Geophysical Fluid Dynamics

Lecture IV

1. Deformation or Strain
2. Vorticity and Circulation
3. Streamfunction
4. Relative motion near a point

Kinematics

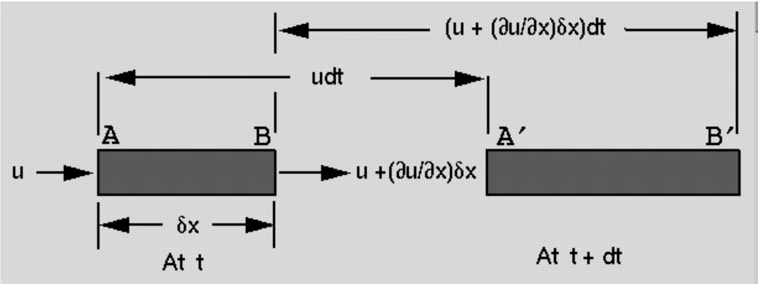
- Concerns with displacements, velocity, accelerations, deformation, rotation of fluid elements.
- Does not refer to the FORCES responsible for such a motion.
- Kinematics describes the appearance of a motion.

Deformation or Strain

- The study of the dynamics of fluid flows involves determination of the FORCES on an element, which depend on the amount of its deformation, or STRAIN.
- Deformation in a fluid is similar to that of a solid.
- NORMAL STRAIN = the change in length per unit length of a linear element.
- SHEAR STRAIN = change of a 90° angle.
- In fluids we talk about STRAIN **RATES**, because it continues to deform.

Linear Strain Rate

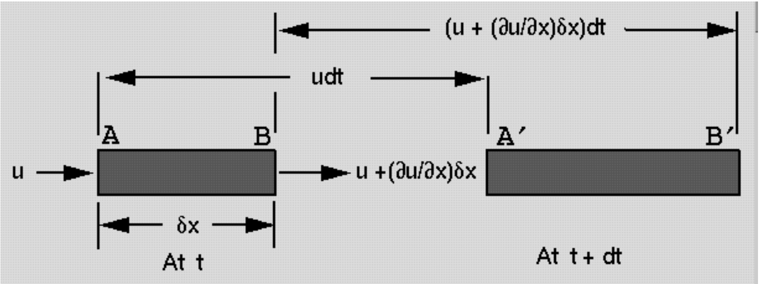
The strain rates express the deformation of a volume element, occurring both in fluids and in solids. The linear strain rate is defined as change in length per unit time and unit length of a linear element.



Linear Strain Rate

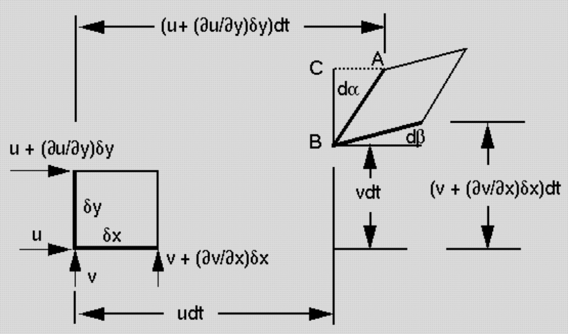
$$\frac{1}{\delta x}\frac{d\delta x}{dt} = \frac{1}{dt}\frac{A'B' - AB}{AB} = \frac{1}{dt}\frac{1}{\delta x}(\delta x + \frac{\partial u}{\partial x}\delta x dt - \delta x) = \frac{\partial u}{\partial x} \quad (1)$$

Here the material derivative D/Dt has been used to "track" the particle. Above we have derived the expression for the linear strain rate in the x direction. The expressions for the y and z direction respectively are of the same form.



Shear Strain Rate

In addition to undergoing normal or linear straining, a volume element (fluid or solid) may also be deformed in shape. The shear strain rate of an element is defined as the decrease of the angle formed by two mutually perpendicular lines on the element per time unit.



Here $d\alpha$ and $d\beta$ are the deformation angles of the figure above,
$d\alpha = CA/CB$

Shear Strain Rate

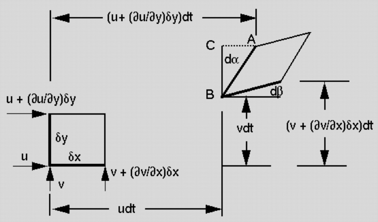
The rate of shear strain is:

$$\frac{d\alpha + d\beta}{dt} = \frac{1}{dt}\left\{\frac{1}{\delta y}\left(\frac{\partial u}{\partial y}\delta y dt\right) + \frac{1}{\delta x}\left(\frac{\partial v}{\partial x}\delta x dt\right)\right\} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad (2)$$

We can express the deformation (linear and shear) of a volume element in terms of the *strain rate tensor*

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad (3)$$

The diagonal terms of the tensor **e** are the linear strain rates, while off the diagonal we find *half* the shear strain rates.



