Geophysical Fluid Dynamics

Lecture V, VI, VII: Conservation Laws

1. **Leibniz Theorem for time derivative of volume integrals**
2. Conservation of Mass - Continuity Equation
   - **Conservation Equation for a tracer**
   - **Advection-Diffusion**
   - **Diffusion**
3. Conservation of Momentum
   - **Cauchy**
   - **Navier-Stokes Equations**
   - **Euler Equation**
   - **The case of a rotating frame (towards the GFD Eq.)**
4. Conservation of Energy
   - **Kinetic, Mechanical, Potential and Total Energy**
   - **First and Second law of thermodynamics**
   - **Bernoulli's Equation - Principle**

Geophysical Fluid Dynamics

Lecture VII: Conservation Laws

1. Conservation of Energy
   - **Kinetic, Mechanical, Potential and Total Energy**
   - **First and Second law of thermodynamics**
   - **Bernoulli's Equation - Principle**

Energy

$$\begin{aligned} \text{Potential Energy} &= mgh = \rho gh = \rho gz \\ \text{Kinetic Energy} &= \frac{1}{2} m\mathbf{u}\cdot\mathbf{u} = \frac{1}{2} m(u^2+v^2+w^2) = \frac{1}{2}\rho\mathbf{u}^2 \end{aligned}$$

Equation for Conservation of Mechanical Energy will be

$$\begin{aligned} \text{rate of change in} (E_k + E_p) = \text{rate of change in E} + \\ + \text{rate of Work} - \text{rate of viscous dissipation} \end{aligned}$$

Mechanical Energy Equation

The mechanical energy equation can be obtained from the scalar product of the momentum equation and the velocity vector. Remember Cauchy's Eq.:

$$\boxed{\rho \frac{Du_i}{Dt} = \rho f_i + \frac{\partial \tau_{ij}}{\partial x_j}}$$

where f_i is the body force (gravitational and Coriolis) and we keep the stress tensor (the constitutive equation will be used only later on).

Mechanical Energy Equation

Now let's multiply the i-momentum equation by u_i, and we'll get to

$$\boxed{\frac{D}{Dt}\left(\frac{u_i u_i}{2} + gz\right) = \frac{1}{\rho} u_i \frac{\partial \tau_{ij}}{\partial x_j}} \qquad (1)$$

The Coriolis force, vanished, does not contribute to any of the energy eq.
That the rate of increase in E_m at any point equals the rate of work done by net surface force $\nabla \cdot \boldsymbol{\tau}$ per unit volume.
Or, if you prefer, let's rewrite it as

$$\boxed{\frac{D}{Dt}\left(\frac{u_i u_i}{2}\right) = g_i u_i + \frac{1}{\rho} u_i \frac{\partial \tau_{ij}}{\partial x_j}} \qquad (2)$$

so that the rate of increase in E_k at any point equals the sum of the rate of work done by net surface force $\nabla \cdot \boldsymbol{\tau}$ and the rate of work done by the body force g.

Deformation work and viscous dissipation

The term $u_i \frac{\partial \tau_{ji}}{\partial x_j}$ is *velocity times net force imbalance at a point*. It means that the net force accelerates the local fluid and increases its Kinetic Energy. However, this in not the total rate of work done by the stress, there is also deformation of the element with no acceleration. So, the total rate of work done by surface forces on a fluid element is

$$\boxed{\frac{\partial u_i \tau_{ij}}{\partial x_j} = u_i \frac{\partial \tau_{ij}}{\partial x_j} + \tau_{ij} \frac{\partial u_i}{\partial x_j}} \tag{3}$$

Total work $=$ increase of E_k $+$ deformation work.
Hence, the second term is work that only deforms the element, and increases its internal energy.

