MEDICAL PHYSICS LAB
LECTURE 6 –
APPLIED LINEAR-SYSTEMS THEORY

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Applied linear-systems theory

- Part I - Impulse response function
- Part II – System characteristic functions
- Part III – Measuring system characteristic functions

- Sources:
  - Ian A. Cunningham, Chapter 2 in Handbook of medical imaging.
    Volume 1, Physics and psychophysics.
    Richard Van Metter, Jacob Beutel, Harold Kundel, editors.
  - Hasegawa, B. H. - The physics of medical X-ray imaging (or the photon and me: how I saw the light) - 1990

Part I - Impulse response function

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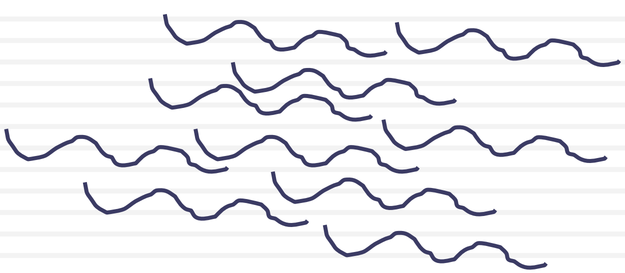
1D or 2D?

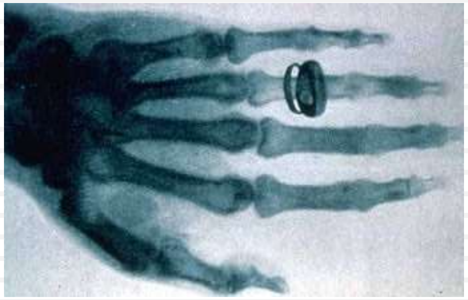
- So far we have introduced a number of mathematical tools
  - Fourier Transforms
  - Convolution, Cross Correlation and Autocorrelation
  - Useful functions and generalized functions
- All of them have been introduced with reference to 1D, 1-dimensional functions ($f(x)$, $\delta(x)$, $III(x)$, etc.)
- In principle, to deal with images we should consider 2D functions ($f(x,y)$, ${}^2\delta(x,y)$, ${}^2III(x,y)$, etc.)
- However, to keep notation simple, we will stick to 1D functions, and generalize to 2D only when really necessary

Imaging System

- In the following, we will consider a medical imaging system as a “black box” that receives an input signal and produces an output image
- The theory will be developed with reference to a planar x-ray radiographic system, which is the oldest and possibly the simplest system.
- The same approach can however be applied also to more complex systems.



Input image
(distribution of x-ray quanta)



output image
(either analog or digital)

Images

- We will distinguish among 3 different image types:
  - Analog image $d(x)$
    - Expressed as a function of the position variable x
    - Arbitrary units (optical density in a film, intensity on a monitor, etc.)
  - Digital image d_n
    - Represents image intensity at a particular pixel identified by n
    - Dimensionless (just numbers)
  - Quantum image $q(x)$
    - Spatial distribution of quanta (function of the position variable x)
    - Dimensions:
      - 1/length (for a 1D quantum image)
      - 1/area (for a 2D quantum image)
    - Statistical properties (Poisson statistics)

Linear Systems

- We will assume the imaging system *S*{ } be a linear system, i.e.

input

h(x)

Imaging system

S{ }

output

S{h(x)}

$$S\{h_1(x)+h_2(x)\}=S\{h_1(x)\}+S\{h_2(x)\}$$

and

$$S\{ah(x)\}=aS\{h(x)\}.$$

for any real constant *a*, which is sometimes summarized as "the output is proportional to the input".