Vorticity of a fluid element

The angular velocities are $\frac{d\beta}{dt}$ and -$\frac{d\alpha}{dt}$ so the average is $\frac{1}{2}\left(-\frac{d\alpha}{dt}+\frac{d\beta}{dt}\right)$.
the Vorticity is 2x the above

$$\omega_3 = \frac{1}{dt}\left\{\frac{1}{\delta y}\left(-\frac{\partial u}{\partial y}\delta y dt\right)+\frac{1}{\delta x}\left(\frac{\partial v}{\partial x}\delta x dt\right)\right\} = \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y} \quad (4)$$

from the curl of a vector, the Vorticity vector is related to the velocity vector $\omega_i = \nabla \times v_i$. For rotation of a stiff body, the vorticity is constant in space. It is twice the angular velocity. The fluid motion is called irrotational if $\omega = 0$. In irrotational flows the velocity vector can be written as a gradient of a scalar function $u_i = \frac{\partial \phi}{\partial x_i}$. This is due to the fact that the curl(grad(·)) operator is identical to zero. Since the velocity field in the case of irrotational fluids can be written as the gradient of a scalar, it is also called potential flow.

Circulation

Related to vorticity is the concept of circulation. The circulation around a closed contour C is defined as the line integral of the tangential component of velocity and is given by:

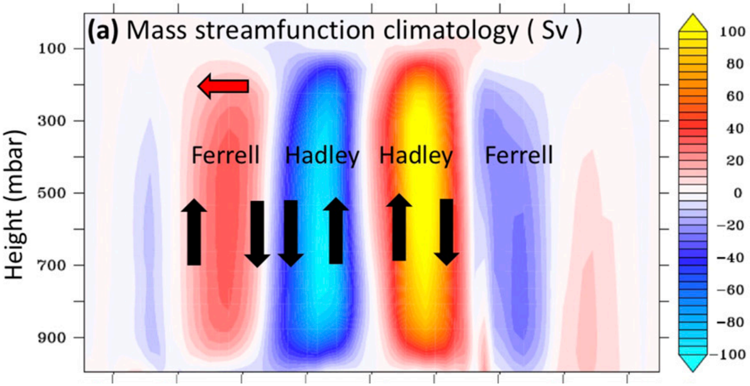
$$\Gamma = \oint_C \mathbf{v} \cdot \mathrm{d}\mathbf{s} \tag{5}$$

According to Stokes theorem the line integral of **v** around a closed contour C equals the "flux" of the vorticity through an arbitrary surface A bounded by C.

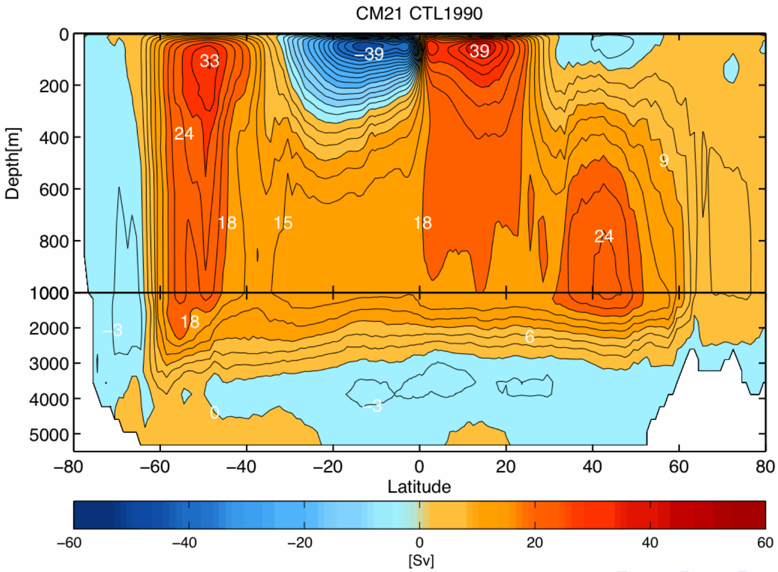
$$\oint_C \mathbf{v} \cdot \mathrm{d}\mathbf{s} = \int_A (\nabla \times \mathbf{v}) \cdot \mathrm{d} = \int_A \omega \cdot \mathrm{d}\mathbf{A} \tag{6}$$

The circulation around a closed curve is equal to the surface integral of the vorticity (or the flux of vorticity).

Streamfunction

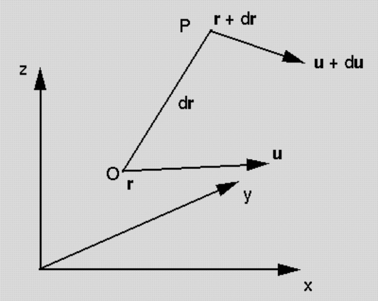


Streamfunction



Relative motion near a point

We now know that a fluid element can (i) deform and (ii) rotate. We can show that the relative motion between two neighbouring points can be written as the sum of the motion due to local rotation plus local deformation.



$$du_i = \frac{\partial u_i}{\partial x_j} dx_j \tag{7}$$

Relative motion near a point

The Velocity Gradient Tensor

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) = \varepsilon_{ij} + \frac{1}{2} r_{ij}. \qquad (8)$$

Is made of the Strain Rate Tensor $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$.

The Rotation Tensor $r_{ij} = \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$

The rotation tensor is

$$\mathbf{r} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \qquad (9)$$

The relative velocity = deformation + rotation

$$\boxed{\mathrm{d}u_i = \varepsilon_{ij}\,\mathrm{d}x_j + \frac{1}{2}(\omega \times \mathrm{d}\mathbf{x})_i} \qquad (10)$$

Next

1. Conservation of Mass, Continuity
2. Conservation of Momentum
3. Navier-Stokes Equation
4. Energy Equations
5. Bernoulli Eq.