Deformation work and viscous dissipation

Substituting the total rate of work into the Mechanical Energy Equation, we see that the **deformation work** is

$$\tau_{ij}\frac{\partial u_i}{\partial x_j} = \tau_{ij} e_{ij} \tag{4}$$

the strain rate tensor! and we now have:

$$\rho\frac{D}{Dt}\left(\frac{u_i u_i}{2} + gz\right) = \frac{\partial u_i \tau_{ij}}{\partial x_j} - \tau_{ij} e_{ij}. \tag{5}$$

and now we make use of the Newtonian constitutive equation:

$$\tau_{ij} = -(p + \frac{2}{3}\mu\nabla\cdot\mathbf{u})\delta_{ij} + 2\mu e_{ij} \tag{6}$$

linearly relating the stress to the rate of strain in a fluid.

Deformation work and viscous dissipation

so now $\tau_{ij}e_{ij}$ becomes

$$\tau_{ij}e_{ij} = -p(\nabla\cdot\mathbf{u}) + 2\mu e_{ij}e_{ij} - \frac{2}{3}\mu(\nabla\cdot\mathbf{u})^2 \tag{7}$$

$$\tau_{ij}e_{ij} = -p(\nabla\cdot\mathbf{u}) + \phi \tag{8}$$

where we have denoted the viscous terms by ϕ.
This is the **deformation work**.

Mechanical Energy Equation

going back to our Mechanical Energy Equation:

$$\rho \frac{D}{Dt}\left(\frac{u_i u_i}{2} + gz\right) = \frac{\partial u_i \tau_{ij}}{\partial x_j} - \tau_{ij} e_{ij}$$

Mechanical Energy Equation

going back to our Mechanical Energy Equation:

$$\rho \frac{D}{Dt}\left(\frac{u_i u_i}{2} + gz\right) = \frac{\partial u_i \tau_{ij}}{\partial x_j} - \tau_{ij} e_{ij}$$

$$\rho \frac{D}{Dt}\left(\frac{u_i u_i}{2} + gz\right) = \frac{\partial u_i \tau_{ij}}{\partial x_j} + p(\nabla \cdot \mathbf{u}) - \phi$$

Mechanical Energy Equation

going back to our Mechanical Energy Equation:

$$\rho \frac{D}{Dt}\left(\frac{u_i u_i}{2} + gz\right) = \frac{\partial u_i \tau_{ij}}{\partial x_j} - \tau_{ij} e_{ij}$$

$$\rho \frac{D}{Dt}\left(\frac{u_i u_i}{2} + gz\right) = \frac{\partial u_i \tau_{ij}}{\partial x_j} + p(\nabla \cdot \mathbf{u}) - \phi$$

or better:

$$\boxed{\rho \frac{D}{Dt}\left(\frac{u_i u_i}{2}\right) = \rho g w + \frac{\partial u_i \tau_{ij}}{\partial x_j} + p(\nabla \cdot \mathbf{u}) - \phi}$$

Mechanical Energy Equation

$$\boxed{\rho \frac{D}{Dt}\left(\frac{u_i u_i}{2}\right) = \rho g w + \frac{\partial u_i \tau_{ij}}{\partial x_j} + p(\nabla \cdot \mathbf{u}) - \phi}$$

$\rho g w$	rate of work by body force
$\frac{\partial u_i \tau_{ij}}{\partial x_j}$	total rate of work by τ
$p(\nabla \cdot \mathbf{u})$	rate of work by volume expansion/contraction
ϕ	rate of viscous dissipation

Mechanical Energy Equation

$$\boxed{\rho \frac{D}{Dt}\left(\frac{u_i u_i}{2}\right) = \rho g w + \frac{\partial u_i \tau_{ij}}{\partial x_j} + p(\nabla \cdot \mathbf{u}) - \phi}$$

the term $\tau_{ij} e_{ij} = -p(\nabla \cdot \mathbf{u}) + \phi$ is the total deformation work rate (per unit volume).

$p(\nabla \cdot \mathbf{u})$	reversible conversion to internal energy by volume changes
ϕ	irreversible conversion to internal energy by viscous effects

ϕ, always positive, a rate of loss of mechanical energy, the rate of viscous dissipation of Kinetic Energy per unit volume, is proportional to μ and the square of the velocity gradient. It is therefore very important in regions of high shear, usually resulting in heat.