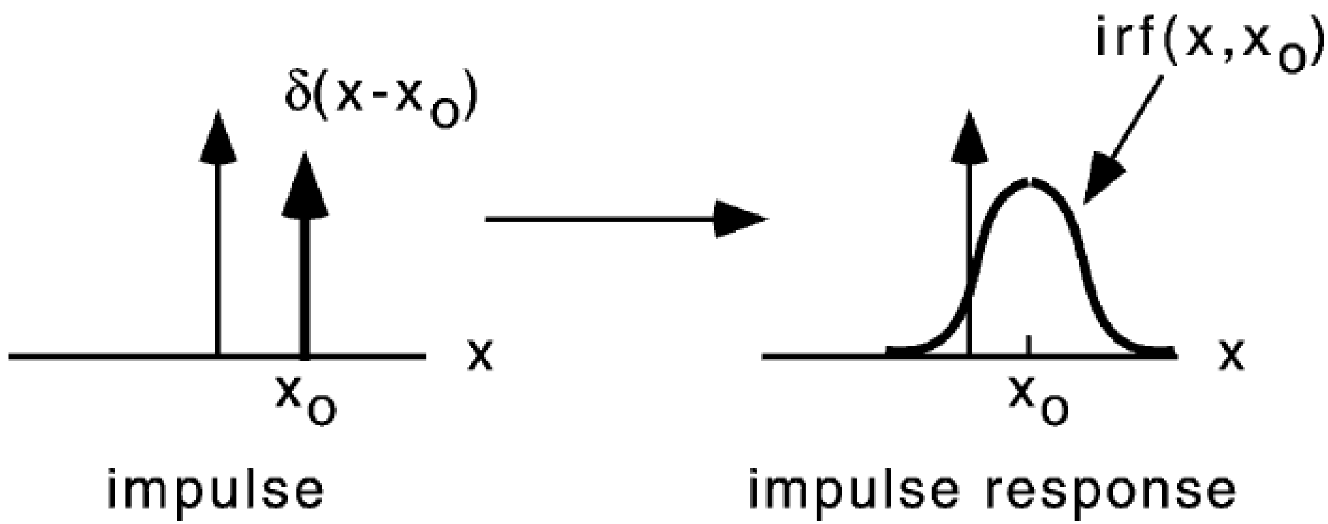
Linear systems as an approximation

- Generally speaking, no real imaging system is actually linear, and the linear system approach must be considered as an approximation
- However, many systems which are not strictly linear
  - Can be linearized by means of an appropriate calibration
  - Can be considered linear provided the amplitude of the input signal is sufficiently small

Impulse response function *irf*(x, x_0)

When a linear system is presented with the input $\delta(x - x_0)$, an impulse located at $x = x_0$, the corresponding output will be $S\{\delta(x - x_0)\}$ which is called the impulse-response function (IRF), i.e.,

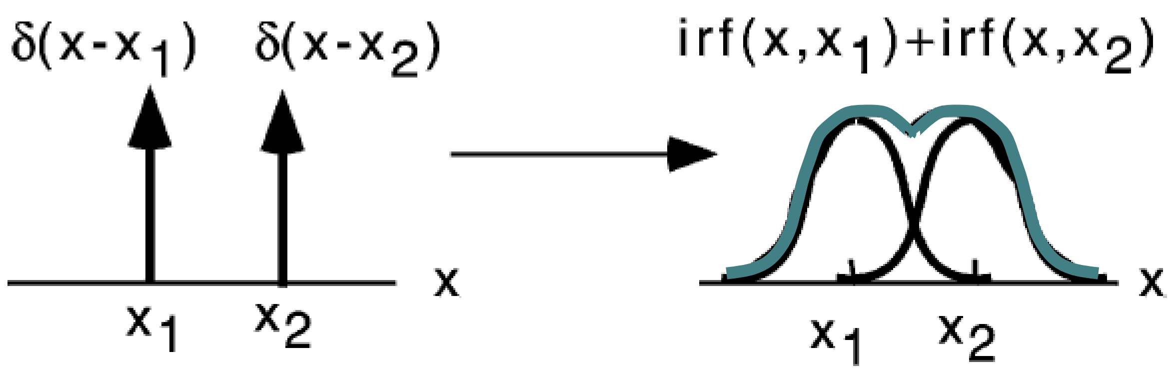
$$\boxed{\mathrm{irf}(x, x_0) = S\left\{\delta(x - x_0)\right\}.} \tag{2.20}$$



