Geophysical Fluid Dynamics

Lecture V, VI, VII: Conservation Laws

1. **Leibniz Theorem for time derivative of volume integrals**
2. Conservation of Mass - Continuity Equation
   - **Conservation Equation for a tracer**
   - **Advection-Diffusion**
   - **Diffusion**
3. Conservation of Momentum
   - **Cauchy**
   - **Navier-Stokes Equations**
   - **Euler Equation**
   - **The case of a rotating frame (towards the GFD Eq.)**
4. Conservation of Energy
   - **Kinetic, Mechanical, Potential and Total Energy**
   - **First and Second law of thermodynamics**
   - **Bernoulli's Equation - Principle**

Lecture VI: Conservation Laws

1. Conservation of Momentum
   - **Cauchy**
   - **Navier-Stokes Equations**
   - **Euler Equation**
   - **The case of a rotating frame (towards the GFD Eq.)**

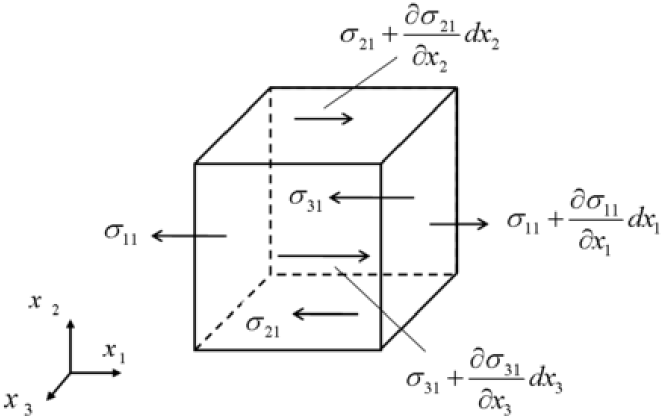
Conservation of Momentum: Forces in Fluids

1. BODY FORCES: without physical contact. They exist because the medium is in a force field (gravitational, magnetic, electrostatic, electromagnetic, ...). We denote a body force by **f**.
2. SURFACE FORCES: forces exerted on an area element by the surrounding through direct contact. They can be normal or tangential to the area (we know this).

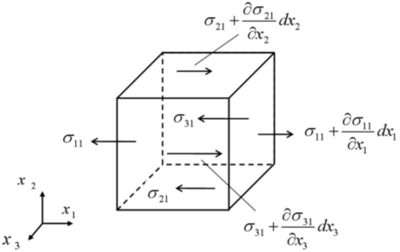
$$\tau_n \equiv \frac{dF_n}{dA}, \tau_s \equiv \frac{dF_s}{dA} \tag{1}$$

Conservation of Momentum

We simply apply Newton's law of motion to an infinitesimal fluid element. (*The net force on the element must equal mass times the acceleration of the element*)



Conservation of Momentum



Considering the torque on a centroid axis and the motion for an infinitesimal fluid element, the sum of the Surface Forces in the x_1 direction then becomes
$(\tau_{11} + \frac{\partial \tau_{11}}{\partial x_1} \frac{dx_1}{2} - \tau_{11} + \frac{\partial \tau_{11}}{\partial x_1} \frac{dx_1}{2}) dx_2 dx_3 + (\ldots) dx_1 dx_3 + (\ldots) dx_1 dx_2$.
Reducing to

$$\boxed{\left(\frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} \right) dx_1 dx_2 dx_3 = \frac{\partial \tau_{j1}}{\partial x_j} dV} \tag{2}$$

Conservation of Momentum: Cauchy's Equation

$$\left(\frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3}\right) dx_1 dx_2 dx_3 = \frac{\partial \tau_{j1}}{\partial x_j} dV \quad (3)$$

So generalizing, the *i*-component of the SURFACE FORCE per unit volume is $\frac{\partial \tau_{ij}}{\partial x_j}$

[We can prove that the stress tensor is symmetric, so that $\tau_{ji} = \tau_{ij}$]

Newton's law gives

$$\boxed{\rho \frac{Du_i}{Dt} = f_i + \frac{\partial \tau_{ij}}{\partial x_j}} \quad (4)$$

Conservation of Momentum: Cauchy's Equation

Equation of motion relating acceleration to the net force at a point and holds for any continuum, solid or fluid, no matter how the stress tensor τ_{ij} is related to the deformation field.

$$\boxed{\rho \frac{Du_i}{Dt} = f_i + \frac{\partial \tau_{ij}}{\partial x_j}} \tag{5}$$

where **f** is a BODY FORCE per unit mass (like Newtonian gravity), so that $\rho\mathbf{f}$ is the body force per unit volume. This is the CAUCHY'S EQUATION.

Constitutive Equation for Newtonian Fluids

A relation between stress and deformation in a continuum is called a *constitutive equation*. It linearly relates the stress to the rate of strain in a fluid medium.