Mechanical Energy Equation

Now consider the continuity eq. times $\frac{1}{2}\rho u_i^2$

$$\frac{1}{2}\rho u_i^2\left(\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_j}(\rho u_j)\right)=0 \tag{9}$$

we add this to Eq.39:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u_i^2\right)+\frac{\partial}{\partial x_j}\left(u_j\frac{1}{2}\rho u_i^2\right)=\rho u_i g_i+u_i\frac{\partial \tau_{ij}}{\partial x_j} \tag{10}$$

if $E\equiv\frac{1}{2}\rho u_i^2$ is Kinetic Energy per unit volume, then

$$\frac{\partial E}{\partial t}+\nabla\cdot(\mathbf{u}E)=\rho\mathbf{u}\cdot\mathbf{g}+\mathbf{u}\cdot(\nabla\cdot\tau) \tag{11}$$

Mechanical Energy Equation

$$\boxed{\frac{\partial E}{\partial t} + \nabla \cdot (\mathbf{u}E) = \rho \mathbf{u} \cdot \mathbf{g} + \mathbf{u} \cdot (\nabla \cdot \tau)} \qquad (12)$$

the second term is the divergence of the Kinetic energy flux.
– If source terms on rhs are zero, then E will increase in time if $\nabla \cdot (\mathbf{u}E)$ is negative.
– Flux divergence terms are also called transport terms, because they transfer quantities (but no net contribution).
– If integrated over the entire volume, their contribution vanishes without any sources at the boundaries. Through the divergence theorem

$$\int_V \nabla \cdot (\mathbf{u}E) dV = \int_A E\mathbf{u} \cdot d\mathbf{A} \qquad (13)$$

which vanishes if the flux is zero at the boundaries.

Conservation of Mechanical energy

$$\boxed{\rho \frac{D}{Dt}\left(E_k + E_p\right) = \frac{\partial u_i \tau_{ij}}{\partial x_j} + p(\nabla \cdot \mathbf{u}) - \phi}$$

Conversion of **grativational potential energy** into **kinetic energy**
Video of MIT Prof. Water Lewin

First law of Thermodynamics: Thermal Energy Equation

The mechanical energy equation was derived from the momentum equation. This means that it is not a separate principle.
We need an independent equation, for flows with varying temperature $\rightarrow$ first law of thermodynamics.
Let $\mathbf{q}$ be a heat flux vector per unit area.
Let e be the internal energy per unit mass (for a perfect gas $e = C_v T$).
The *stored* energy per unit mass is then $(e + \frac{1}{2} u_i u_i)$
The rate of change of stored energy equals the sum of rate of total work done and rate of heat addition to a material volume.

$$\boxed{de = dw + dQ} \tag{14}$$

First law of Thermodynamics: Thermal Energy Equation

the First law of Thermodynamics can also be written as

$$\rho \frac{D}{Dt}\left(e + \frac{u_i u_i}{2}\right) = \rho g w + \frac{\partial u_i \tau_{ij}}{\partial x_j} - \frac{\partial q_i}{\partial x_i}$$

This expression has both the mechanical and the thermal energy terms. If we subtract the mechanical energy equation

$$\rho \frac{D}{Dt}\left(\frac{u_i u_i}{2}\right) = \rho g w + \frac{\partial u_i \tau_{ij}}{\partial x_j} + p(\nabla \cdot \mathbf{u}) - \phi$$

we obtain the thermal **energy equation** (aka **heat equation**)

$$\boxed{\rho \frac{De}{Dt} = -\nabla \cdot \mathbf{q} - p(\nabla \cdot \mathbf{u}) + \phi} \tag{15}$$

First law of Thermodynamics: Thermal Energy Equation

$$\boxed{\rho \frac{De}{Dt} = -\nabla \cdot \mathbf{q} - p(\nabla \cdot \mathbf{u}) + \phi}$$

this is saying that internal energy increases because of convergence of heat, volume compression, and heating due to viscous dissipation.