The superposition principle

- For any input expressed as a superposition of many impulse functions, the output of a linear system will consist of the superposition of one *irf* for each input impulse
- A simple example:

$$\mathrm{S}\left\{\delta(x-x_1)+\delta(x-x_2)\right\}=\mathrm{irf}(x,x_1)+\mathrm{irf}(x,x_2)$$



The superposition principle

- More generally, let us consider the case in which the input is represented by an arbitrary function *h(x)*
- Since $h(x)=\int_{-\infty}^{+\infty} h(x')\delta(x-x')dx'$ it is readily shown that:

$$S\{h(x)\} = S\left\{\int_{-\infty}^{+\infty} h(x')\delta(x-x')dx'\right\}$$

$$= \int_{-\infty}^{+\infty} h(x')S\{\delta(x-x')\}dx'$$

$$= \int_{-\infty}^{+\infty} h(x')irf(x,x')dx'$$

- The latter is said the *superposition integral*
- Thus the *irf* contains all the information about an imaging system necessary to describe its response to any input *h(x)*

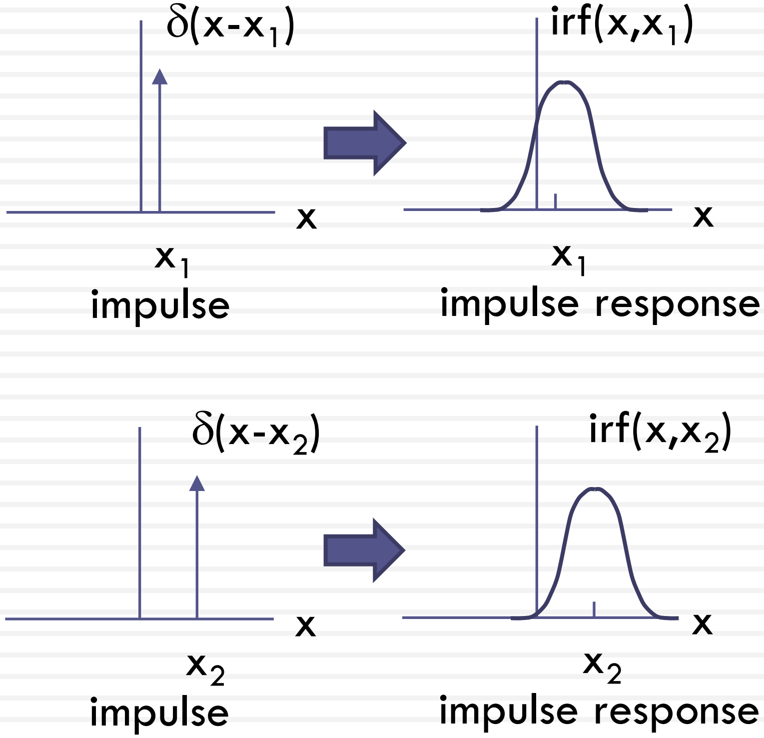
Linear and shift-invariant systems

- The imaging system will be assumed also to be shift-invariant (isoplanatic), which means that a particular structure will appear the same, regardless of where in the image it is placed
- Virtually all imaging systems are (approximately) shift-invariant, and if they are not, it is always possible to restrict the analysis to a central region where they are reasonably so.
- Shift-invariant imaging systems must have a shift-invariant *irf*, which means that the shape of the *irf* is independent of the position x_0, i.e. it only depends on the distance of x from x_0:

