MEDICAL PHYSICS LAB

LECTURE 3 – THE FOURIER TRANSFORM & SOME USEFUL FUNCTIONS

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The Fourier Transform & some useful functions

- Organized in 3 parts:
  - Part 1 – the Fourier Transform
  - Part 2 – convolution
  - Part 3 – some useful functions
- Source:
  - Ronald N. Bracewell, The Fourier Transform and its applications, 3rd edition, McGraw-Hill, 2000

Part 1 – the Fourier Transform

Source: Ronald N. Bracewell, The Fourier Transform and its applications, 3rd edition, McGraw-Hill, 2000
Chapter 2

The Fourier Transform

- Suppose we have a one-dimensional function *f(x)*, representing a physical quantity
- We will define *F(s)*, the Fourier transform of *f(x)*, as:

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi xs}\, dx$$

- This is sometimes written as:
  - $\mathcal{F}[f(x)] = F(s)$, or
  - $f(x) \supset F(s)$
- Notice that *s* describes the frequency of the oscillating term and is equal to the number of cycles per unit of *x*

The Fourier Transform

- The Fourier Transform is somehow reversible, i.e. :

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi xs}\, dx$$

$$f(x) = \int_{-\infty}^{\infty} F(s) e^{i2\pi xs}\, ds.$$

- The second equation can be written as:
  - $f(x) = \mathcal{F}^{-1}[F(s)]$, or
  - $F(s) \subset f(x)$
- Regarding these definitions, sometimes we shall say that:
  - $F(s)$ is the *minus-i* transform of $f(x)$
  - $f(x)$ is the *plus-i* transform of $F(s)$

Oddness and Evenness

- Let us remind that
  - a function *E*(*x*) such that $E(x) = E(-x)$ is a symmetrical, or *even* function
  - a function O(*x*) such that $O(x) = -O(-x)$ is a antisymmetrical, or *odd* function

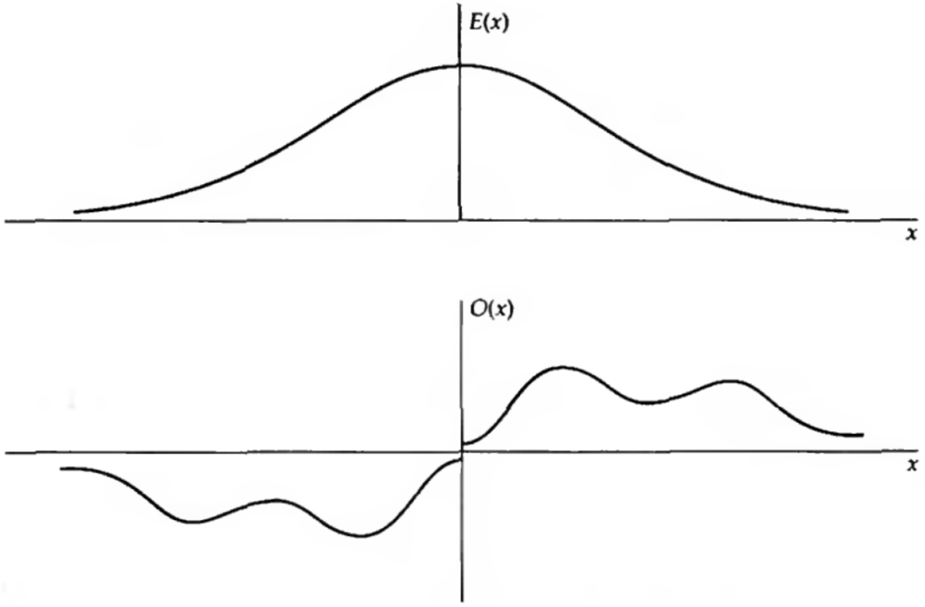


Fig. 2.2 An even function *E*(*x*) and an odd function *O*(*x*).

Oddness and Evenness

- Any function f(x) can be split unambiguously into its even [$E(x)$] and odd [O(x)] parts, $f(x) = E(x) + O(x)$, where:

$$E(x) = \tfrac{1}{2}[f(x) + f(-x)] \qquad O(x) = \tfrac{1}{2}[f(x) - f(-x)]$$

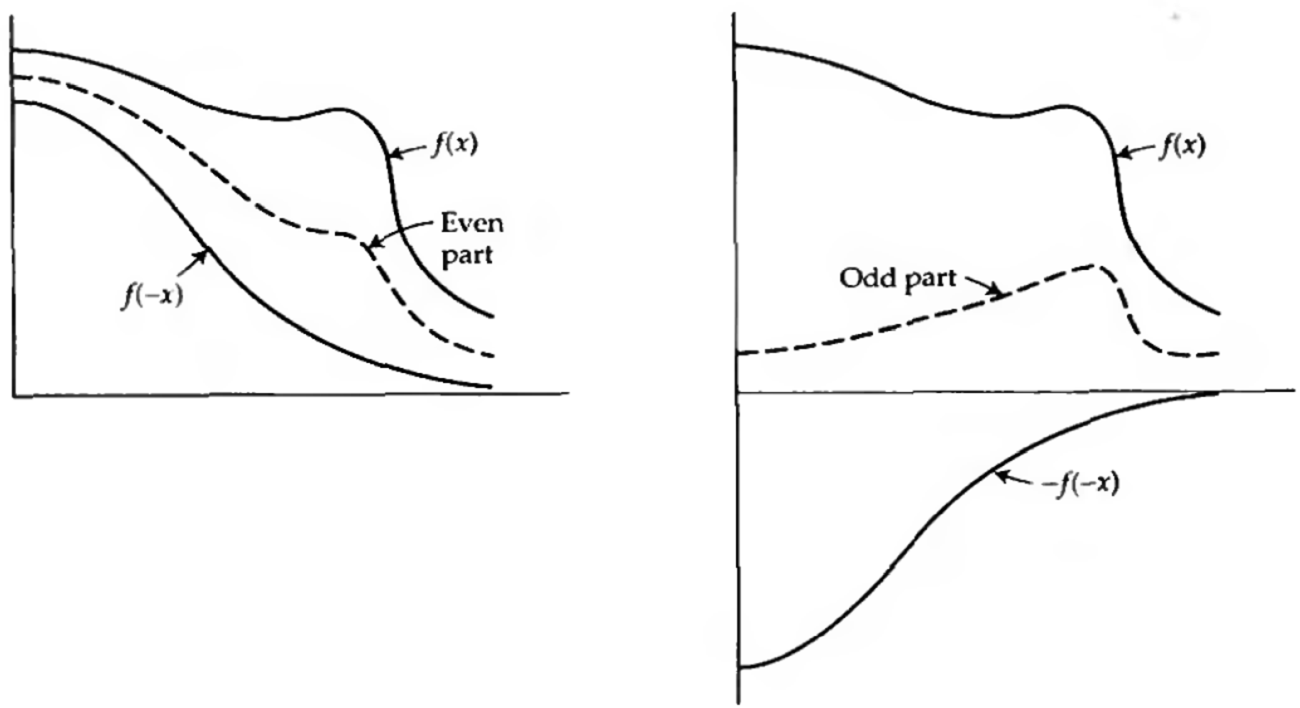


Fig. 2.4 Constructions for the even and odd parts of a given function $f(x)$.

Oddness and Evenness

- Since $f(x) = E(x) + O(x)$, then $F(s) = \mathcal{F}[f(x)]$ can be written as:

$$2\int_0^\infty E(x)\cos(2\pi xs)\,dx - 2i\int_0^\infty O(x)\sin(2\pi xs)\,dx$$

cosine transform of $E(x)$ (defined for $s>0$)

$i\cdot$ sine transform of $O(x)$ (defined for $s>0$)

- It is immediately noted that if $f(x)$ is even, then $F(s)$ is even and if $f(x)$ is odd, then $F(s)$ is odd
- Another important fact is that if $f(x)$ is even, then the $+i$ transform is actually the same as the $-i$ transform

Oddness and Evenness

- In general $f(x)$ [and thus $E(x)$ and $O(x)$] can have complex values. Thus, the following scheme applies:

$f(x)$	$F(s)$
Real and even	Real and even
Real and odd	Imaginary and odd
Imaginary and even	Imaginary and even
Complex and even	Complex and even
Complex and odd	Complex and odd
Real and asymmetrical	Complex and hermitian*
Imaginary and asymmetrical	Complex and antihermitian**
Real even plus imaginary odd	Real
Real odd plus imaginary even	Imaginary
Even	Even
Odd	Odd

- * hermitian = real part is even, imaginary part is odd
- ** antihermitian = real part is odd, imaginary part is even

Part 2 – convolution

Source: Ronald N. Bracewell, The Fourier Transform and its applications, 3rd edition, McGraw-Hill, 2000
Chapter 3

Convolution

The convolution of two functions $f(x)$ and $g(x)$ is

$$\int_{-\infty}^{\infty} f(u)g(x-u)\,du,$$

or briefly,

$$f(x) * g(x).$$

The convolution itself is also a function of x, let us say $h(x)$.

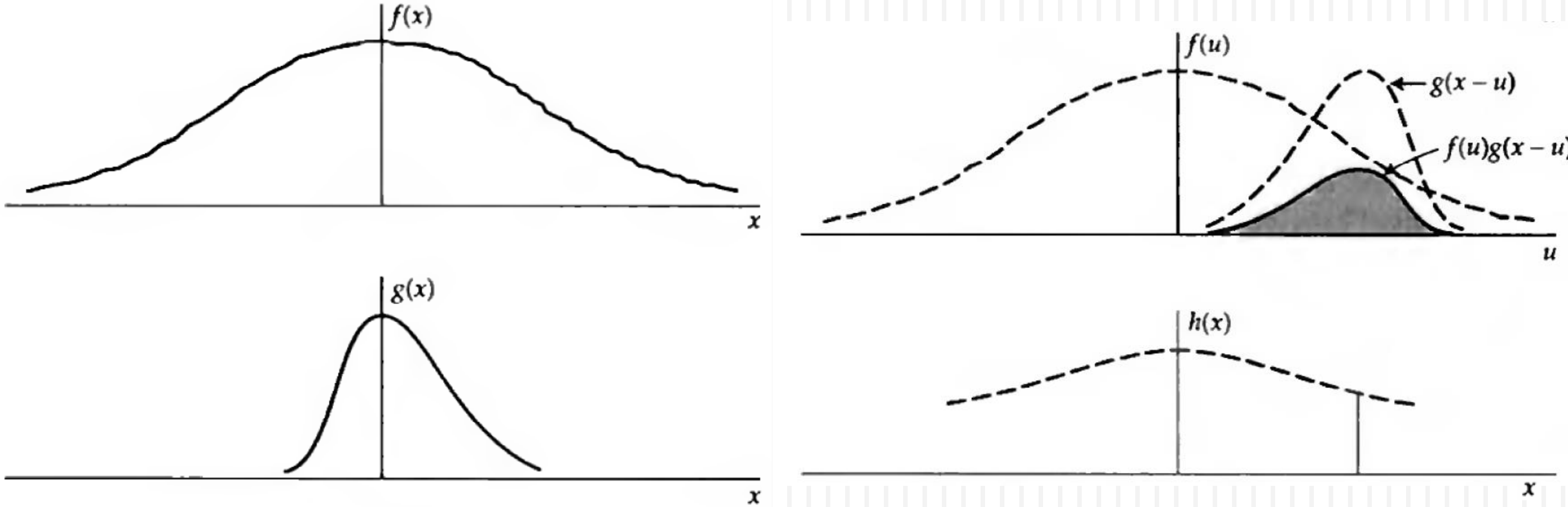
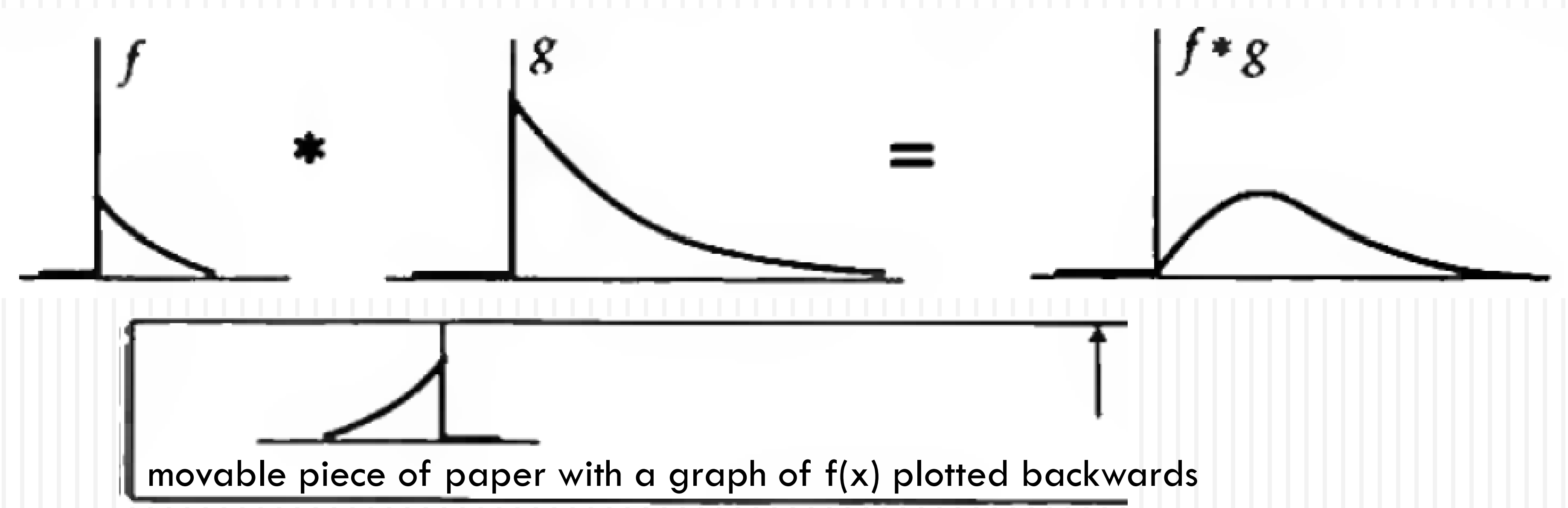


Fig. 3.1 The convolution integral $h(x) = f(x) * g(x)$ represented by a shaded area.

Graphical construction for convolution



Convolution as smoothing

- In some contexts, convolution means smoothing
- In the following example *h(x)* is
  - smoother
  - more spread out
  - with less total variation

  as compared with *f(x)*

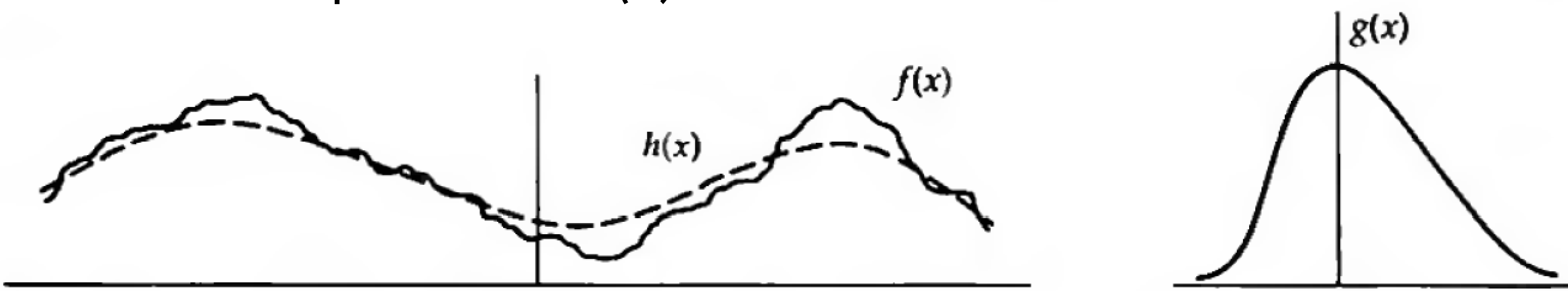


Fig. 3.2 **Illustrating the smoothing effect of convolution ($h = f * g$).**

Properties of convolution

convolution is commutative; that is,

$$f * g = g * f,$$

or

$$\int_{-\infty}^{\infty} f(u)g(x-u)\,du = \int_{-\infty}^{\infty} g(u)f(x-u)\,du.$$

Convolution is also associative (provided that all the convolution integrals exist),

$$f * (g * h) = (f * g) * h,$$

and distributive over addition,

$$f * (g + h) = f * g + f * h.$$

The abbreviated notation with asterisks (*) thus proves very convenient in formal manipulation, since the asterisks behave like multiplication signs.

