Geophysical Fluid Dynamics

Lecture V, VI, VII: Conservation Laws

1. **Leibniz Theorem for time derivative of volume integrals**
2. Conservation of Mass - Continuity Equation
   - **Conservation Equation for a tracer**
   - **Advection-Diffusion**
   - **Diffusion**
3. Conservation of Momentum
   - **Cauchy**
   - **Navier-Stokes Equations**
   - **Euler Equation**
   - **The case of a rotating frame (towards the GFD Eq.)**
4. Conservation of Energy
   - **Kinetic, Mechanical, Potential and Total Energy**
   - **First and Second law of thermodynamics**
   - **Bernoulli's Equation - Principle**

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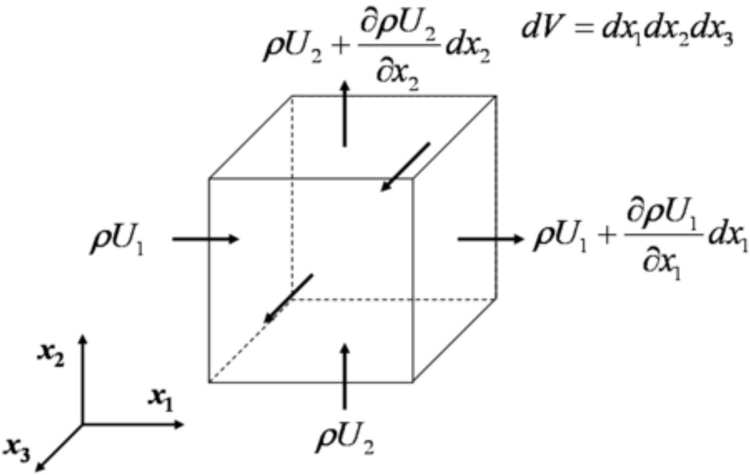
Conservation of Mass



Conservation of Mass

Conservation of Mass

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0} \tag{1}$$

or

$$\boxed{1/\rho \frac{D\rho}{Dt} + \nabla \cdot \mathbf{u} = 0}. \tag{2}$$

For a steady flow, it reduces to

$$\boxed{\nabla \cdot (\rho \mathbf{u}) = 0} \tag{3}$$

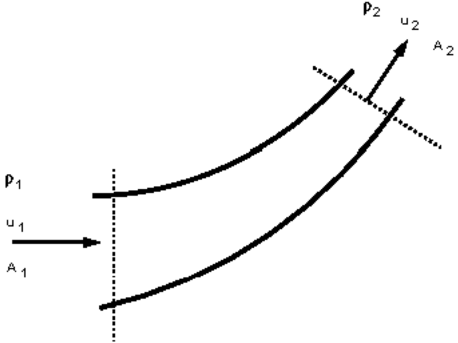
For an incompressible flow, it reduces to

$$\boxed{\nabla \cdot \mathbf{u} = 0} \tag{4}$$

Continuity

$$\rho_1 u_1 A_1 = \rho_2 u_2 A_2 \quad (5)$$

so that, in this case, $u_2 > u_1$.



Conservation law for a substance/tracer

The molecular coefficient

$$\boxed{K = U^c \times l}, \tag{6}$$

where U^c is the characteristic velocity and l is the free molecular distance. Units are $m^2 s^{-1}$.
If we have advection and diffusion the balance is

$$\boxed{\frac{\partial C}{\partial t} = (\text{Advection in - Advection out}) + (\text{Diffusion in - Diffusion out})} \tag{7}$$

Conservation law for a substance/tracer

$$\boxed{\frac{DC}{Dt} = -C\nabla\mathbf{u} - K\nabla^2 C} \tag{8}$$

For incompressible / steady flows, since
$\frac{\partial \rho}{\partial t} = 0 \rightarrow \nabla \cdot \mathbf{u} = 0$
so we have the *Advection-Diffusion Equation*

$$\boxed{\frac{DC}{Dt} = -K\nabla^2 C} \tag{9}$$

If we assume no advection ($\mathbf{u} = 0$), then we have the *Diffusion Equation*

$$\boxed{\frac{\partial C}{\partial t} = -K\nabla^2 C} \tag{10}$$