Fluid Mechanics

Lecture X: Vorticity Dynamics

Vorticity

- A vortex line is a curve in the fluid such that its tangent at any point gives the direction of the local vorticity.
- A vortex line is related to the vorticity the same way a streamline is related to the velocity vector.
- A vortex line satisfies the equations:

$$\frac{dx}{\omega_x} = \frac{dy}{\omega_y} = \frac{dz}{\omega_z} \tag{1}$$

Vorticity

- A group of vortex lines bound a vortex tube.
- The circulation around a narrow vortex tube is

$$d\Gamma = \omega \cdot dA \tag{2}$$

- The strength of a vortex tube is defined as the circulation around a closed circuit taken on the surface of the tube.

$$\Gamma = \oint_c u_i dx_i \tag{3}$$

Vorticity

- IRROTATIONALITY does not imply the absence of viscous stresses.
- In fact, viscous stresses must always exist in irrotational flows for real fluids, because the fluid elements deform in such a flow.

Vorticity

- If the flow is irrotational then the net viscous forces vanish on the element.
- The only example of vorticity and no viscous stresses is that of solid-body rotation.
- In solid-body rotation the fluid elements do not deform.
- Viscous stresses are proportional to the deformation rate, and hence they are zero for this flow.

$$\sigma_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = 0 \tag{4}$$

Kelvin's Circulation Theorem

In an inviscid, barotropic flow with conservative body forces, the circulation around a closed curve C moving with the fluid remains constant in time (if the motion is observed from a nonrotating frame).

$$\frac{D\Gamma}{Dt} = 0 \qquad (5)$$

Assumptions

- inviscid flow: $\mu = 0$
- conservative body forces: $f_i = \nabla(-gz)$
- Barotropic flow, Density is a function of pressure only: $\rho = \rho(p)$

Reversible forms of compressibility are OK (pressure) but mixing is irreversible and therefore $\rho \neq \rho(T, S)$.

Kelvin's Circulation Theorem

Integrating around a closed contour:

$$\frac{D\Gamma}{Dt} = \oint_c [-dP + dg + 1/2d(u_i u_i)] = 0 \qquad (6)$$

- P and g are single valued since they are reversible forms of work
- $u_i u_i$ is single valued since u_i is continuous

Kelvin's Circulation Theorem

Circulation is the surface integral of vorticity. Integrating around a closed contour:

$$\frac{D\Gamma}{Dt} = \frac{D}{Dt}\int_A \omega_i dA = 0 \qquad (7)$$

or $\int_A \omega_i dA$ is constant as we follow the flow.
If the flow is irrotational $\omega_i = 0$ the flow will remain irrotational under the four assumptions:

- Inviscid flow
- Conservative body forces
- Barotropic flow
- Nonrotaing frame

Kelvin's Circulation Theorem

- **Inviscid flow:** Circulation is preserved if there are no net viscous forces along the path followed by C. If C moves into viscous regions such as boundary layers along solid surface, then circulation changes. Viscous effects cause diffusion of vorticity in or out of the fluid circuit, thereby changing the circulation.
- **Conservative body forces:** gravity acts through the centre of mass of a fluid particle and therefore does not rotate it.

Kelvin's Circulation Theorem

- **Barotropic flow:** in a baroclinic flow, lines of constant p and ρ are not parallel, and the net pressure force does not pass through the centre of mass creating a torque which changes the vorticity and circulation. Geophysical flows, which are dominated by baroclinicity, are full of vorticity. Examples: heating from below creates a buoyant force generating a plume ($\rho = \rho(T)$), cooling from above will generate rolls and vorticity.
- **Nonrotating frame:** motions observed with respect to a rotating frame of reference can developed vorticity through Coriolis (shown later).

Vorticity Equation in a Nonrotating frame

- The flow is barotropic
- we retain viscous effects
- baroclinicity and the effect of a rotating frame of reference will be dealt in the next derivation.

Vorticity is $\omega = \nabla x \mathbf{u}$ and its curl is zero $\nabla \cdot \omega = 0$. The rate of change of vorticity is:

$$\boxed{\frac{D}{Dt}\omega = (\omega \cdot \nabla)\mathbf{u} + \nu \nabla^2 \omega} \tag{8}$$

where the first term on the r.h.s. is the rate of change of vorticity due to stretching and tilting of vortex lines, and the second term is the rate of change of vorticity due to diffusion of vorticity.

Vorticity Equation in a Rotating frame

- The flow is nonbarotropic
- We use a rotating frame of reference.
- We still approximate to a nearly incompressible Boussinesq fluid so that the contonuity equation is $\nabla \cdot \mathbf{u} = o$
- Continuity is $u_{i,i} = 0$
- The momentum equation is

$$\frac{\partial u_i}{\partial t} + u_j u_{i,j} + 2\varepsilon_{ijk}\Omega_j u_k = -(1/\rho)p_{,i} + g_i + \nu u_{i,jj} \tag{9}$$

- after some manipulation we get to

$$\boxed{\frac{\partial u_i}{\partial t} + (1/2u_j^2 + \Pi)_{,i} - \varepsilon_{ijk} u_j(\omega_k + 2\Omega_k) = -(1/\rho)p_{,i} - \nu\varepsilon_{ijk}\omega_{k,j}} \tag{10}$$

This is a form of the N-S equation, and we now take its curl.

Vorticity Equation in a Rotating frame

$$\boxed{\frac{D}{Dt}\omega = (\omega + 2\Omega)\cdot\nabla\mathbf{u} + (1/\rho^2)\nabla\rho\wedge\nabla p + \nu\nabla^2\omega} \qquad (11)$$

This is the Vorticity equation for a nearly incompressible fluid (Boussinesq) in rotating coordinates.

1. 1st term is the rate of change of relative vorticity following a fluid particle
2. 2nd term is Absolute Vorticity
3. 3rd term is the rate of change due to baroclinicity of the flow
4. 4th term is the rate of change due to diffusion

$(\omega + 2\Omega)\cdot\nabla\mathbf{u}$ is a crucial term in the vorticity dynamics.

Vorticity Equation in a Rotating frame

$$\boxed{(\omega \cdot \nabla)\mathbf{u} = \omega \frac{\partial \mathbf{u}}{\partial s}} \tag{12}$$

$=$ the magnitude of ω times the derivative of $\mathbf{u}$ in the direction of ω

$$\boxed{(\omega \cdot \nabla)\mathbf{u} = \omega \frac{\partial u_s}{\partial s} + \omega \frac{\partial u_n}{\partial s} + \omega \frac{\partial u_m}{\partial s}} \tag{13}$$

.

The first is the increase of u_s along the vortex line s (stretching of vortex line). The second and third represent the change of normal velocity components along s: rate of turning and tilting of vortex lines about the m and n axes.

Vorticity Equation in a Rotating frame

$$\boxed{2(\Omega \cdot \nabla)\mathbf{u} = 2\Omega\left(\frac{\partial \mathbf{u}}{\partial z}\right)} \tag{14}$$

The third term says that stretching of fluid lines in the z direction increases ω_z.

The first and second say that turning and tilting of fluid lines increase the relative vorticity along x and y. Here only fluid lines have to tilt/turn. This is because vertical fluid lines contain planetary vorticity 2Ω.