Part 3 – some useful functions

Source: Ronald N. Bracewell, The Fourier Transform and its applications, 3rd edition, McGraw-Hill, 2000
Chapter 4

The rect function $\Pi(x)$

- the rectangle function of unit height and base is defined as:

$$\Pi(x) = \begin{cases} 0 & |x| > \frac{1}{2} \\ (\frac{1}{2} & |x| = \frac{1}{2}) \\ 1 & |x| < \frac{1}{2}. \end{cases}$$

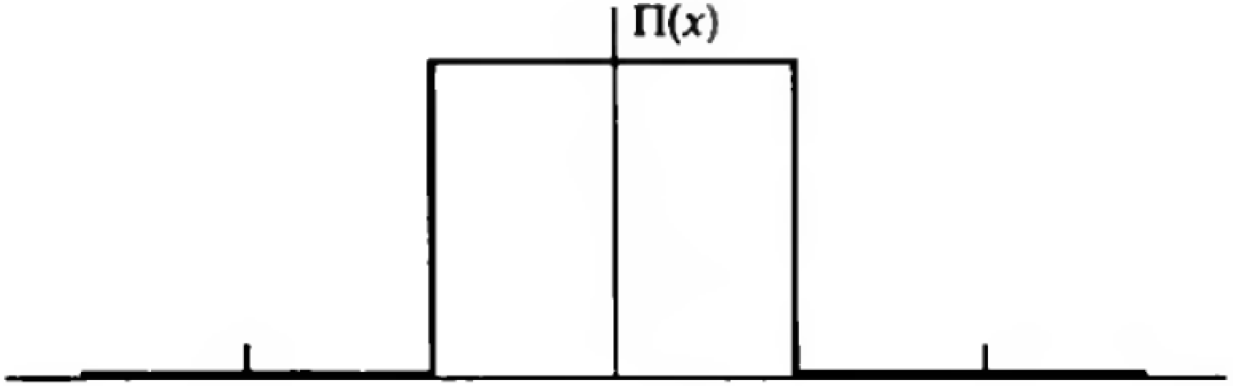


Fig. 4.1 **The rectangle function of unit height and base, $\Pi(x)$.**

The rect function $\Pi(x)$

It provides simple notation for segments of functions which have simple expres-
sions, for example, $f(x) = \Pi(x)\cos\pi x$ is compact notation for

$$f(x) = \begin{cases} 0 & x < -\frac{1}{2} \\ \cos\pi x & -\frac{1}{2} < x < \frac{1}{2} \\ 0 & \frac{1}{2} < x \end{cases}$$

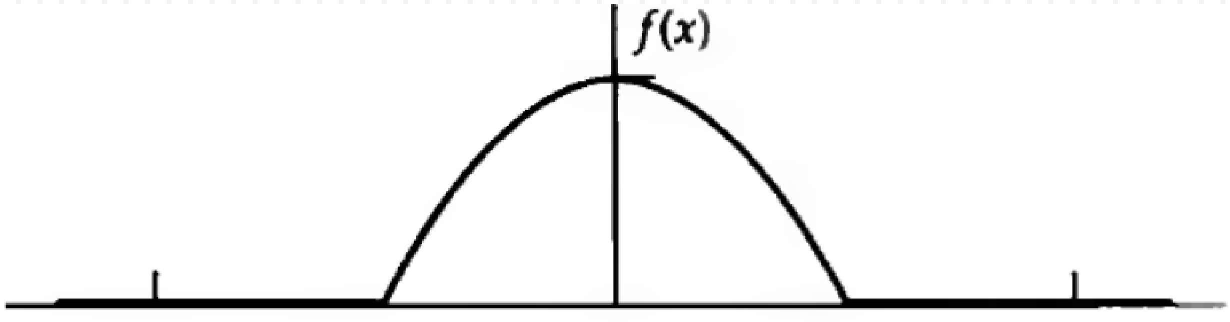


Fig. 4.2 **A segmented function expressed by $\Pi(x)\cos\pi x$.**

The rect function $\Pi(x)$

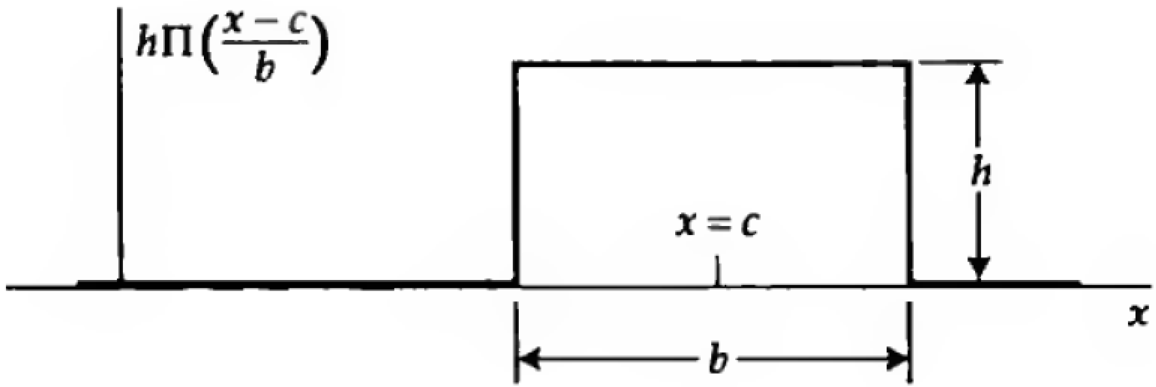


Fig. 4.3 **A displaced rectangle function of arbitrary height and base expressed in terms of $\Pi(x)$.**

The triangle function $\Lambda(x)$

- the triangle function of unit height and area is defined as:

$$\Lambda(x) = \begin{cases} 0 & |x| > 1 \\ 1 - |x| & |x| < 1. \end{cases}$$

Note that $h\Lambda(x/\tfrac{1}{2}b)$ is a triangle function of height h, base b, and area $\tfrac{1}{2}hb$.

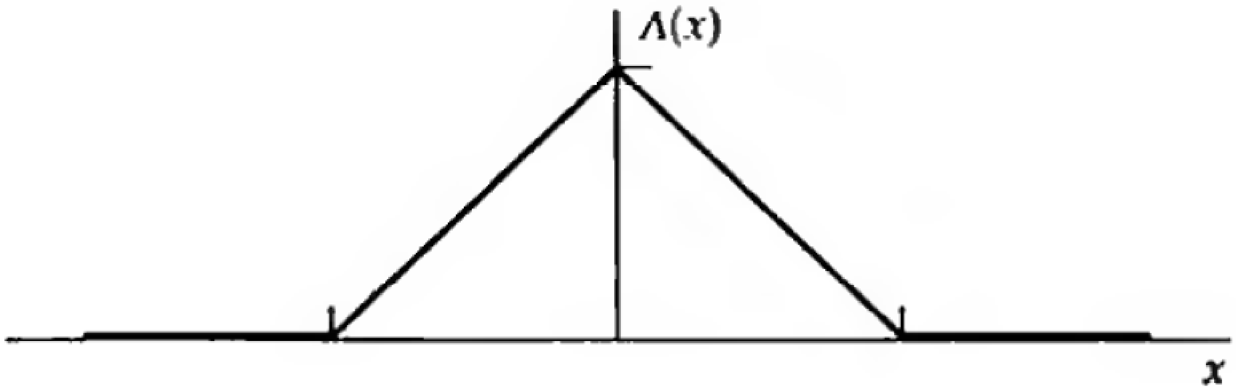


Fig. 4.4 **The triangle function of unit height and area, $\Lambda(x)$.**

The interpolating function *sinc*(x)

We define

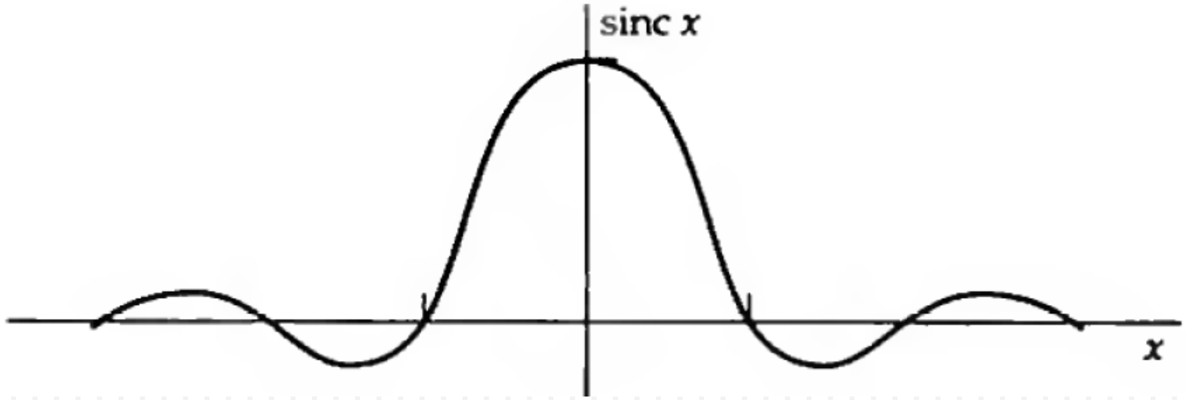
$$\operatorname{sinc} x = \frac{\sin \pi x}{\pi x},$$

a function with the properties that

$$\operatorname{sinc} 0 = 1,$$

$$\operatorname{sinc} n = 0 \qquad n = \text{nonzero integer}$$

$$\int_{-\infty}^{\infty} \operatorname{sinc} x \, dx = 1.$$



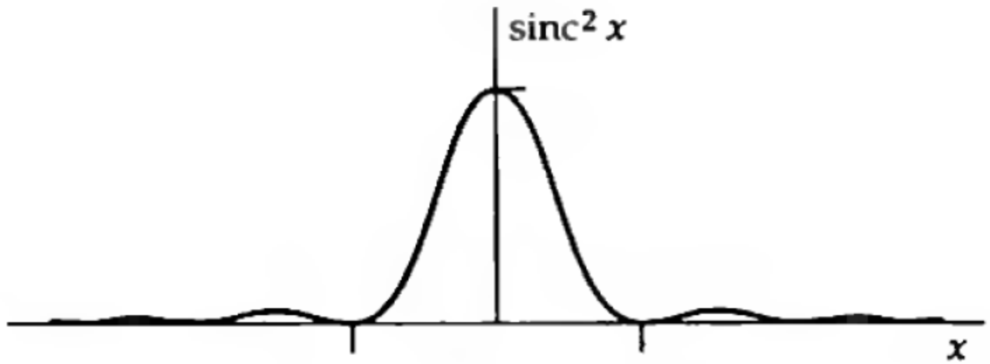
The square of *sinc(x)*

- Of note is also $\mathrm{sinc}^2 x = \left(\frac{\sin \pi x}{\pi x}\right)^2.$
- Among the properties of $\mathrm{sinc}^2 x$ are the following:

$$\mathrm{sinc}^2\, 0 = 1$$

$$\mathrm{sinc}^2\, n = 0 \qquad n = \text{nonzero integer}$$

$$\int_{-\infty}^{\infty} \mathrm{sinc}^2 x \, dx = 1.$$



Homework

- Prove the equations marked with a ☑.
- Show that:
  - if f(x) is even, then its (-i) Fourier Transform and its inverse (+i) Fourier transform coincide
  - if f(x) is real and even, then F(s) is real and even.
  - if f(x) is real and odd, then F(s) is imaginary and odd.
- Calculate the convolution h(x)= $aE(\alpha x) * bE(\beta x)$ where E(x) is a truncated exponential function:

$$E(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & x < 0. \end{cases}$$

- Show that if $f(x)$ is real,

$$\int_{-\infty}^{\infty} f(x)f(-x)\,dx = \int_{-\infty}^{\infty} [E(x)]^2\,dx - \int_{-\infty}^{\infty} [O(x)]^2\,dx,$$

and note that the left-hand side is the central value of the self-convolution of $f(x)$; that is, $f * f|_0$.

- Show that the first derivative of $\Lambda(x)$ is given by

$$\Lambda'(x) = -\Pi\left(\frac{x}{2}\right)\operatorname{sgn} x$$

ErrataCorrige (Bracewell, 3rd edition)

THE FOURIER TRANSFORM AND FOURIER'S INTEGRAL THEOREM

The Fourier transform of $f(x)$ is defined as

$$\int_{-\infty}^{\infty} f(x)e^{-i2\pi xs}\,dx.$$

This integral, which is a function of s, may be written $F(s)$. Transforming $F(s)$ by the same formula, we have

F(s)

$$\int_{-\infty}^{\infty} \cancel{f(s)}e^{-i2\pi ws}\,ds.$$

f(x)

When ~~$F(x)$~~ is an even function of x, that is, when $f(x) = f(-x)$, the repeated transformation yields $f(w)$, the same function we began with. This is the cyclical property of the Fourier transformation, and since the cycle is of two steps, the reciprocal property is implied: if $F(s)$ is the Fourier transform of $f(x)$, then $f(x)$ is the Fourier transform of $F(s)$.

The cyclical and reciprocal properties are imperfect, however, because when $f(x)$ is odd—that is, when $f(x) = -f(-x)$—the repeated transformation yields $f(-w)$. In general, whether $f(x)$ is even or odd or neither, repeated transformation yields $f(-w)$.

Correct (Bracewell, 2nd edition)

The Fourier transform and Fourier's integral theorem

The Fourier transform of $f(x)$ is defined as

$$\int_{-\infty}^{\infty} f(x)e^{-i2\pi xs}\, dx.$$

This integral, which is a function of s, may be written $F(s)$. Transforming $F(s)$ by the same formula, we have

$$\int_{-\infty}^{\infty} F(s)e^{-i2\pi ws}\, ds.$$

When $f(x)$ is an even function of x, that is, when $f(x) = f(-x)$, the repeated transformation yields $f(w)$, the same function we began with. This is the cyclical property of the Fourier transformation, and since the cycle is of two steps, the reciprocal property is implied: if $F(s)$ is the Fourier transform of $f(x)$, then $f(x)$ is the Fourier transform of $F(s)$.

The cyclical and reciprocal properties are imperfect, however, because when $f(x)$ is odd—that is, when $f(x) = -f(-x)$—the repeated transformation yields $f(-w)$. In general, whether $f(x)$ is even or odd or neither, repeated transformation yields $f(-w)$.

MEDICAL PHYSICS LAB
LECTURE 4 – MORE USEFUL FUNCTIONS & FOURIER TRANSFORMS

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More useful functions & Fourier Transforms

- Organized in 5 parts:
  - Part 1 – the impulse symbol $\delta(x)$
  - Part 2 – the sampling or replicating symbol $III(x)$
  - Part 3 – some useful Fourier transforms
  - Part 4 – theorems on the Fourier transform
  - Part 5 – more transforms involving $\delta(x)$
- Source:
  - Ronald N. Bracewell, The Fourier Transform and its applications, 3rd edition, McGraw-Hill, 2000

Part 1 – the impulse symbol $\delta(x)$

Source: Ronald N. Bracewell, The Fourier Transform and its applications, 3rd edition, McGraw-Hill, 2000
Chapter 5

The Dirac delta

- Often in physics we need to represent an impulse infinitely concentrated in time (or space), such that

$$\delta(x) = 0 \qquad x \neq 0$$

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1.$$

- the impulse symbol δ(x) is commonly referred to as the "Dirac delta"
- Please note that the impulse symbol δ(x) does not represent a function, since its value is undefined for x=0
- Dirac coined the term "improper function" while today we speak of a generalized function

The Dirac delta and the Heaviside step function

- Considering the definition of the Dirac delta the following identity is found:

$$\int_{-\infty}^{x} \delta(x')\, dx' = H(x)$$

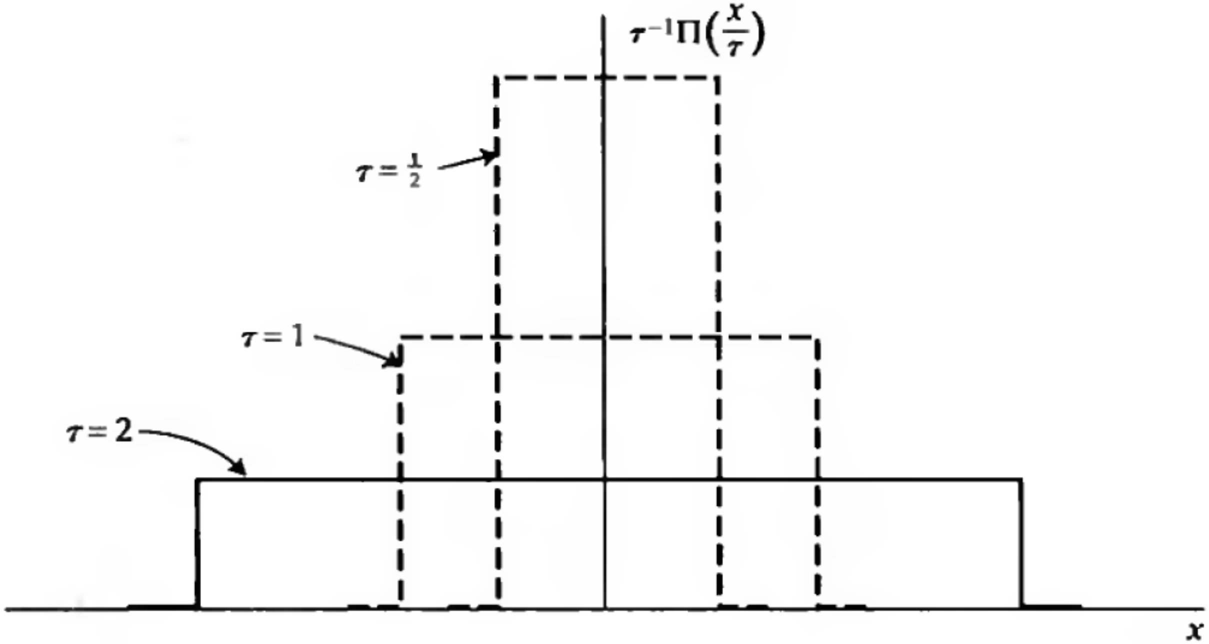
- Thus, one can state that

$$\delta(x) = \frac{d}{dx} H(x)$$

i.e. the Dirac delta "is the derivative of" the Heaviside step function

The Dirac delta as a generalized function

- Consider $\tau^{-1}\Pi(x/\tau)$
  - It is a rectangle function of height τ^{-1}, base τ and unit area
  - As τ tends to small values, this function gets taller and narrower, while keeping unit area



The Dirac delta as a generalized function

- The Dirac delta can be considered as the limiting case of this unit-area rectangle functions sequence:

$$\lim_{\tau \to 0} \quad \tau^{-1} \Pi\left(\frac{x}{\tau}\right)$$

- Other sequences that lead to the Dirac delta when $\tau \to 0$ are for instance:
  - unit-area Gaussian profiles $\quad \tau^{-1} e^{-\pi x^2/\tau^2}$
  - unit-area triangle functions $\quad \tau^{-1} \Lambda\left(\frac{x}{\tau}\right)$
  - unit-area sinc functions $\quad \tau^{-1} \operatorname{sinc} \frac{x}{\tau}$

(check that all the functions above have unit area)

Graphical representation of δ(x)

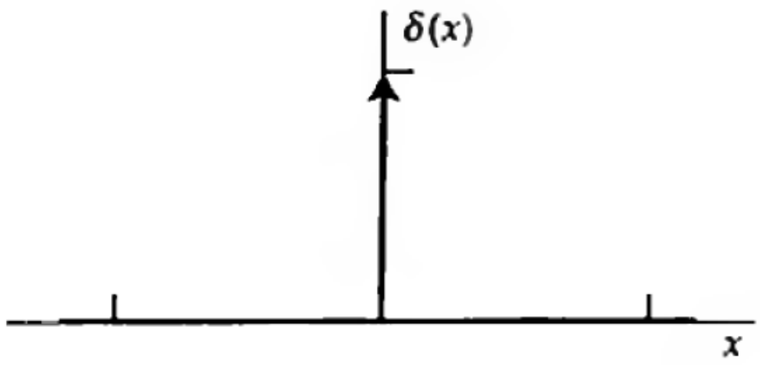
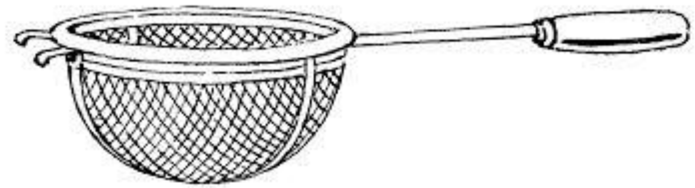


Fig. 5.3 Graphical representation of the impulse symbol $\delta(x)$ as a spike of unit height.

