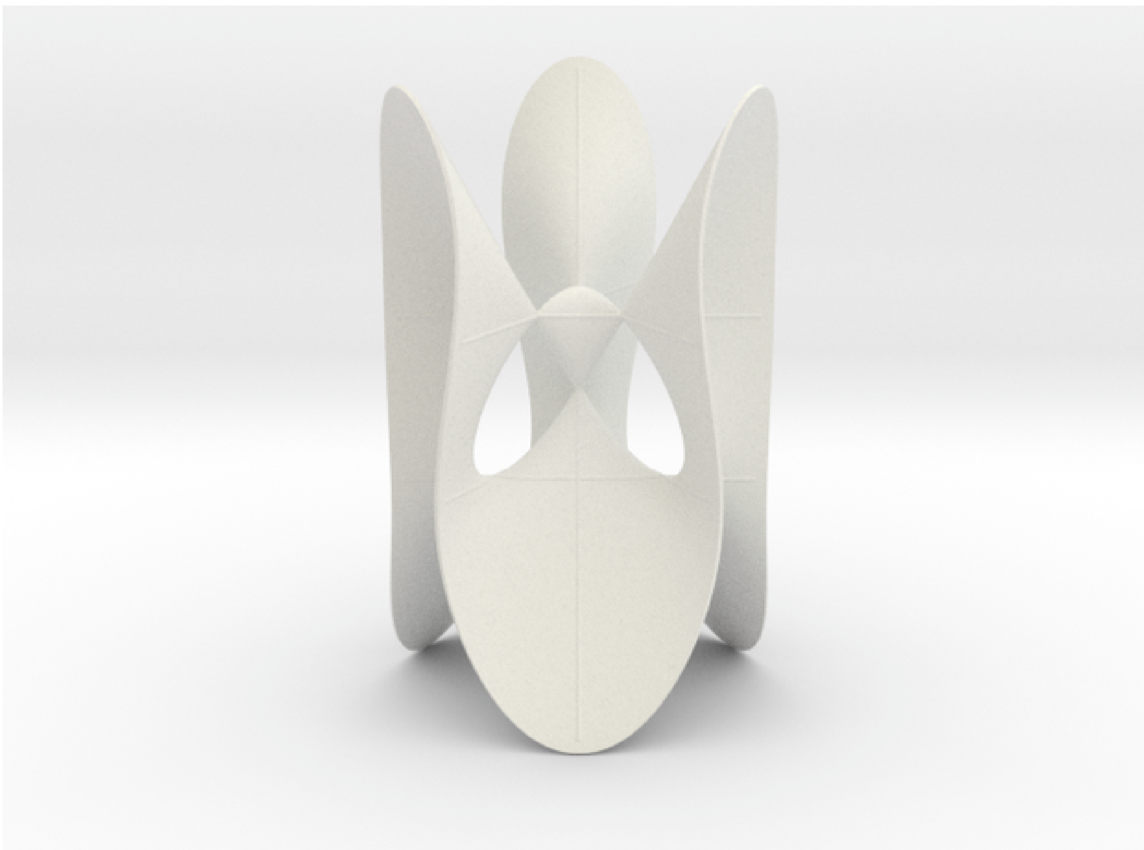


Figure 19.3: The Cayley cubic surface

A list of all possible types of singularities of cubic surfaces, with figures, can be found in the following web page: https://singsurf.org/parade/Cubics.php

Bibliography

[AM] M.F. Atiyah and I.G. MacDonald, *Introduction to Commutative Algebra*, Addison-Wesley, 1969.

[C] Steven D. Cutkosky, *Introduction to Algebraic Geometry*, American Mathematical Society, Providence, 2018.

[F] William Fulton, *Algebraic curves*, http://www.math.lsa.umich.edu/ wfulton/

[rH] Robin Hartshorne, *Algebraic Geometry.* Graduate Texts in Mathematics, vol. 52, Springer-Verlag New York, 1977.

[jH] Joe Harris, *Algebraic Geometry. A first course*. Graduate Texts in Mathematics, vol. 133, Springer–Verlag New York, 1992.

[K] Ernst Kunz, *Introduction to Commutative Algebra and Algebraic Geometry*, Birkhaüser Basel, 2013.

[L] Serge Lang, *Algebra*, 3rd ed., Springer-Verlag New York, 2002.

[N] Emmy Noeher, *Der Endlichkeitssatz der Invarianten endlicher Gruppen der Charakteristik p*, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen (1926)

[P] Christian Peskine, *An algebraic introduction to Complex Projective Geometry*, Cambridge University Press, 1996.

[R] Miles Reid, *Undergraduate Algebraic Geometry*, Cambridge University Press, 1988.

[S] Igor R. Šafarevič, *Basic algebraic geometry 1*, Springer-Verlag Berlin Heidelberg, 3rd ed. 2013.

[jS] Judith Sally, *Noether Normalization*, in Emmy Noether in Bryn Mawr, Proceedings of a Symposium Sponsored by the Association for Women in Mathematics in Honor of Emmy Noether's 100th Birthday, Springer-Verlag New York, 1983, 41–45

[W] Robert J. Walker, *Algebraic curves*, Springer-Verlag New York, 1978.

[ZS] Oscar Zariski and Pierre Samuel, *Commutative Algebra*, Springer-Verlag, 1975.