In a fluid at rest, there are only normal forces / stresses. And we know that the stress tensor is isotropic. A second-order isotropic tensor is the Kronecker delta δ. Any isotropic tensor must be proportional to δ. Therefore, in static, the stress must take the form

$$\boxed{\tau_{ij} = -p\delta_{ij},} \tag{6}$$

where p is thermodynamic pressure, $f(\rho, T)$. The negative sign is because the normal components of τ are considered positive if they indicate tension, rather than compression.

Constitutive Equation for Newtonian Fluids

A **moving fluid** develops more components of stress due to *viscosity*, and shear stresses develop. The diagonal terms of τ_{ij} are now unequal.

$$\boxed{\tau_{ij} = -p\delta_{ij} + \sigma_{ij}.} \tag{7}$$

σ_{ij} is the **deviatoric stress tensor**, nonisotropic, related to the velocity gradient tensor $\partial u_i/\partial x_j$:

$$\partial u_i/\partial x_j = e_{ij} + r_{ij} \tag{8}$$

The rotation tensor, the antisymmetric part, can not generate stress as it represents fluid rotation and not deformation. Stresses are generated by the strain rate tensor e_{ij}.

$$e_{ij} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{9}$$

... Now a few assumptions (and jumps) ...

Constitutive Equation for Newtonian Fluids

- **Hypothesis of Newtonian Fluid**: We assume a linear relationship between shear stresses and deformations $\tau = \mu(du/dy)$, so that we have

$$\sigma_{ij} = K_{ijmn} e_{mn}. \tag{10}$$

K_{ijmn} is a 4^{th} order tensor (81 components) that depends on the thermodynamic state of the medium. This says that each stress component is linearly related to all nine component of $e_{ij} \rightarrow$ 81 constants are needed for this.

Constitutive Equation for Newtonian Fluids

- **Hypothesis of isotropic medium**: from 81 coefficients of K_{ijmn} we go down to 3!
- **Hypothesis of symmetric σ_{ij}**: from 3 coefficients of K_{ijmn} we go down to 2!
- **Stokes' Hypothesis**: pressure p of the fluid is equal to the thermodynamic pressure or to that of the static fluid, $\hat{p}$. From 2 coefficients of K_{ijmn} we go down to 1!

Constitutive Equation for Newtonian Fluids

So from here:

$$\tau_{ij} = -p\delta_{ij} + \sigma_{ij}.$$

we get to the final form of the constitutive equation, which is

$$\boxed{\tau_{ij} = -(p + \frac{2}{3}\mu\nabla\cdot\mathbf{u})\delta_{ij} + 2\mu e_{ij}} \tag{11}$$

- The linear relation between τ and $\mathbf{e}$ is consistent with Newton's definition of viscosity coefficient in a parallel flow $u(y)$, so that Eq.(**??**) gives a shear stress of $\tau = \mu(du/dy)$. Hence, this only applies to *Newtonian fluids*.
- The nondiagonal terms of Eq.(**??**) relate the shear stress to the shear strain rate

$$\tau_{12} = \mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

Navier-Stokes Equations

Now let's plug the constitutive equation into the Cauchy's equation to get the **Equation of Motion for a Newtonian Fluid**

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and the constitutive Eq.:

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$$\tau_{ij} = -(p + \frac{2}{3}\mu \nabla \cdot \mathbf{u})\delta_{ij} + 2\mu e_{ij}$$

The two give the general form of the **Navier-Stokes equation**:

$$\boxed{\rho \frac{Du_i}{Dt} = f_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[2\mu e_{ij} - 2/3\mu(\nabla \cdot \mathbf{u})\delta_{ij}\right]}$$

Navier-Stokes Equations

$$\boxed{\rho \frac{Du_i}{Dt} = f_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[2\mu e_{ij} - 2/3\mu(\nabla \cdot \mathbf{u})\delta_{ij}\right]}$$

we have noted that $(\partial p/\partial x_j)\delta_{ij} = (\partial p/\partial x_i)$.
Viscosity μ is generally a function of the thermodynamic state. For fluids, generally μ depends strongly on Temperature:

$$\begin{cases} \mu & \text{decreases as T increases} \quad \text{for Liquids} \\ \mu & \text{increases as T increases} \quad \text{for Gases} \end{cases}$$

Navier-Stokes Equations

- If temperature differences are small enough within the fluid, then $\mu \sim constant$.
- Also, for incompressible fluids: $\nabla \cdot \mathbf{u} = 0$, and so we can write:

$$\rho \frac{Du_i}{Dt} = f_i - \frac{\partial p}{\partial x_i} + 2\mu \frac{\partial}{\partial x_j} e_{ij}$$

which is

$$\boxed{\rho \frac{Du_i}{Dt} = f_i - \frac{\partial p}{\partial x_i} + \mu \nabla^2 u_i}$$

or

$$\boxed{\rho \frac{D\mathbf{u}}{Dt} = \mathbf{f} - \nabla p + \mu \nabla^2 \mathbf{u}}$$