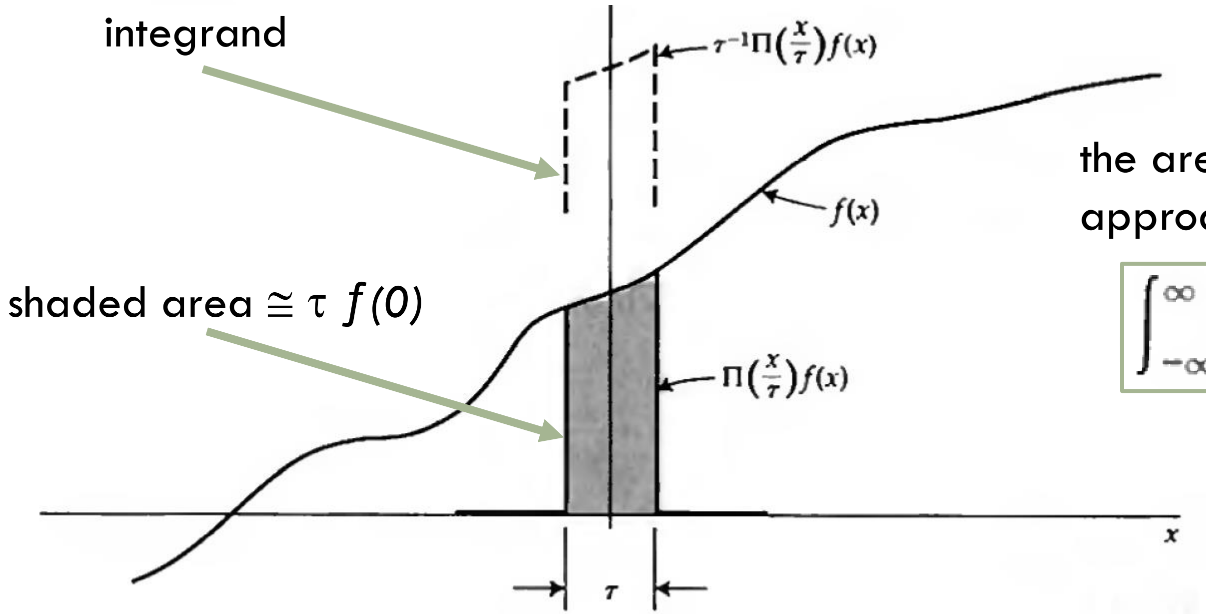
The sifting property



- Let us now consider $\int_{-\infty}^{\infty} \delta(x) f(x)\, dx.$

Thus we substitute the sequence $\tau^{-1}\Pi(x/\tau)$ for $\delta(x)$, perform the multiplication and integration, and finally take the limit of the integral as $\tau \to 0$:

$$\lim_{\tau \to 0} \int_{-\infty}^{\infty} \tau^{-1}\Pi\left(\frac{x}{\tau}\right) f(x)\, dx.$$



the area under the integrand approaches *f (0)* as $\tau \to 0$, i.e.:

$$\int_{-\infty}^{\infty} \delta(x) f(x)\, dx = f(0)$$

Fig. 5.2 **Explaining the sifting property. The shaded area is approximately $\tau f(0)$.**

The sifting property as a convolution

- The sifting property implies also that

$$\int_{-\infty}^{\infty} \delta(x-a) f(x)\, dx = f(a)$$

$$\int_{-\infty}^{\infty} \delta(x) f(x-a)\, dx = f(-a)$$

- The same concept can be written as a convolution

$$\int_{-\infty}^{\infty} \delta(x') f(x-x')\, dx' = \int_{-\infty}^{\infty} \delta(x-x') f(x')\, dx' = f(x)$$

or, using the asterisk notation

$$\delta(x) * f(x) = f(x) * \delta(x) = f(x)$$

Other properties of δ(x)

- Considering δ(x) as the limiting case of a sequence, it's easy to show that $\delta(ax) = \frac{1}{|a|}\delta(x)$.
- In the particular case $a = -1$ this reduces to $\delta(-x) = \delta(x)$
- if $f(x)$ is continuous at $x = 0$,

$$f(x)\,\delta(x) = f(0)\,\delta(x)$$

- From the sifting property, putting $f(x) = x$, we have

$$\int_{-\infty}^{\infty} x\,\delta(x)\,dx = 0.$$

One generally writes

$$x\,\delta(x) = 0,$$

Part 2 – the sampling or replicating symbol *III*(*x*)

Source: Ronald N. Bracewell, The Fourier Transform and its applications, 3rd edition, McGraw-Hill, 2000
Chapter 5

The *shah* symbol *III(x)*

- Let us consider an infinite sequence of unit impulses spaced at unit interval. This can be described by the *shah* symbol *III(x)*:

$$\mathrm{III}(x) = \sum_{n=-\infty}^{\infty} \delta(x-n)$$

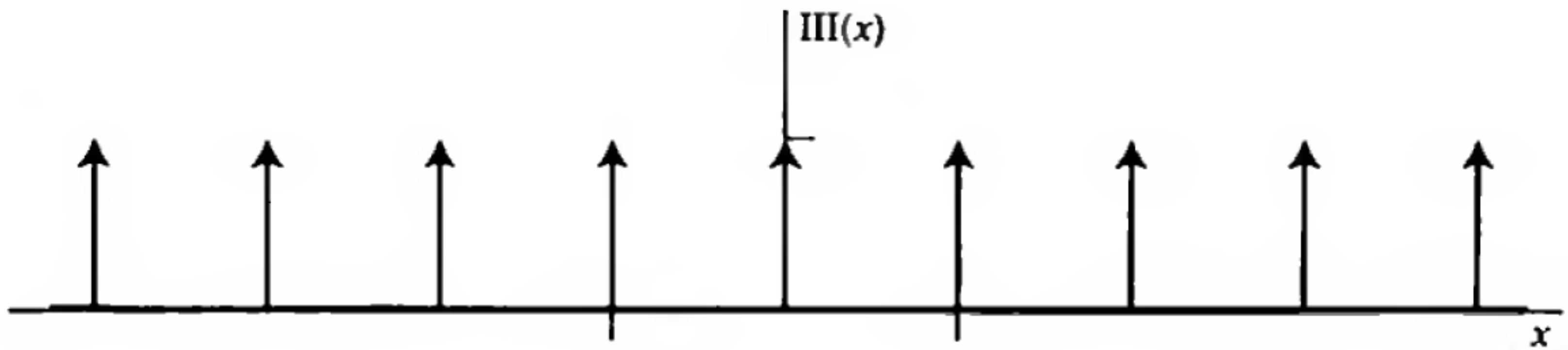
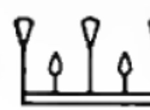


Fig. 5.4 **The shah symbol III(x).**

[2]The symbol III is pronounced *shah* after the Cyrillic character Ш, which is said to have been modeled on the Hebrew letter ש (*shin*), which in turn may derive from the Egyptian , a hieroglyph depicting papyrus plants along the Nile.

Properties of *III*(*x*)

Various obvious properties may be pointed out:

$$\text{III}(ax) = \frac{1}{|a|}\sum \delta\left(x - \frac{n}{a}\right)$$

$$\text{III}(-x) = \text{III}(x)$$

$$\text{III}(x + n) = \text{III}(x) \qquad n \text{ integral}$$

$$\text{III}(x - \tfrac{1}{2}) = \text{III}(x + \tfrac{1}{2})$$

$$\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \text{III}(x)\, dx = 1$$

$$\text{III}(x) = 0 \qquad x \neq n.$$

Evidently, III(x) is periodic with unit period.

The sampling property of $III(x)$

A periodic *sampling* property follows as a generalization of the sifting integral already discussed in connection with the impulse symbol. Thus multiplication of a function $f(x)$ by $\text{III}(x)$ effectively samples it at unit intervals:

$$\text{III}(x)f(x) = \sum_{n=-\infty}^{\infty} f(n)\,\delta(x-n).$$

The information about $f(x)$ in the intervals between integers where $\text{III}(x) = 0$ is not contained in the product; however, the values of $f(x)$ at integral values of x are preserved (see Fig. 5.5).

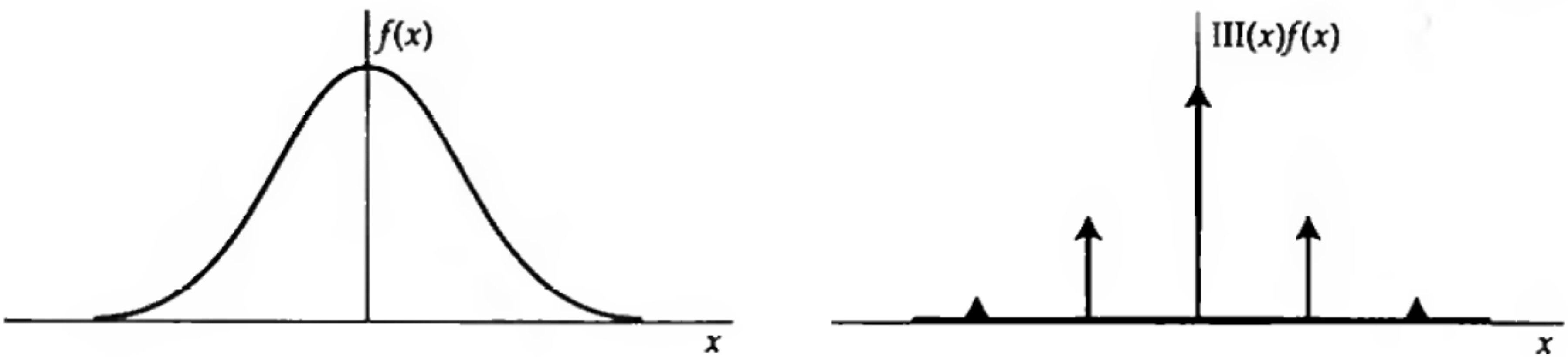


Fig. 5.5 **The sampling property of III(x).**

The replicating property of *III(x)*

Just as important as the sampling property under multiplication is a *replicating* property exhibited when III(x) enters into convolution with a function $f(x)$. Thus

$$\mathrm{III}(x) * f(x) = \sum_{n=-\infty}^{\infty} f(x-n);$$

as shown in Fig. 5.6, the function $f(x)$ appears in replica at unit intervals of x *ad infinitum* in both directions. Of course, if $f(x)$ spreads over a base more than one unit wide, there is overlapping.

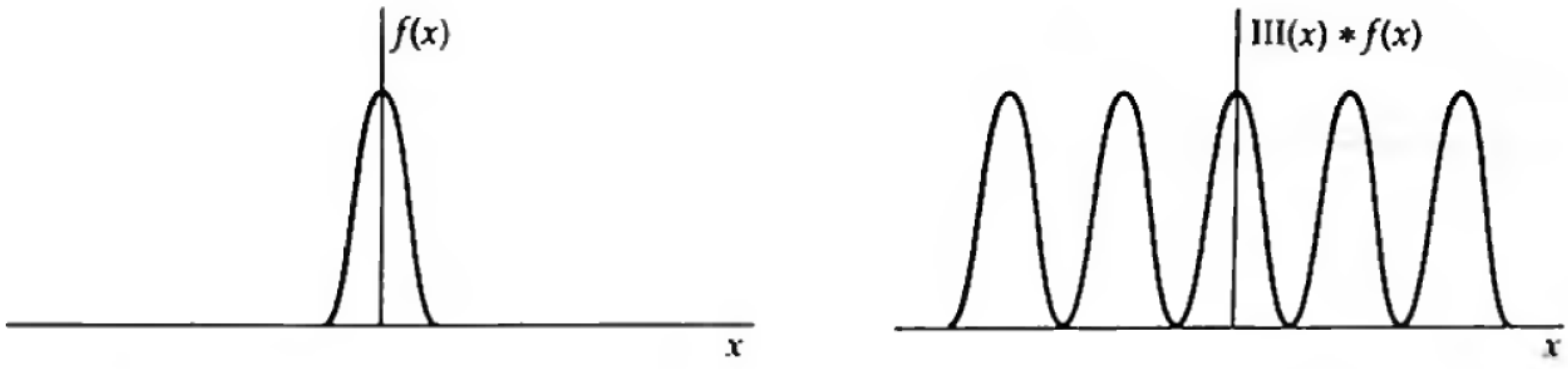


Fig. 5.6 **The replicating property of III(x).**

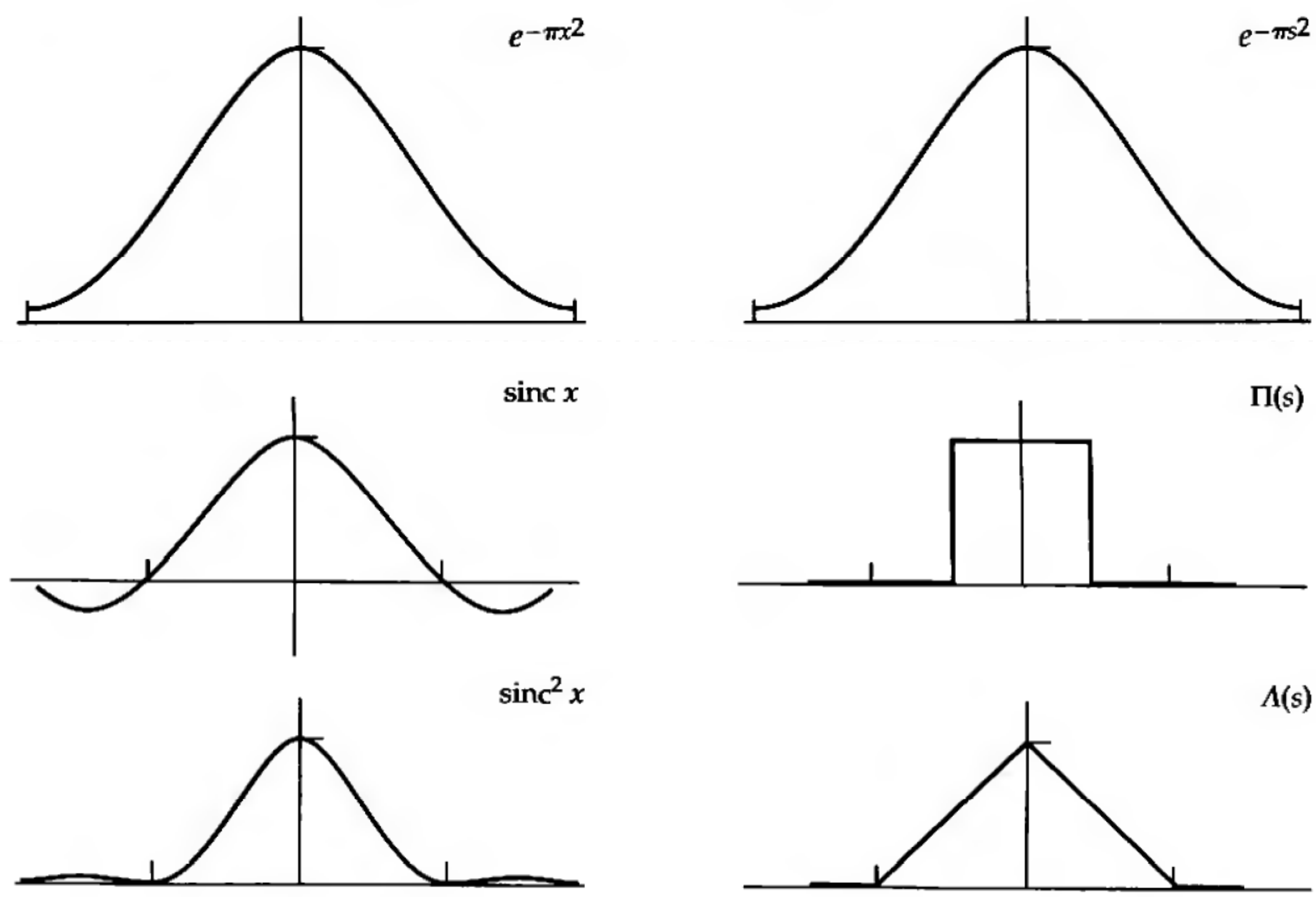
Part 3 – Fourier pairs

Source: Ronald N. Bracewell, The Fourier Transform and its applications, 3rd edition, McGraw-Hill, 2000
Chapters 6-7