Bernoulli Equation (or the conservation of energy)

The Bernoulli equation is derived from the inviscid momentum equation, the Euler equation.

$$\rho \frac{Du_i}{Dt} = \rho f_i - \nabla p$$

We will assume flows that are

1. steady
2. inviscid
3. no heat conduction

Bernoulli Equation (or the conservation of energy)

Let's start from the total energy equation:

$$\rho \frac{D}{Dt}\left(e + \frac{u_i u_i}{2} + gz\right) = \frac{\partial u_i \tau_{ij}}{\partial x_j} + \frac{\partial q_i}{\partial x_i}$$

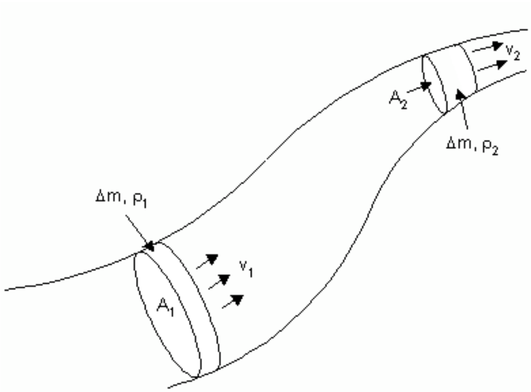
to get to

$$\boxed{B \equiv \frac{1}{2}\mathbf{u}\cdot\mathbf{u} + gz + \frac{1}{\rho}p = \text{constant}}$$

Bernoulli Equation (or the conservation of energy)

The Bernoulli function will be:

$$B \equiv \frac{1}{2}\mathbf{u}\cdot\mathbf{u} + gz + \frac{1}{\rho}p$$



Bernoulli Equation (or the conservation of energy)

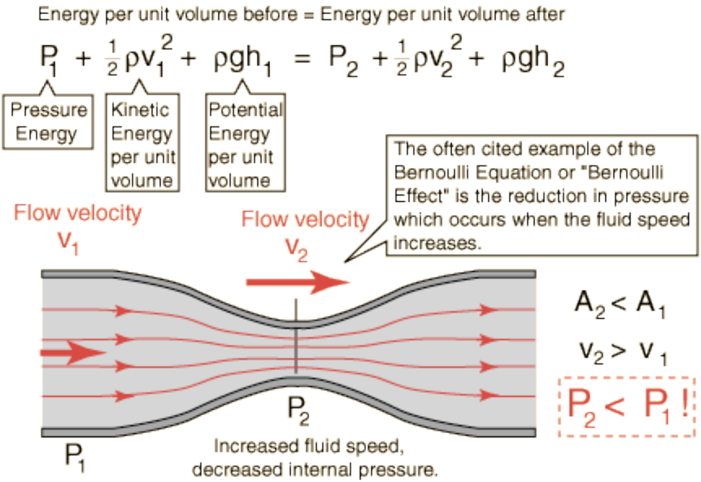
$$\boxed{B \equiv \frac{1}{2}\mathbf{u}\cdot\mathbf{u} + gz + \frac{1}{\rho}p = \text{constant}}$$

The Bernoulli equation is a statement of the principle of conservation of energy along a streamline:
Kinetic + Potential + Pressure energy = Total Energy = Constant
Strong Restrictions

1. Flow is steady
2. Density is constant
3. Friction losses are negligible
4. relating the states at two points along a single streamline

Very hard to satisfy all these conditions, but many real situations are close enough and this equation is a reasonable approximation to the fluid behaviour.

Bernoulli Equation (or the conservation of energy)



Bernoulli Equation (or the conservation of energy)

$$\boxed{B \equiv \frac{1}{2}\mathbf{u}\cdot\mathbf{u} + gz + \frac{1}{\rho}p = \text{constant}}$$

$$p_1 + \frac{\rho U_1^2}{2} = p_2 \tag{16}$$

1. static pressure: does not include any dynamic effects. It represents thermodynamic pressure.
2. Dynamic pressure: is the pressure rise when a fluid in motion is brought to rest isentropically (no viscous effects).
3. stagnation pressure: is the sum of static and dynamic pressures.
4. ρgz is the hydrostatic pressure.
5. $p + \rho U^2/2 + \rho gz$ is the total pressure.

Bernouilli Eq. says that total pressure is constant along a streamline.

Fluid Mechanics

Conservation Laws

1. Conservation of Mass - Continuity Equation
2. Conservation of Momentum
3. Conservation of Energy