Navier-Stokes Equations

If we neglect viscous effects , which is true far from boundaries, we obtain the **Euler Equation**:

$$\boxed{\rho \frac{Du_i}{Dt} = f_i - \frac{\partial p}{\partial x_i}}$$

or

$$\boxed{\rho \frac{D\mathbf{u}}{Dt} = \mathbf{f} - \nabla p}$$

Body forces on a Rotating frame

1. In a non-rotating frame: $\mathbf{f} = \mathbf{g}$
2. In a rotating frame: $\mathbf{f} = \mathbf{g} + \{\text{apparent force due to rotation}\}$

It can be easily proved that, in a rotating frame of reference (such as the Earth), the velocities between a fixed frame of reference and a frame of reference rotating at a uniform angular velocity $\mathbf{\Omega}$ are related by:

$$\mathbf{u_f} = \mathbf{u_r} + \mathbf{\Omega} \times \mathbf{r}, \tag{12}$$

where $\mathbf{r}$ is the position vector. This means that, after some manipulation (see any textbook ...),

$$\boxed{\mathbf{a_f} = \mathbf{a} + 2\mathbf{\Omega} \times \mathbf{u}.} \tag{13}$$

So that the (inertial) acceleration equals the acceleration in a rotating system plus the **Coriolis acceleration**.

The Coriolis force / acceleration / effect

We now have

$$\boxed{\frac{D\mathbf{u}}{Dt} + 2\Omega \times \mathbf{u} = \mathbf{g} - \frac{1}{\rho}\nabla p + \nu \nabla^2 u_{ij}},$$

where $\nu = \mu/\rho$ is the kinematic viscosity.
The earth rotates at a rate

$$\Omega = 2\pi\, rad/day = 0.73 \times 10^{-4} s^{-1}$$

We decompose the components of the angular velocity of the earth:

$$2\Omega \times \mathbf{u} = 2\left[(\Omega_y w - \Omega_z v)\hat{i} + (\Omega_z u - \Omega_x w)\hat{j} + (\Omega_x v - \Omega_y u)\hat{k}\right] \quad (14)$$

Thin layer approximation on a rotating sphere

For atmosphere and oceans the depth scale of flow $\sim$ few kilometers. The horizontal scale is instead $\sim$ hundreds/thousands of kilometeres.

So we can make the *thin layer approximation*: $\boxed{w << u}$. We can understand this from the continuity equation.

So the Coriolis components reduce to:

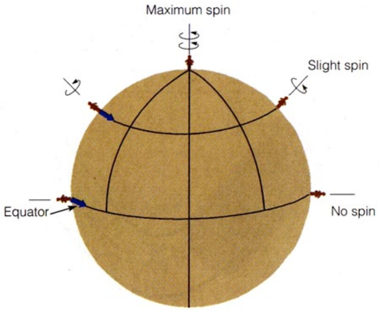
$$2\Omega \times \mathbf{u} = 2\left[-\Omega_z v\hat{i} + \Omega_z u\hat{j} + (\Omega_x v - \Omega_y u)\hat{k} \right]. \tag{15}$$

Note that:

$$\Omega_z = \Omega sin\theta$$
$$\Omega_y = \Omega cos\theta$$
$$\mathbf{g} = -g\hat{k}$$

and let's define $\boxed{f = 2\Omega sin\theta}$ (twice the vertical component of Ω).

The Coriolis force / acceleration / effect



Planetary Vorticity (or Coriolis parameter, or Coriolis frequency):

$$\boxed{f = 2\Omega sin\theta} \tag{16}$$

f is max at the poles (max spin) and zero at the Equator (only translation).

The Equations of motion on a rotating earth

$$2\Omega \times \mathbf{u} = 2\left[-fv\,\hat{i} + fu\,\hat{j} - \Omega sin\theta\, u\,\hat{k} \right]. \tag{17}$$

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\nabla^2 u \tag{18}$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\nabla^2 v \tag{19}$$

$$\frac{Dw}{Dt} - (2\Omega cos\theta)u = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\nabla^2 w - g \tag{20}$$

but the vertical component of the Coriolis force is negligible compared to the dominant terms in the vertical equation of motion

The Equations of motion on a rotating earth

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\nabla^2 u \quad (21)$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\nabla^2 v \quad (22)$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\nabla^2 w - g \quad (23)$$

The equation of motion for a thin shell on a rotating earth.
Only the vertical component of the earth's angular velocity appears as a consequence of the flatness of the fluid trajectories.

Conservation laws in fluid mechanics

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu \nabla^2 u \tag{24}$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu \nabla^2 v \tag{25}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu \nabla^2 w - g \tag{26}$$

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0 \tag{27}$$

Next, conservation of Energy and Bernoulli's Equation.