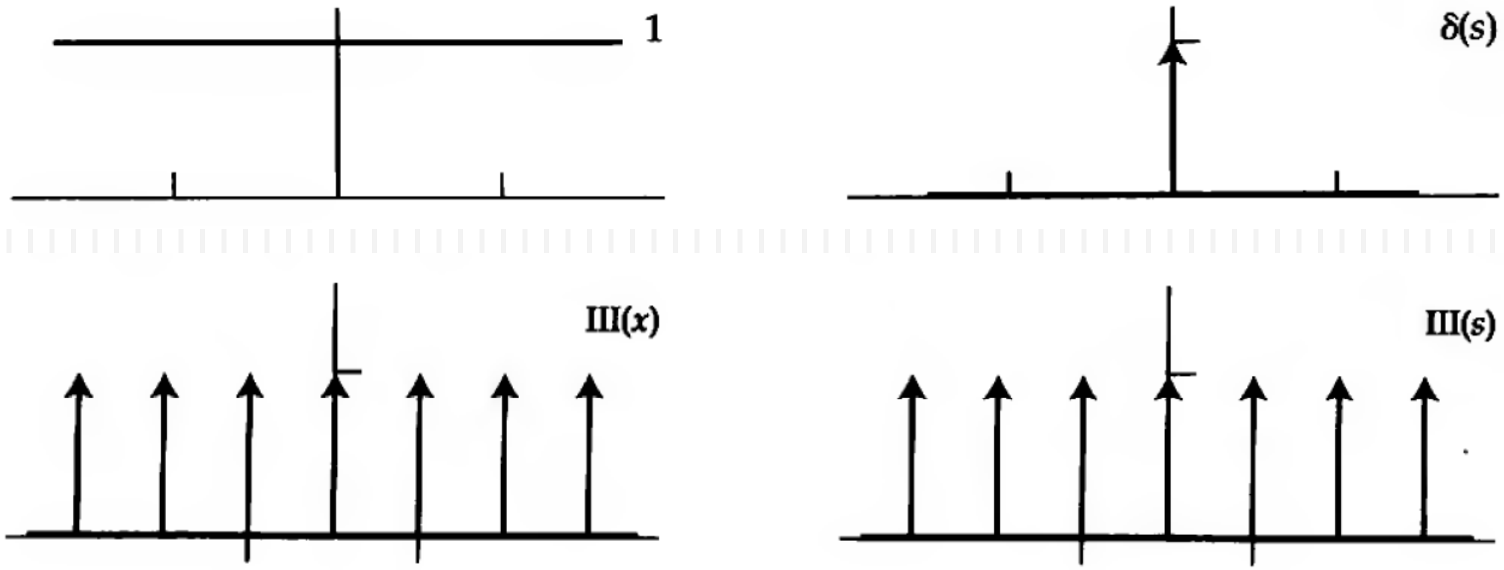
Fourier pairs

- Some useful Fourier Transforms are:
  - $e^{-\pi x^2} \supset e^{-\pi s^2}$
  - $sinc\, x \supset \Pi(s)$
  - $sinc^2\, x \supset \Lambda(s)$
  - $\delta(x) \supset 1 = H(|s|)$
  - $III(x) \supset III(s)$
- Since these are all even functions, the $+i$ transform is actually the same as the $-i$ transform and thus we have also:
  - $e^{-\pi x^2} \supset e^{-\pi s^2}$
  - $\Pi(x) \supset sinc\, s$
  - $\Lambda(x) \supset sinc^2\, s$
  - $1 = H(|x|) \supset \delta(s)$
  - $III(x) \supset III(s)$
- These functions are said to be Fourier pairs

Fourier Pairs



Fourier Pairs



Part 4 – Theorems on the Fourier transform

Source: Ronald N. Bracewell, The Fourier Transform and its applications, 3rd edition, McGraw-Hill, 2000
Chapter 6

SIMILARITY THEOREM

If $f(x)$ has the Fourier transform $F(s)$, then $f(ax)$ has the Fourier transform $|a|^{-1}F(s/a)$.

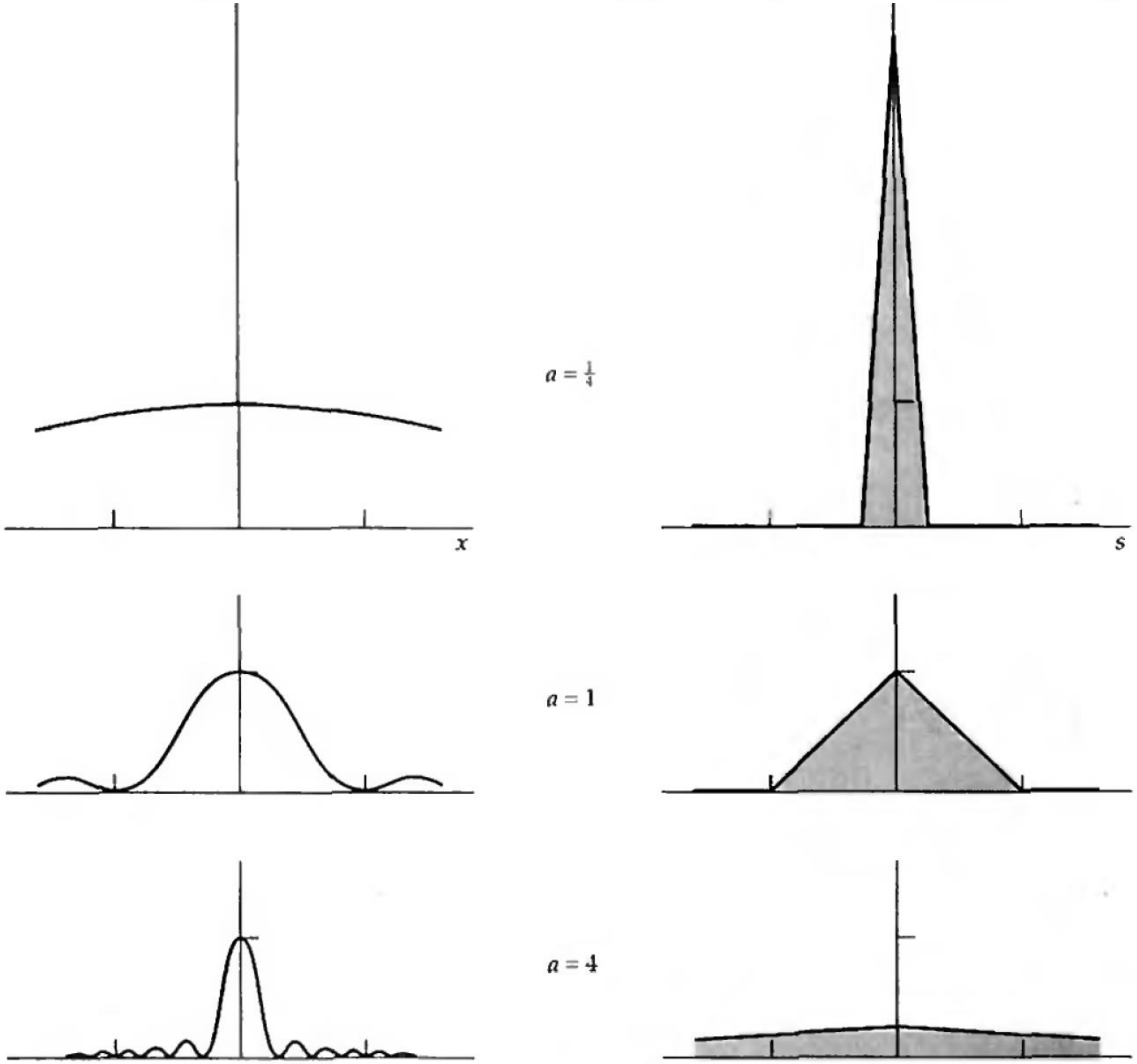


Fig. 6.2 The effect of changes in the scale of abscissas as described by the similarity, or abscissa-scaling, theorem. The shaded area remains constant.

ADDITION THEOREM

If $f(x)$ and $g(x)$ have the Fourier transforms $F(s)$ and $G(s)$, respectively, then $f(x) + g(x)$ has the Fourier transform $F(s) + G(s)$.

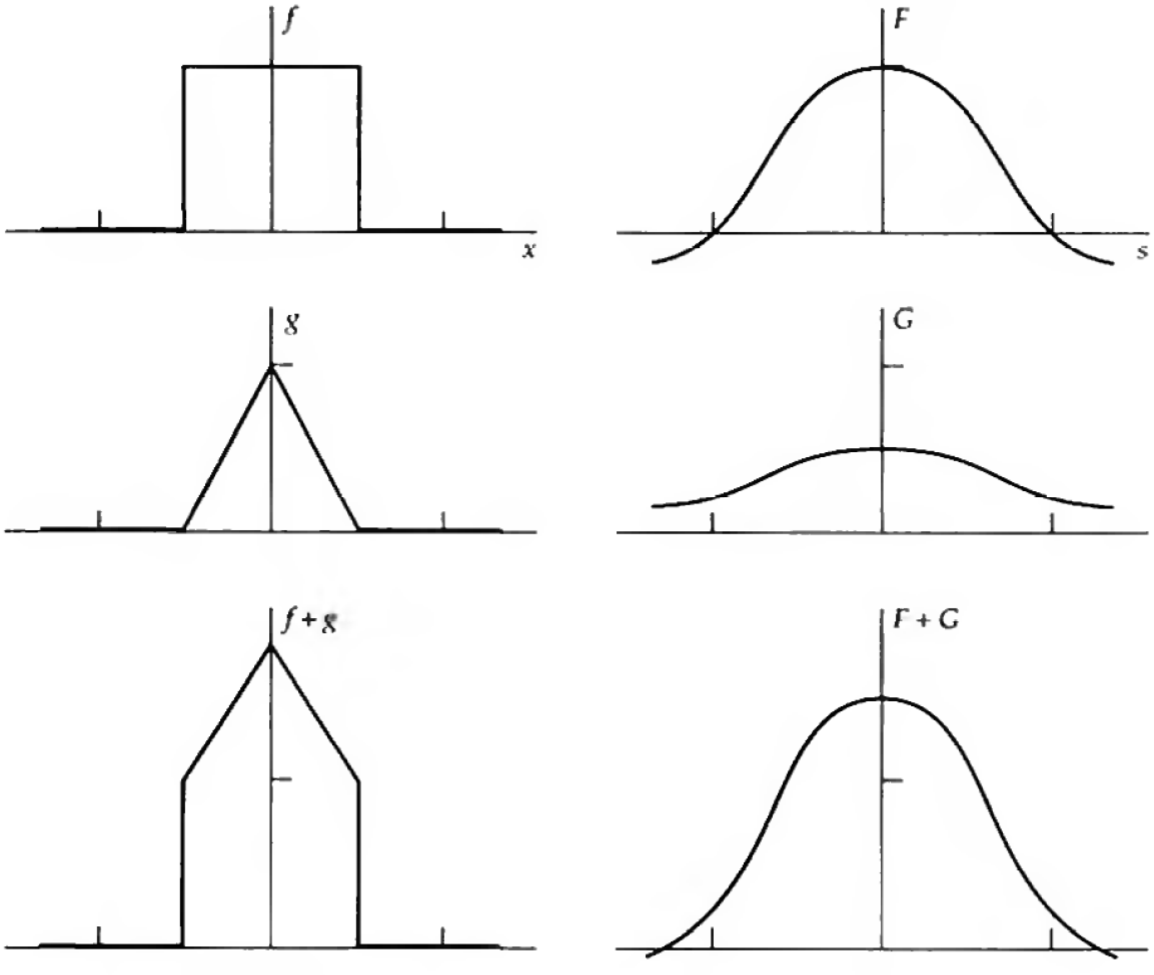
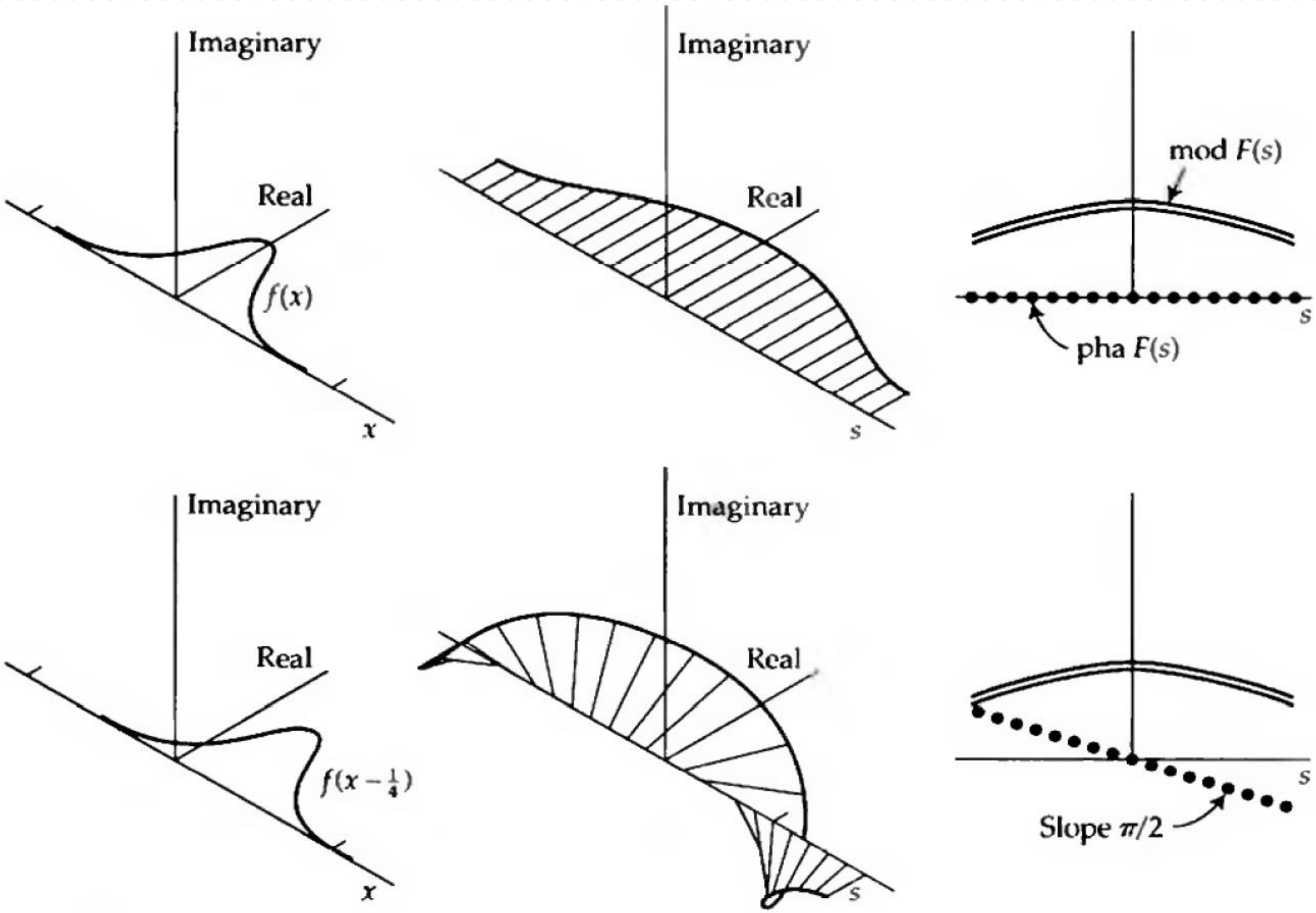


Fig. 6.5 The addition theorem $f + g \supset F + G$.

SHIFT THEOREM

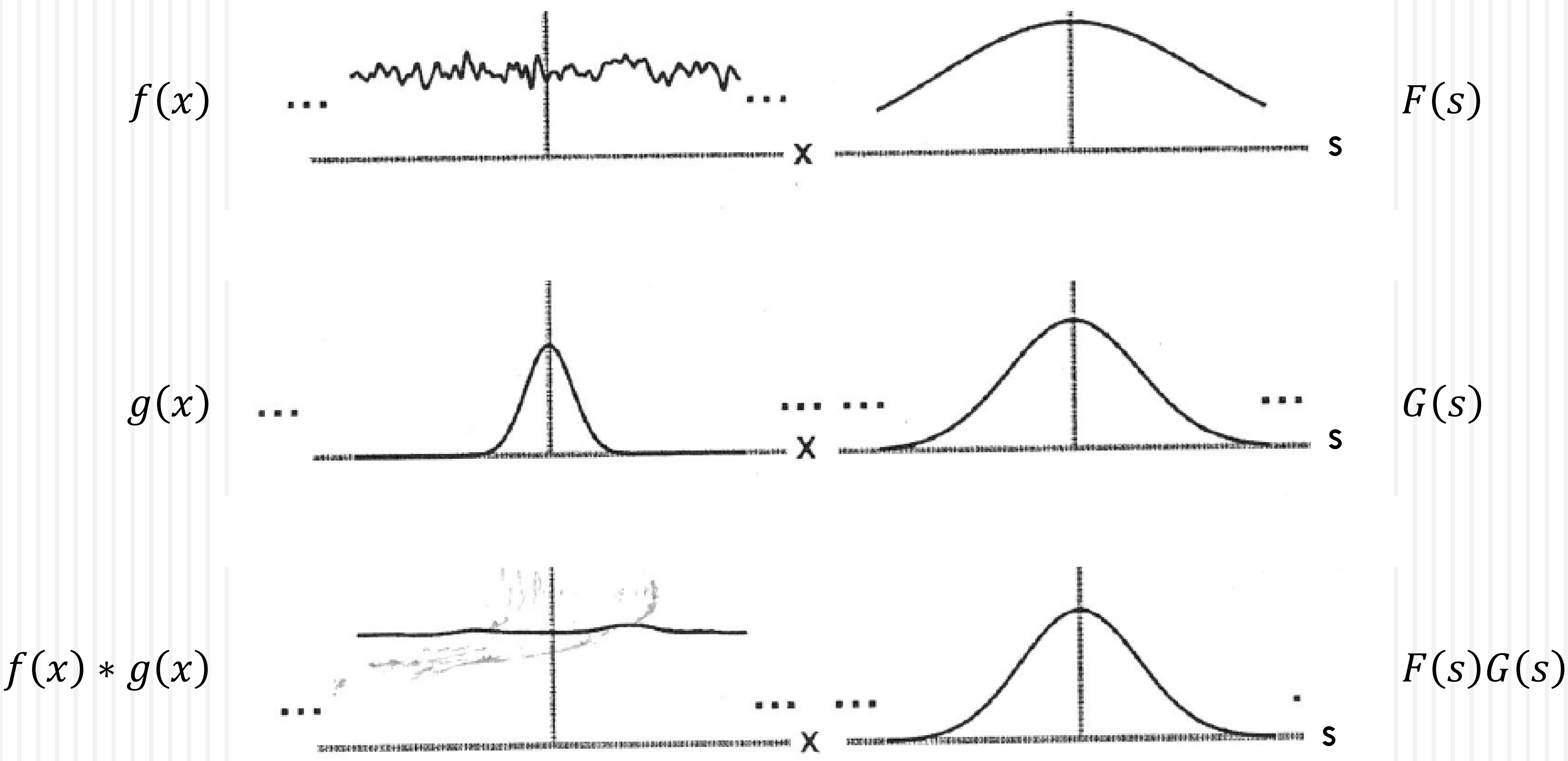
If $f(x)$ has the Fourier transform $F(s)$, then $f(x-a)$ has the Fourier transform $e^{-2\pi ias}F(s)$.



CONVOLUTION THEOREM

*If f(x) has the Fourier transform F(s) and g(x) has the Fourier transform G(s), then f(x) * g(x) has the Fourier transform F(s)G(s); that is, convolution of two functions means multiplication of their transforms.*

- Quite remarkably, the converse holds as well:
  - $f(x) * g(x) \supset F(s)G(s)$ and also
  - $f(x)g(x) \supset F(s) * G(s)$



RAYLEIGH'S THEOREM

The integral of the squared modulus of a function is equal to the integral of the squared modulus of its spectrum; that is,

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |F(s)|^2\, ds.$$

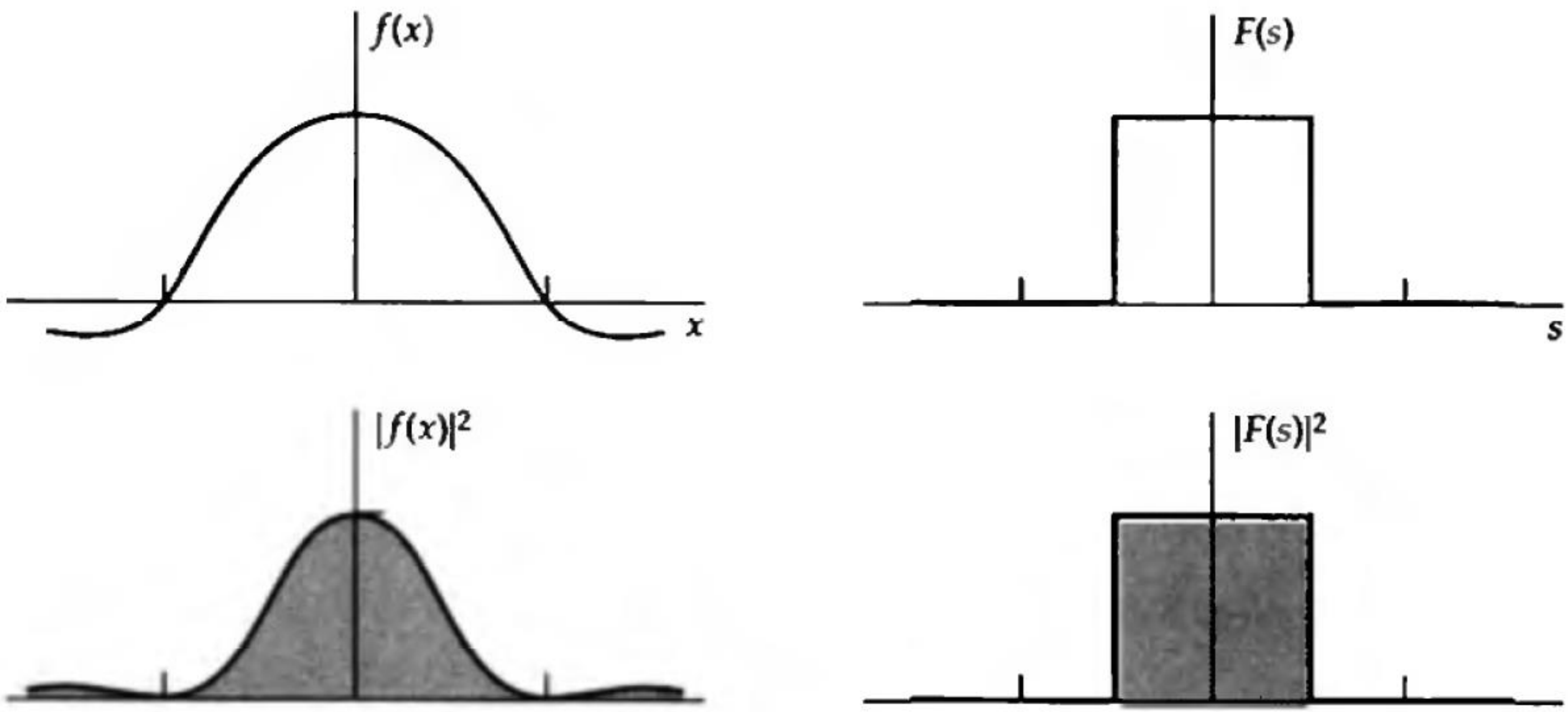


Fig. 6.9 **Rayleigh's theorem: the shaded areas are equal.**

Part 5 – more transforms involving δ(x)

Sources: Ronald N. Bracewell, The Fourier Transform and its applications, 3rd edition, McGraw-Hill, 2000

http://www.thefouriertransform.com/

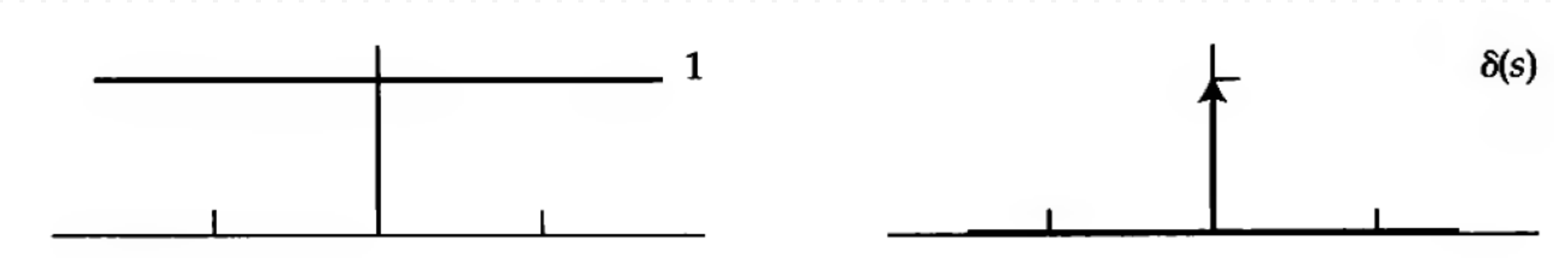
Transforms involving Dirac delta

- Let us consider again the Fourier Pair:
  - $\delta(x) \supset 1$
  - $1 \supset \delta(s)$
- If we write out the second identity using the definition of Fourier transform, we have:
  - $1 \supset \delta(s)$
  - $\delta(s) = \int_{-\infty}^{+\infty} e^{-i2\pi sx} dx$
- By changing the name of the variables, we obtain the so-called integral representation of the Dirac delta:

$$\delta(x) = \int_{-\infty}^{+\infty} e^{-i2\pi xu} du$$

Transforms involving Dirac delta

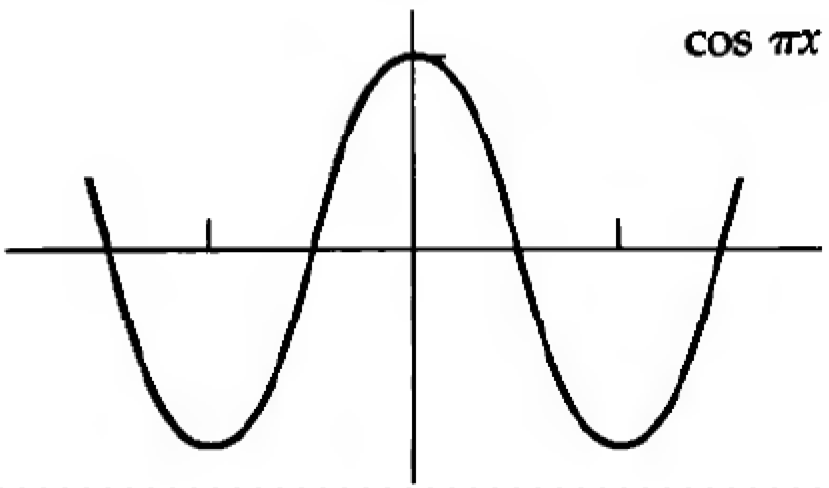
- Let us consider again the Fourier Pair:
  - $\delta(x) \supset 1$
  - $1 \supset \delta(s)$

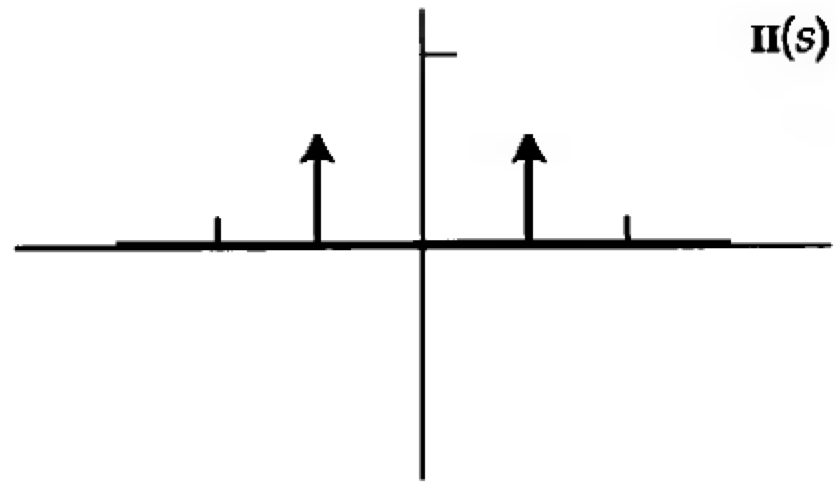


- More general relations involving Dirac delta are:
  - $\delta(x-a) \supset e^{-2\pi ias}$ (by virtue of the shift theorem)
  - $e^{2\pi ibx} \supset \delta(s-b)$ (follows from definition)

Fourier Transform of $\cos(\pi x)$

- Since $\cos(2\pi bx) = \frac{e^{i2\pi bx} + e^{-i2\pi bx}}{2}$, it follows:
  - $\cos(2\pi bx) \supset \frac{1}{2}[\delta(s-b) + \delta(s+b)]$
  - $\cos(\pi x) \supset \frac{1}{2}\left[\delta\left(s - \frac{1}{2}\right) + \delta\left(s + \frac{1}{2}\right)\right] \equiv \mathrm{II}(\mathrm{s})$

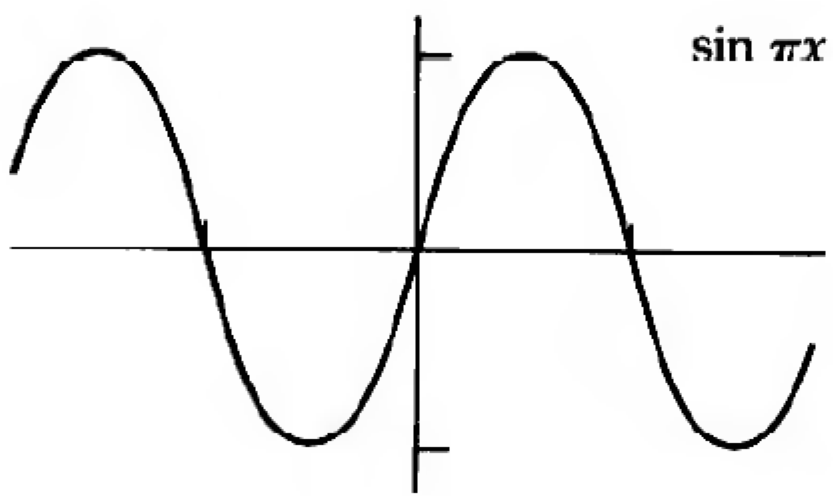


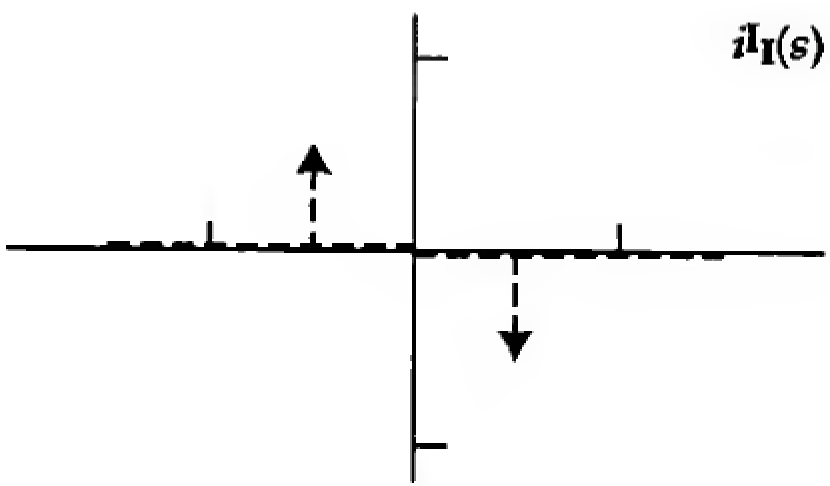


- $\mathrm{II}(x) = \frac{1}{2}\delta(x + \frac{1}{2}) + \frac{1}{2}\delta(x - \frac{1}{2})$ is the even impulse pair
  - note: do not confuse with the rectangle function Π (s)

Fourier Transform of *sin(πx)*

- Similarly since $\sin(2\pi bx) = \frac{e^{i2\pi bx} - e^{-i2\pi bx}}{2i}$, it follows:
  - $\sin(2\pi bx) \supset \frac{1}{2i}[\delta(s-b) - \delta(s+b)]$
  - $\sin(\pi x) \supset \frac{1}{2i}\left[\delta\left(s-\frac{1}{2}\right) - \delta\left(s+\frac{1}{2}\right)\right] \equiv i\,\mathrm{I}_{\mathrm{I}}(s)$





- $\mathrm{I}_{\mathrm{I}}(x) = \frac{1}{2}\delta(x+\frac{1}{2}) - \frac{1}{2}\delta(x-\frac{1}{2})$ is the odd impulse pair

Homework

- Prove the equations and the theorems marked with ☑
- Check that an immediate application of the similarity theorem to $e^{-\pi x^2} \supset e^{-\pi s^2}$ is:

$$\int_{-\infty}^{\infty} e^{-\pi(ax)^2} e^{-i2\pi sx}\, dx = |a|^{-1} e^{-\pi(s/a)^2}.$$

  - verify that in the limit $a \to 0$ this implies $1 \supset \delta(s)$
- By using the definition of the Fourier Transform, show that: $\Pi(x) \supset sinc\ s$
- Try to show the following identity:

$$sinc(x) * sinc(x) = sinc(x)$$

  - 1) by using the definition of convolution (don't waste your time… give up immediately, it's just to realize that's not easy)
  - 2) by using the convolution theorem (that's easy!)

MEDICAL PHYSICS LAB

LECTURE 5 –
THE TWO DOMAINS

Luigi Rigon
University of Trieste and INFN

Practical Use of Fourier Transforms

- Organized in 2 parts:
  - Part 1 – the time and frequency domains
  - Part 2 – transforming images: the space and spatial frequency domains
- Sources:
  - T.A.Gallagher, A.J.Nemeth, L.Hacein-Bey *An Introduction to the Fourier Transform: Relationship to MRI* AJR:190 (2008) 1396-1405
  - http://www.imaios.com/en/e-Courses/e-MRI/The-Physics-behind-it-all/Fourier-transform
  - http://www.thefouriertransform.com/
  - http://www.ysbl.york.ac.uk/~cowtan/fourier/fourier.html

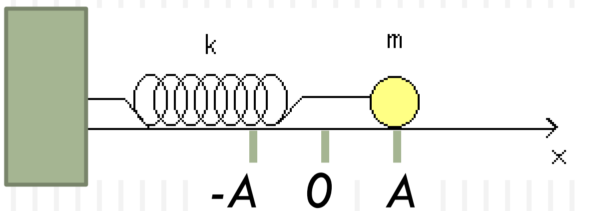
Part 1 – the time and frequency domains

Sources: T.A.Gallagher, A.J.Nemeth, L.Hacein-Bey An Introduction to the Fourier Transform: Relationship to MRI AJR:190 (2008) 1396-1405

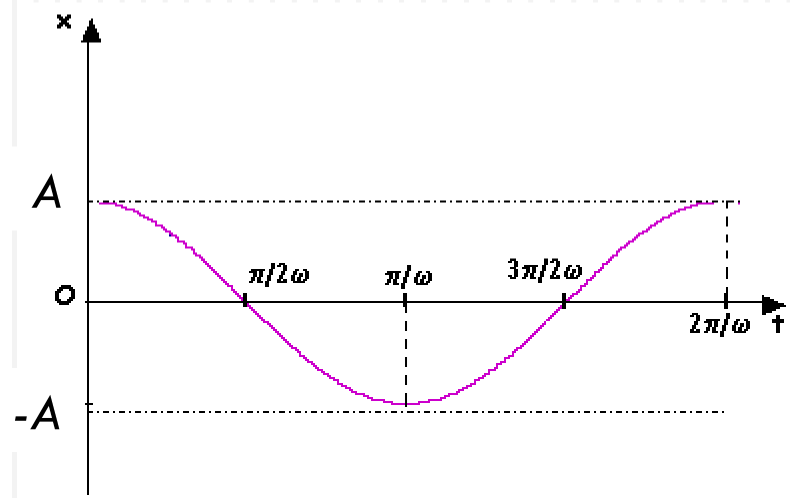
http://www.imaios.com/en/e-Courses/e-MRI/The-Physics-behind-it-all/Fourier-transform

Oscillations

- Often a physical quantity is an oscillating function of time, the most obvious example being the simple harmonic motion:



$$x(t) = A\cos(\omega t)$$



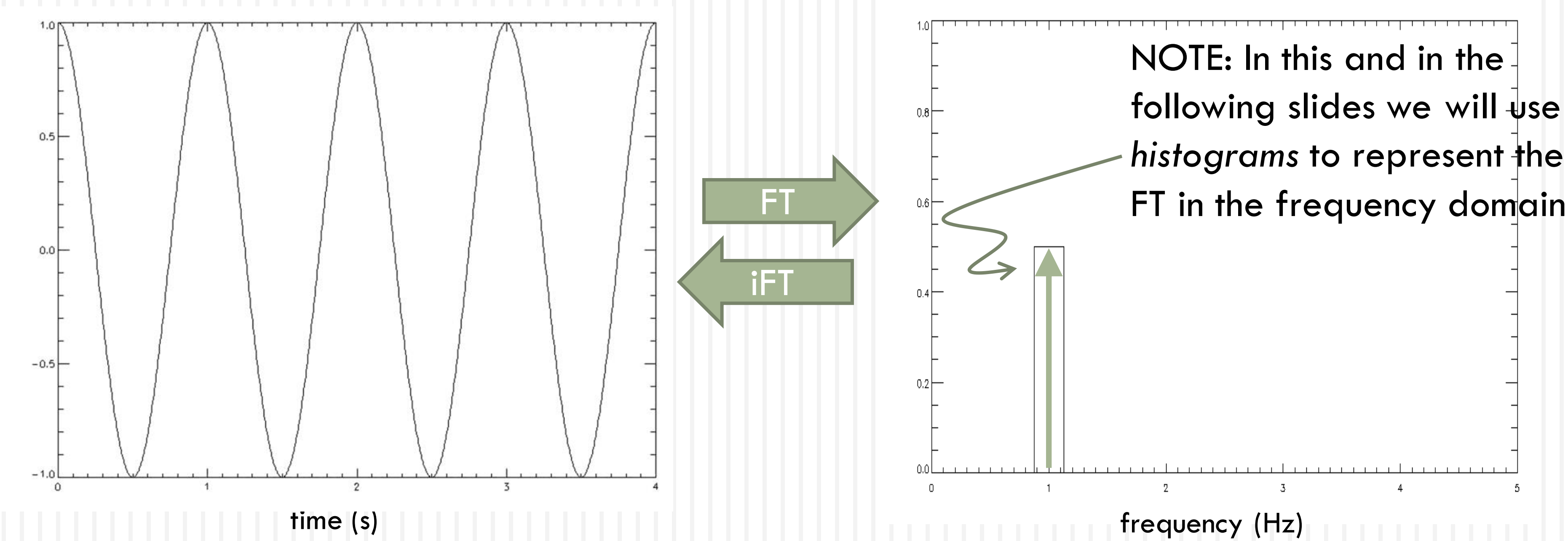
$$\omega = \frac{2\pi}{T} = 2\pi\nu$$

$$\nu = \frac{1}{T}$$

  - A is the amplitude of the oscillation
  - T is the period and it is measured in s
  - ν is the frequency and it is measured in Hertz (1 Hz= 1s^{-1})

The time and frequency domains

- By performing the Fourier Transform of an harmonic oscillation we represent it in a diagram indicating the amplitude as a function of the frequency (here the negative frequencies are disregarded)

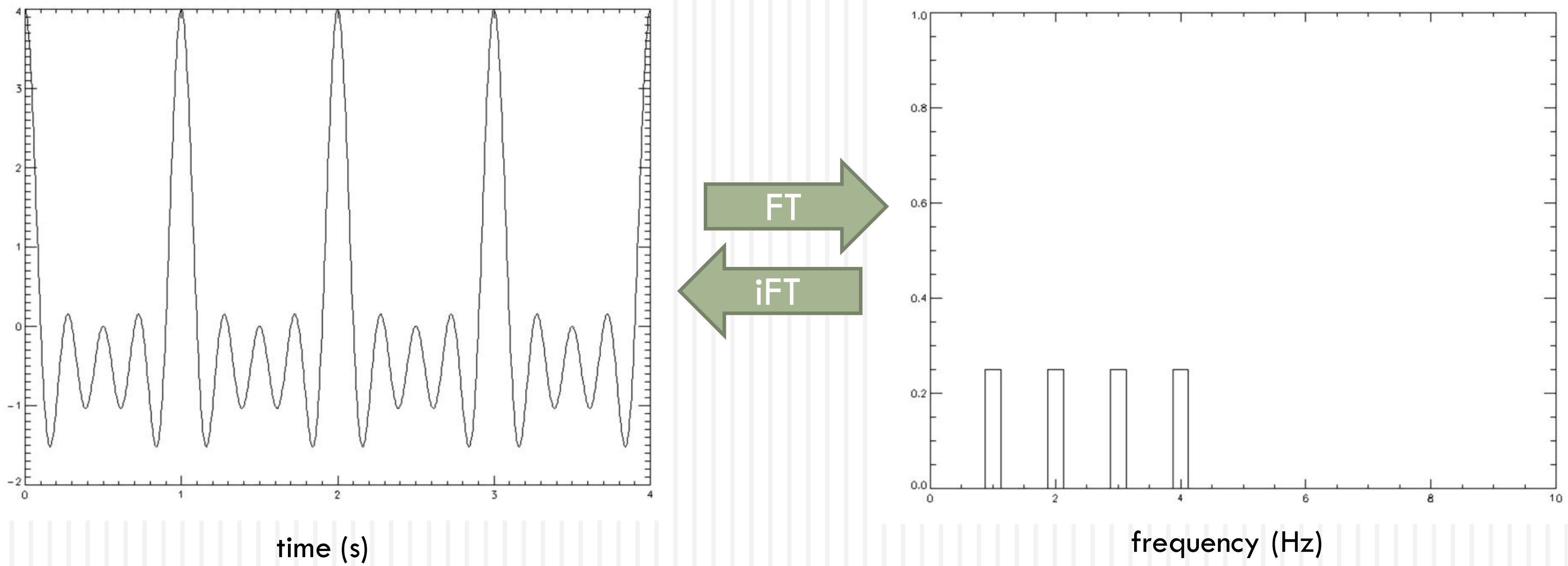


- The descriptions in the two domains (time or frequency) are equivalent, thanks to the reversibility of the Fourier Transform

The time and frequency domains

- The representation in the frequency domain is particularly useful in the case of complex oscillations (sum of harmonic oscillations), eg:

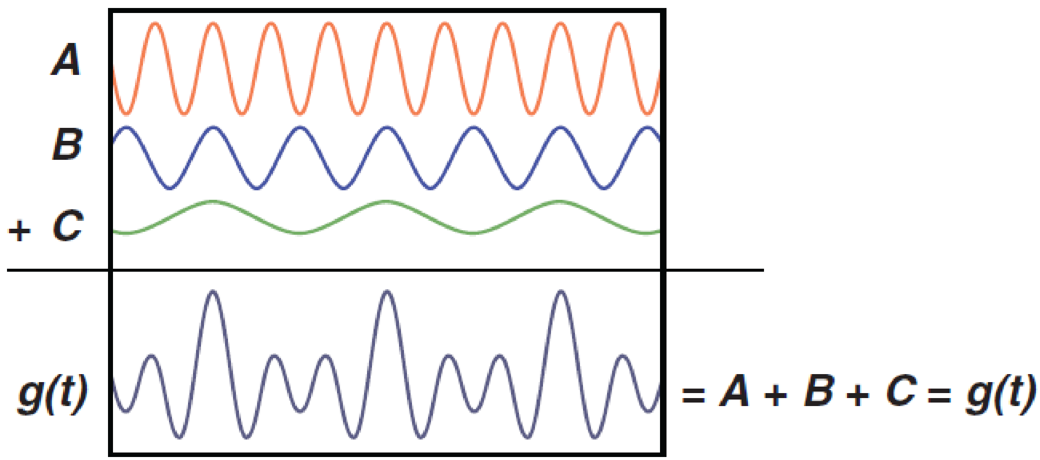
$$x(t) = \cos(2\pi t) + \cos(4\pi t) + \cos(6\pi t) + \cos(8\pi t)$$

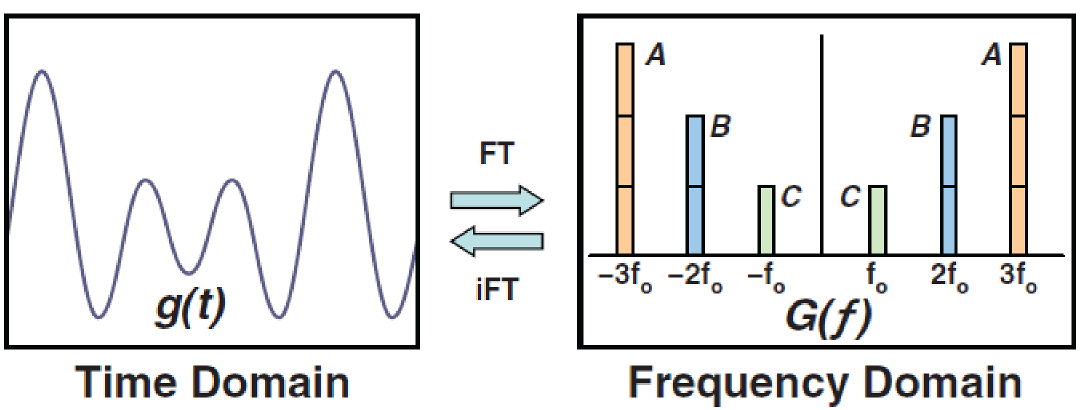


- Also in this case the two diagrams contain exactly the same information (even if the one on the right hand side looks simpler)

The time and frequency domains

- No information is gained or lost in the two domains:
  - we merely change the way we see the same information





Fourier Transform:

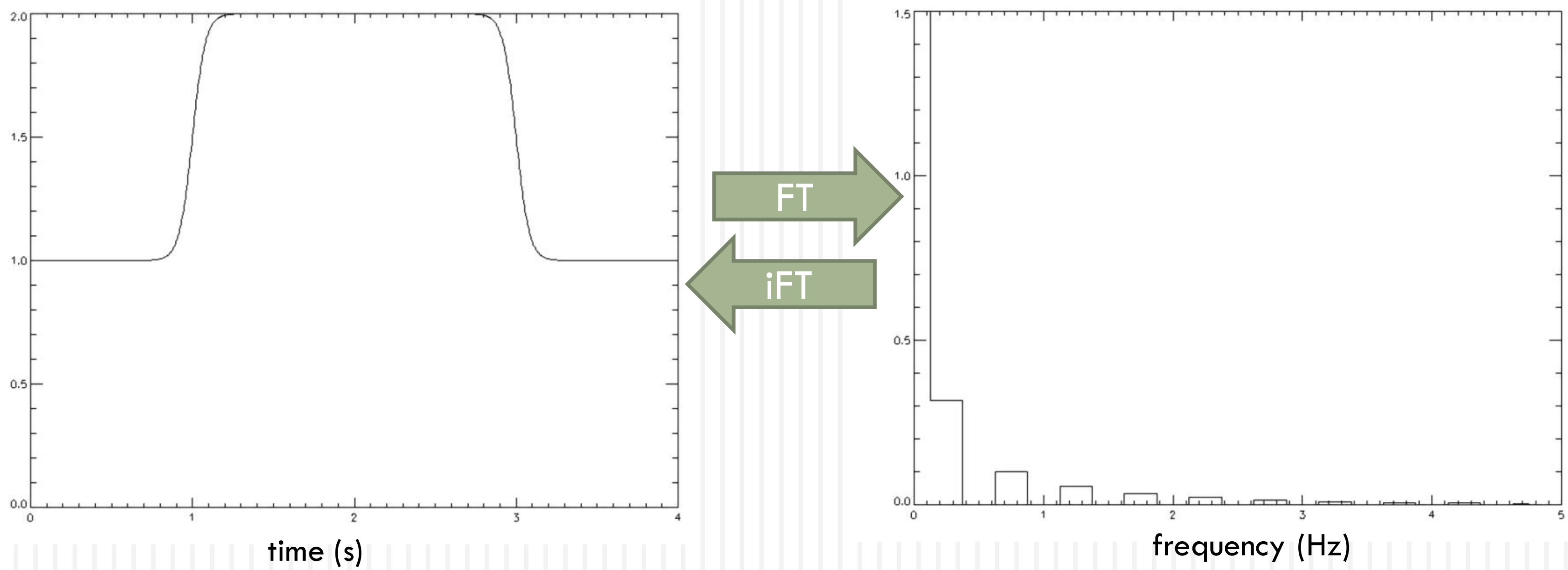
$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-i2\pi ft}dt$$

Inverse Fourier Transform:

$$g(t) = \int_{-\infty}^{\infty} G(f)e^{i2\pi ft}df$$

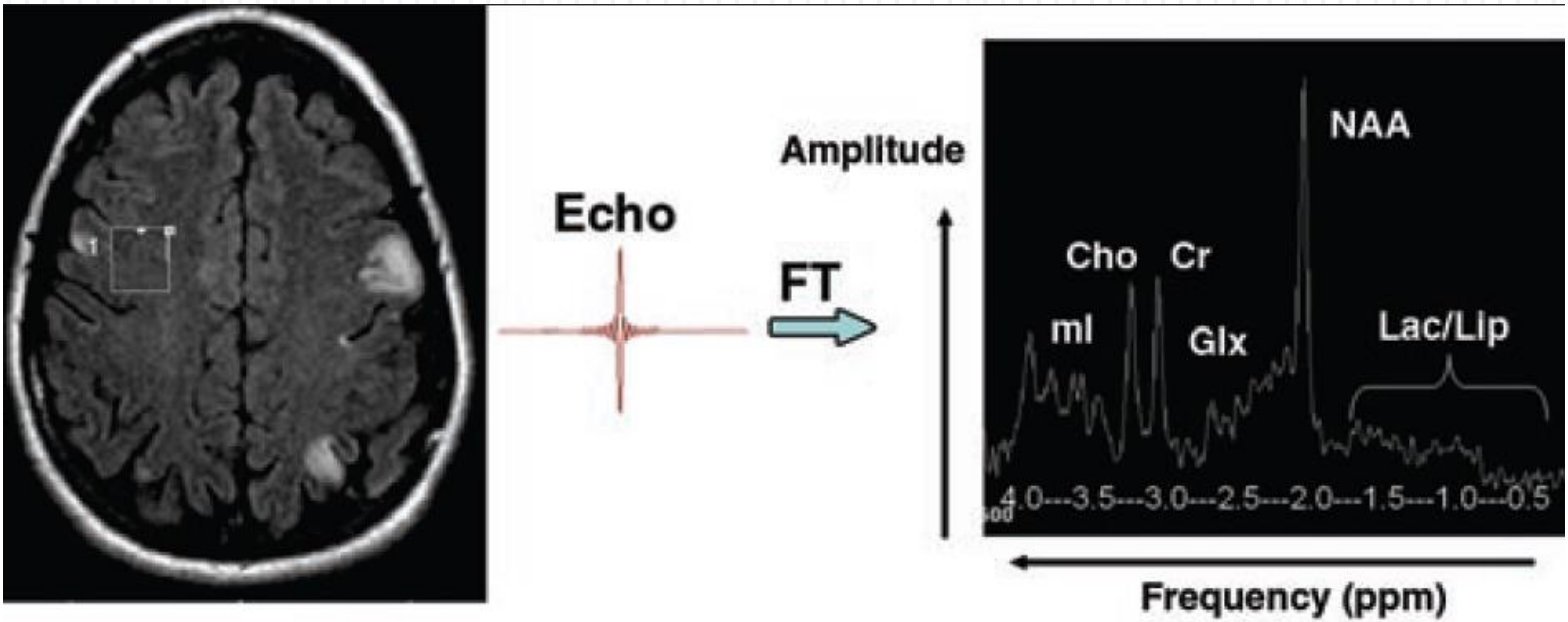
The time and frequency domains

- In general, the Fourier transform allows to represent whatever signal (even non-periodic) in the frequency domain.
  - In a sense, the signal is decomposed as a sum of harmonic oscillations, weighted with appropriate coefficients (Fourier theorem).
  - The Fourier transform allows us to study the frequency content of a variety of complicated signals



Example: MR spectroscopy

- Multiple neuronal metabolites resonate at characteristic frequencies on the basis of their unique chemical structure.
- The so-called echo is a composite signal of many different oscillations from metabolites in the ROI, which is resolved into individual resonance frequencies and their relative amplitudes (abundance) by the Fourier transform



Part 3 – the space and spatial frequency domain

Sources: T.A.Gallagher, A.J.Nemeth, L.Hacein-Bey An Introduction to the Fourier Transform: Relationship to MRI AJR:190 (2008) 1396-1405

http://www.imaios.com/en/e-Courses/e-MRI/The-Physics-behind-it-all/Fourier-transform

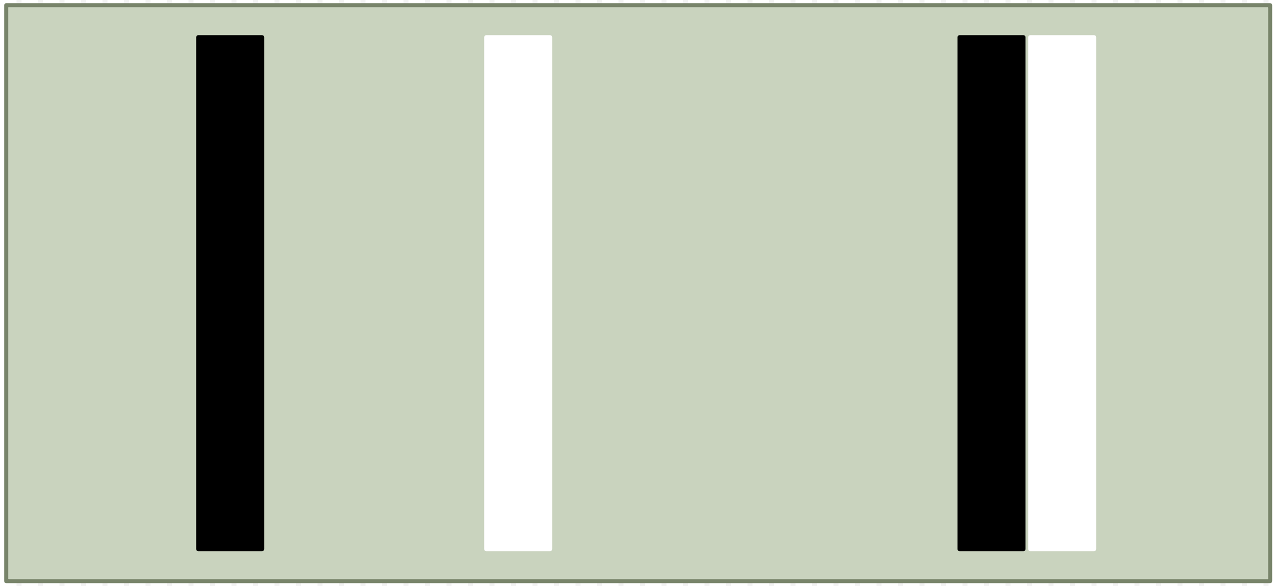
http://www.ysbl.york.ac.uk/~cowtan/fourier/fourier.html

Transforming images

- The possibility to utilize Fourier transforms also applies to:
  - functions of space (rather than functions of time)
  - functions of several variables.
- In particular, Fourier transforms also apply to 2D images, which are defined in the plane (i.e. are functions of 2 space variables)
- In general, the Fourier domain will have the same number of dimensions of the domain where the original function is defined (real space), and reciprocal units (reciprocal space)
- In the case of images, the Fourier domain (reciprocal space) is the domain of the 2D spatial frequencies. If the measurement units in real space are mm (typical for medical images) the measurement units in Fourier space are cycle/mm (in radiology line pair per mm or lp/mm)

Spatial frequencies (lp/mm)

- In medical imaging, the spatial frequencies are often defined in terms of "line pairs per millimeter", lp/mm
- A line is either a dark line or a light line; a line pair comprises a dark line and an adjacent light line.



Spatial frequencies

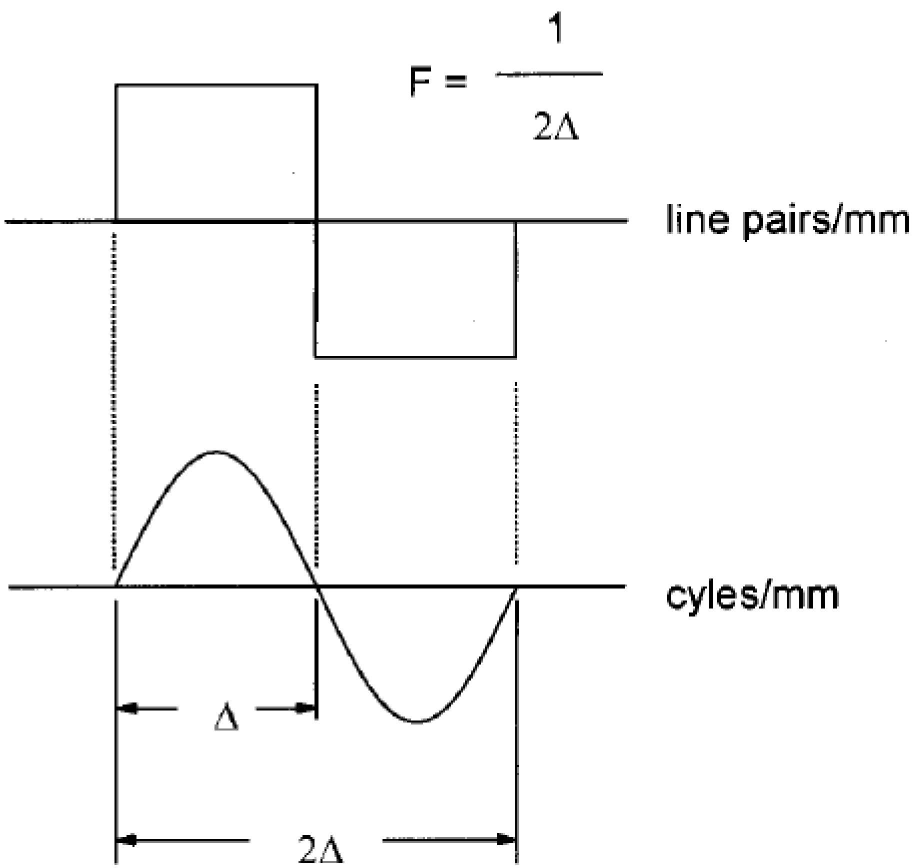
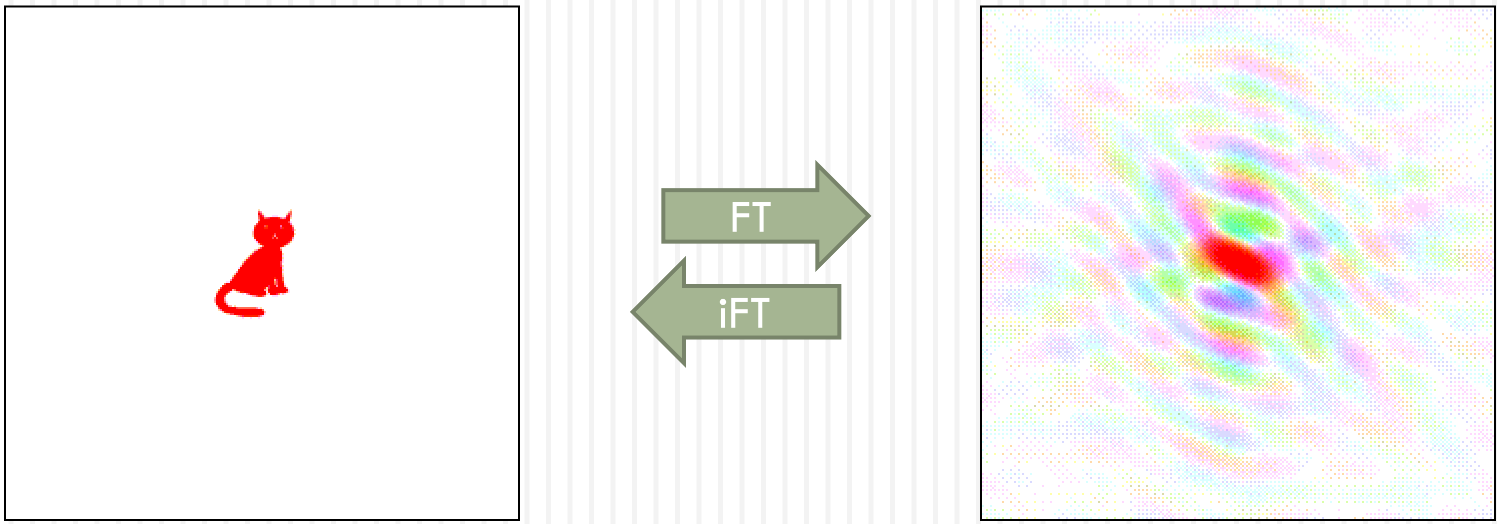


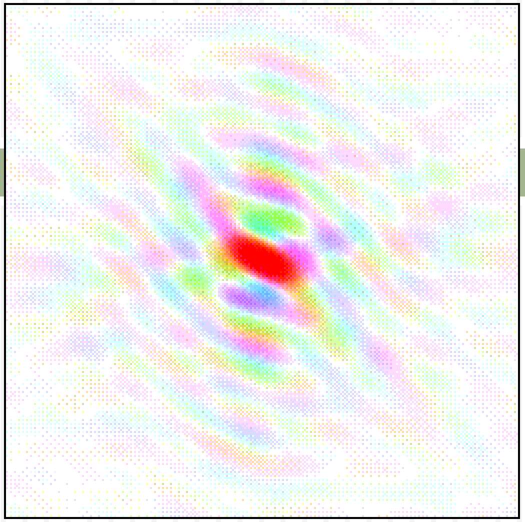
FIGURE 10-17. The concept of *spatial frequency*. A single sine wave (*bottom*) with the width of one-half of the sine wave, which is equal to a distance Δ. The complete width of the sine wave (2Δ) corresponds to one *cycle*. With Δ measured in millimeters, the corresponding spatial frequency is $F = \frac{1}{2\Delta}$. Smaller objects (small Δ) correspond to higher spatial frequencies, and larger objects (large Δ) correspond to lower spatial frequencies. The square wave (top) is a simplification of the sine wave, and the square wave shown corresponds to a single line pair.

Transforming images



- Image
  - space domain
  - 2D real space
  - mm
- Fourier Transform
  - spatial frequency domain
  - 2D reciprocal space (k-space)
  - cycle/mm or lp/mm

Transforming images



- One pixel in k-space, when inverse-transformed, contributes a single, specific spatial frequency (alternating light and dark lines) to the entire image.
- A 2D inverse Fourier transform of the entirety of k-space combines all spatial frequencies, and results in the image we see.
- Depending on where a pixel resides in k-space, the lines will be of varying frequency and orientation. By convention, high spatial frequencies are mapped to the periphery of k-space and low spatial frequencies are mapped near the origin.
- The relative intensity of a pixel reflects its overall contribution to the image, with brighter pixels contributing more of a particular spatial frequency.

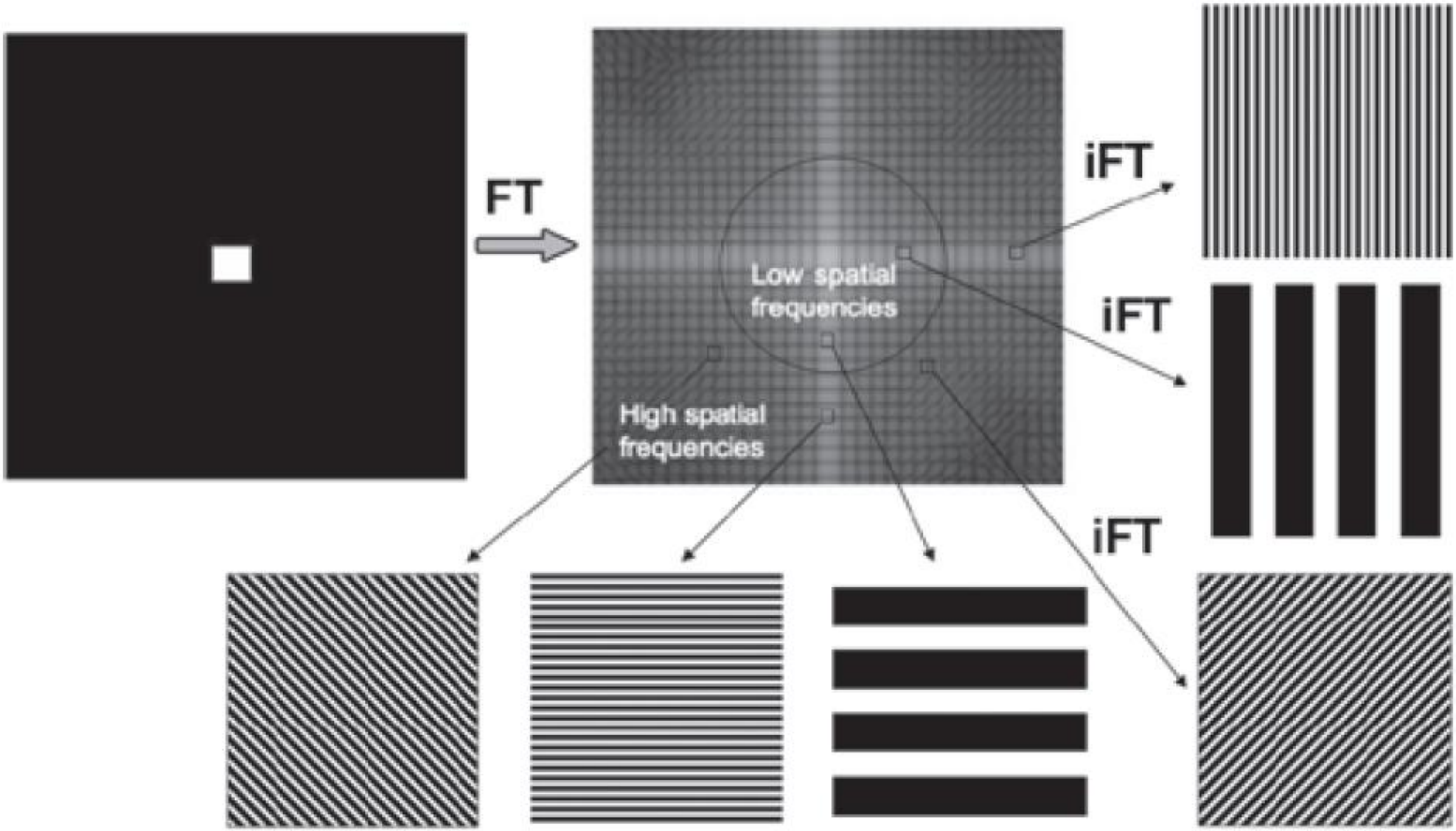
FT of a white canvas

- Fourier transform of a blank canvas (left) is one bright dot at the origin in the Fourier space (right)

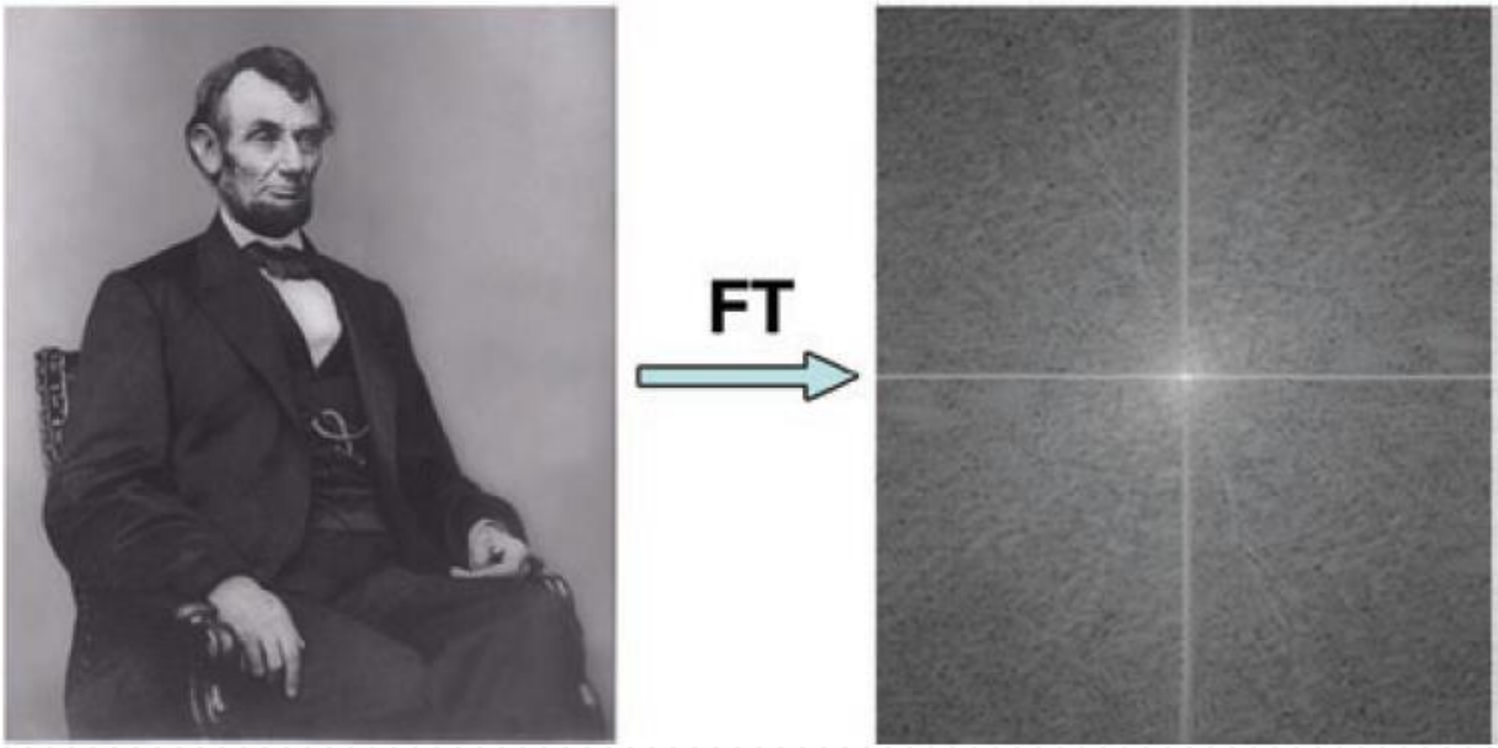
FT of a single spatial frequency image

- Fourier transform of a single spatial frequency in the image domain is simple.
- Three bright dots are seen in the Fourier space as a consequence of symmetry properties inherent to the Fourier transform

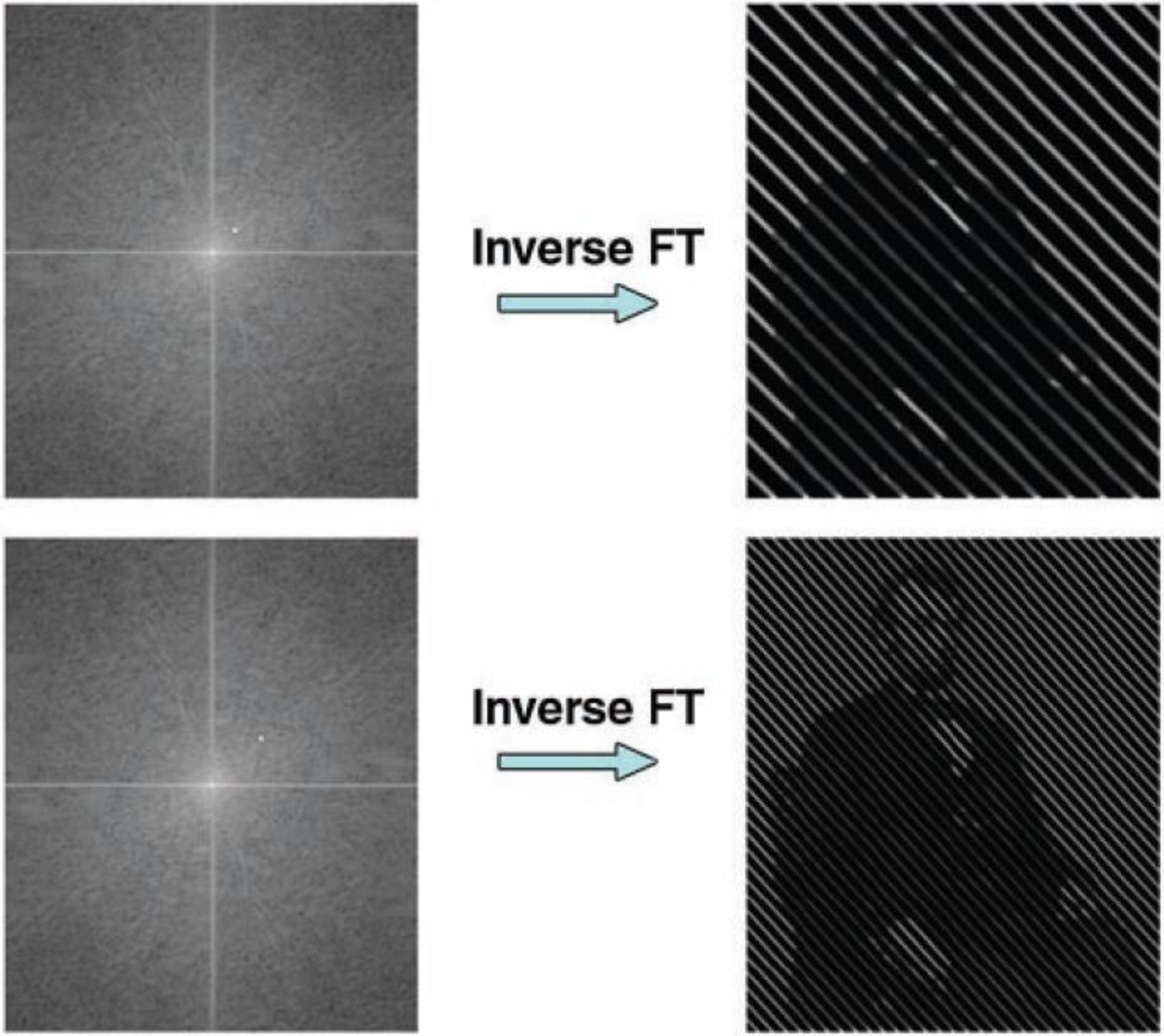
FT of a square



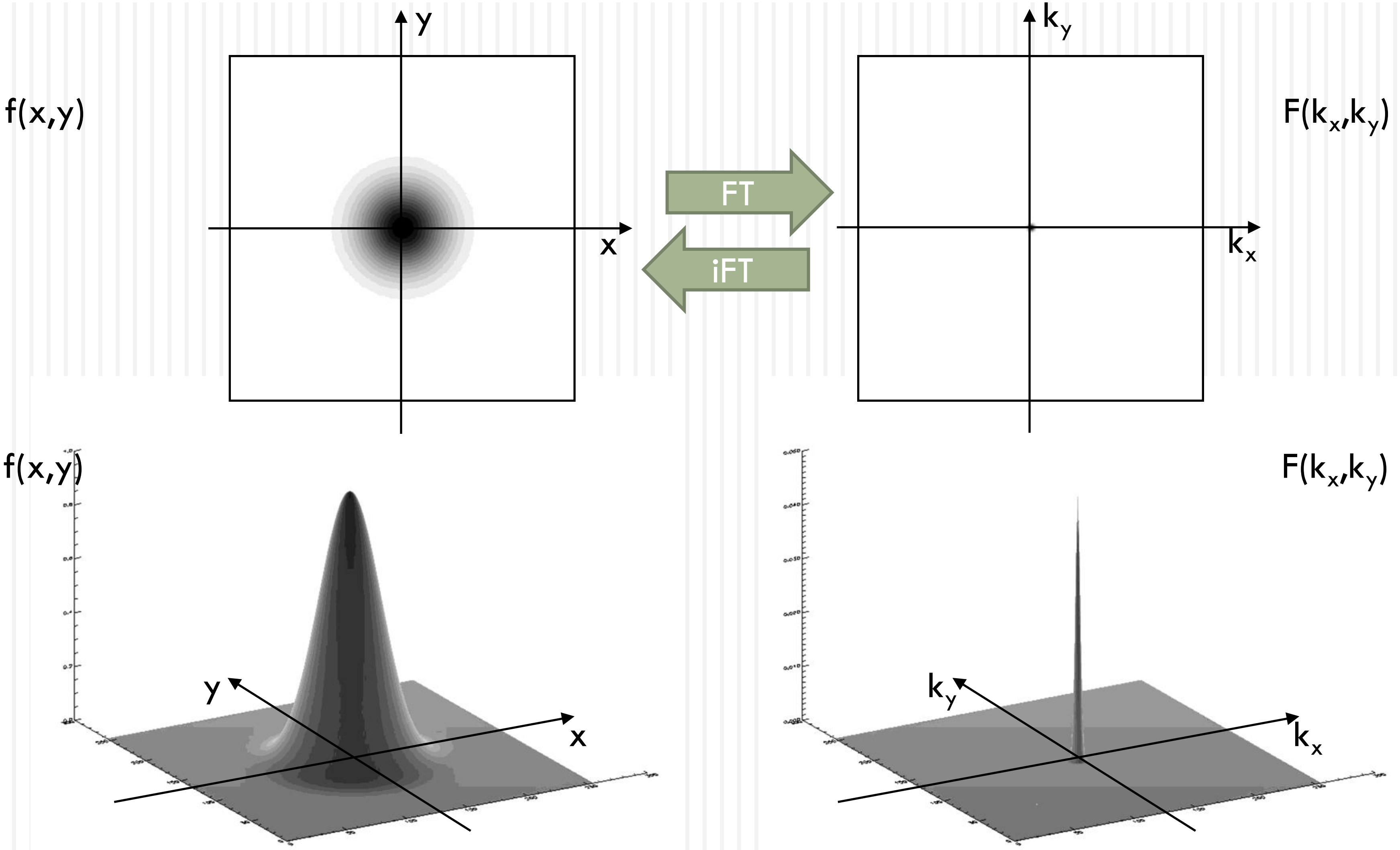
FT of Abraham Lincoln



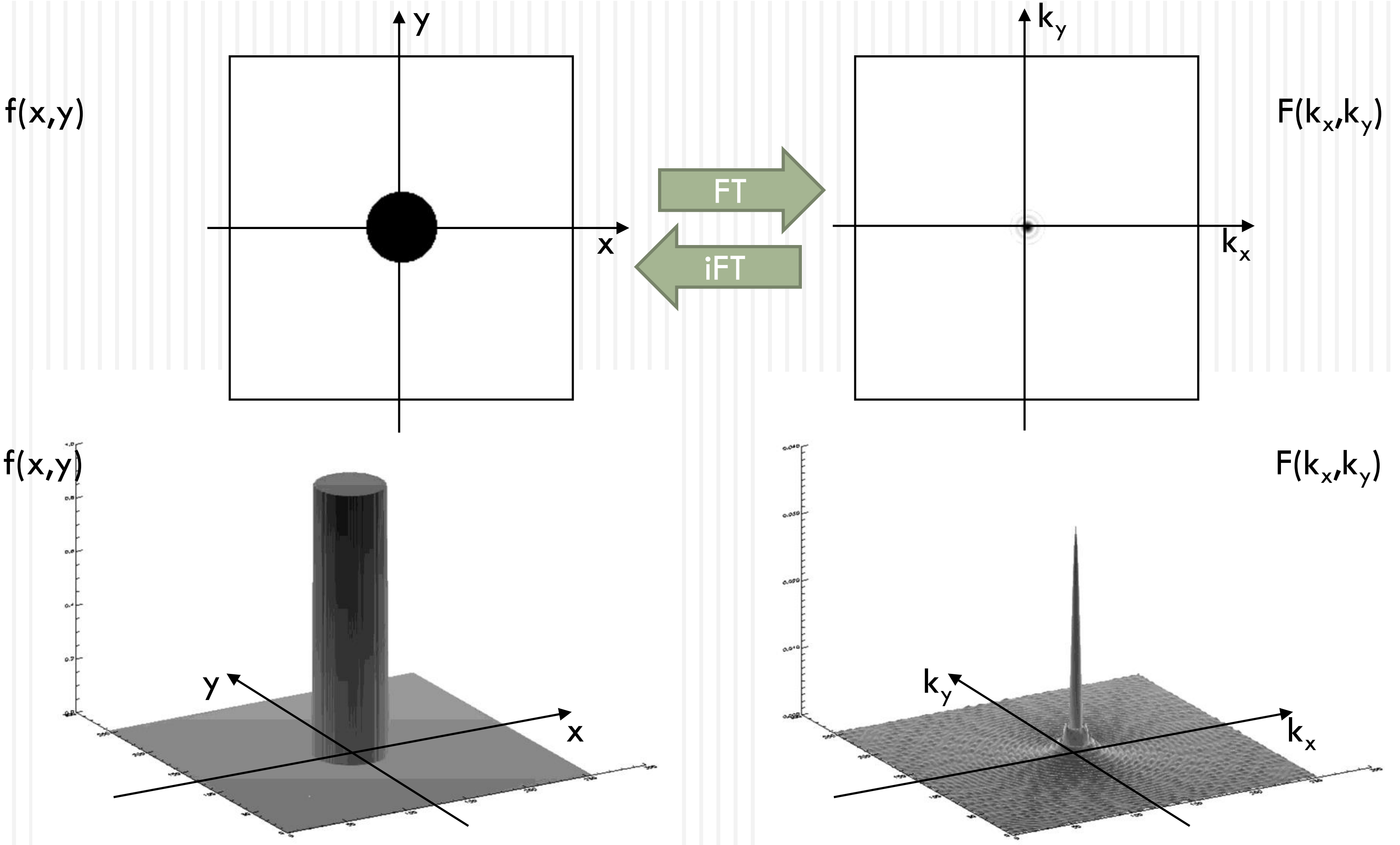
- Remember: a single pixel in the image does not have a single pixel correlate in the Fourier space.
- Rather, <u>each</u> pixel in Fourier space contributes a spatial frequency to the overall image of Lincoln

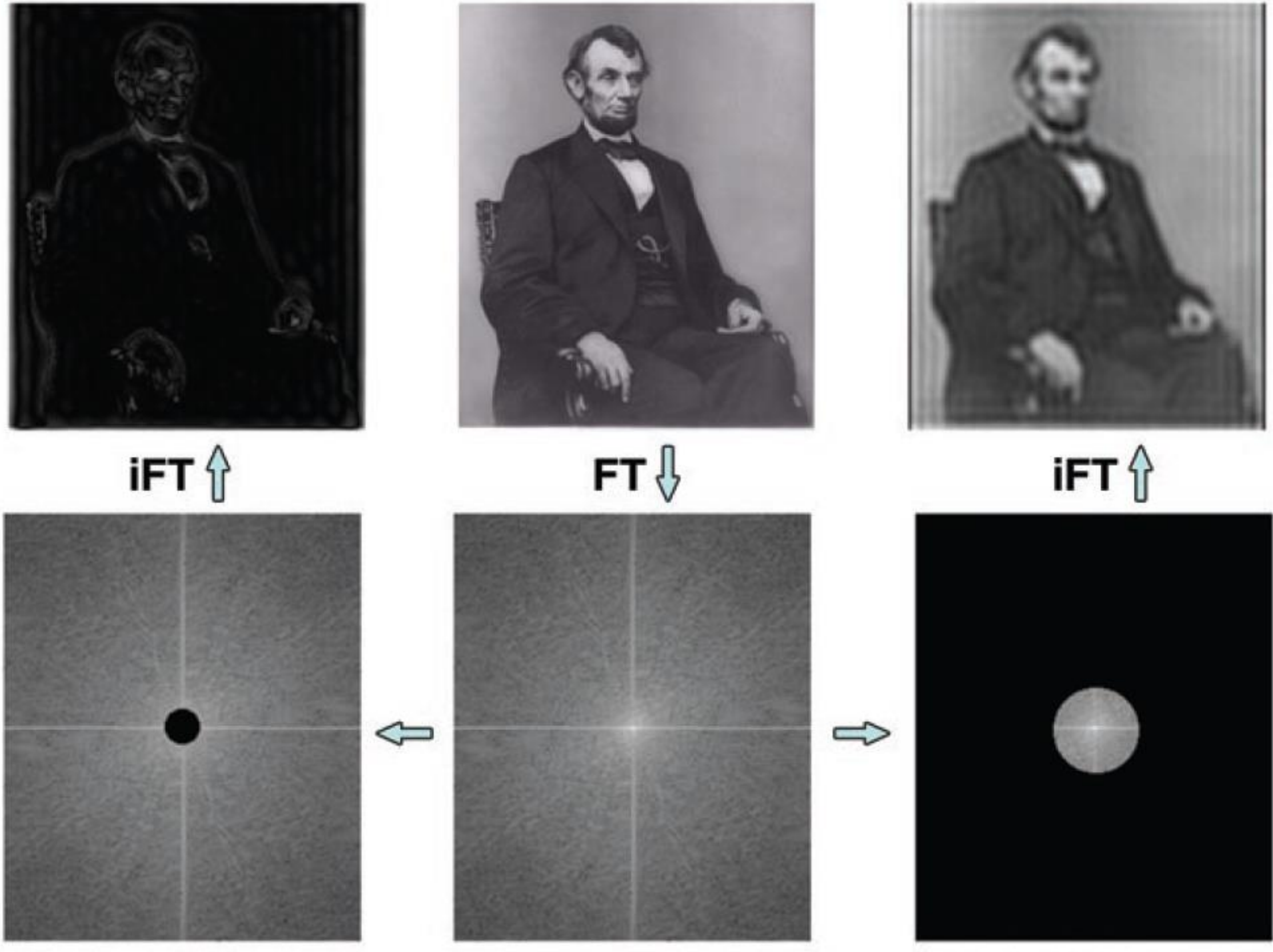

Inverse FT
Inverse FT

FT of a blurred disk



FT of a sharp disk




iFT
FT
iFT

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