$$\boxed{irf(x,x_0) = irf(x-x_0)}$$

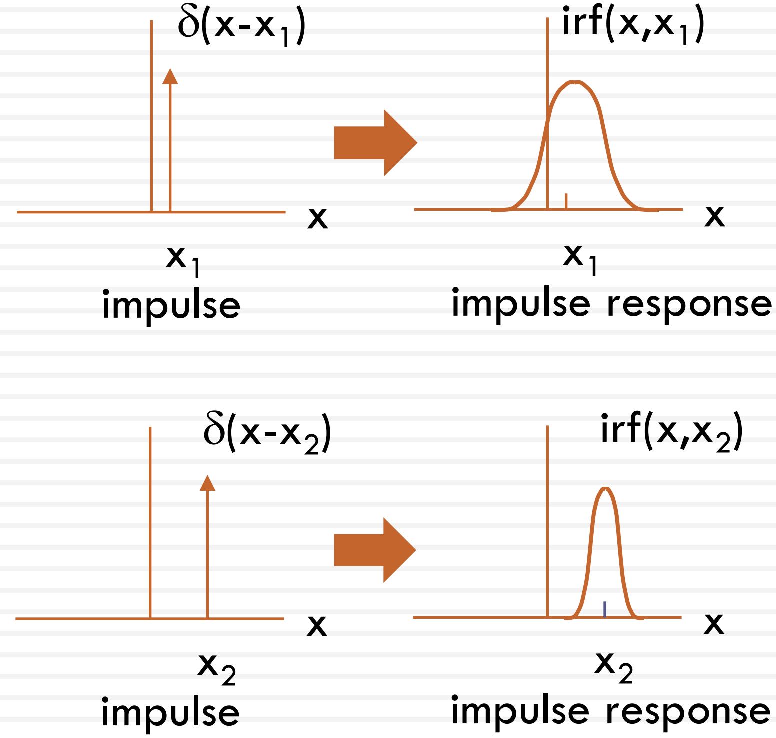
Linear and shift-invariant systems

Shift-invariant system



$irf(x,x_0)=irf(x-x_0)$

Non-Shift-invariant system



$irf(x,x_0)$ *really depends on* x_0

The convolution integral

- When the irf is shift-invariant, the superposition integral

$$S\{h(x)\} = \int_{-\infty}^{+\infty} h(x')irf(x, x')dx'$$

can be written as

$$S\{h(x)\} = \int_{-\infty}^{+\infty} h(x')irf(x - x')dx'$$

which is actually a <u>convolution</u> integral

$$S\{h(x)\} = h(x) * irf(x)$$

Part II – System characteristic functions

- Ian A. Cunningham, Chapter 2 in Handbook of medical imaging.
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  Richard Van Metter, Jacob Beutel, Harold Kundel, editors.

A special case: a sinusoidal input

- Let us consider the <u>special case</u> of an input that varies sinusoidally with the position, i.e.

$$h(x) = e^{i2\pi ux} = \cos(2\pi ux) + i\sin(2\pi ux)$$

where u is the spatial frequency (cycles/mm).
The output $d(x)$ is:

$$d(x) = S\{h(x)\} = h(x) * irf(x) = \int_{-\infty}^{\infty} irf(x')\, e^{i2\pi u(x-x')} dx'$$

$$= e^{i2\pi ux} \underbrace{\int_{-\infty}^{\infty} irf(x')\, e^{-i2\pi ux'}\, dx'}_{T(u)}$$

Fourier Transform of *irf*(*x*)
We call it the system characteristic function, *T*(*u*)

- Thus: $d(x) = \boxed{S\{e^{i2\pi ux}\} = \mathrm{T(u)}e^{i2\pi ux}}$

The system characteristic function *T(u)*

$$S\{e^{i2\pi ux}\} = \mathrm{T(u)}e^{i2\pi ux}$$

- In this particular case, the output is thus proportional to the input, the scaling factor being *T(u)*, which is the Fourier transform of *irf(x)*
- Thus complex exponential of the form $e^{i2\pi ux}$ are eigenfunctions of the imaging system
- *T(u)* describes the eigenvalues and is called the *characteristic function* of the system
- In general *T(u)* has complex values, however:
  - If *irf(x)* is real and even, *T(u)* is also real and even
  - *T(0)* represents the area under *irf(x)* and is always real

The spatial-frequency domain

- Let us consider again the convolution integral

$$d(x) = S\{h(x)\} = h(x) * irf(x)$$

- If we define

$$d(x) \supset D(u)$$
$$h(x) \supset H(u)$$

- As a consequence of the convolution theorem we have that

$$D(u) = H(u)T(u)$$

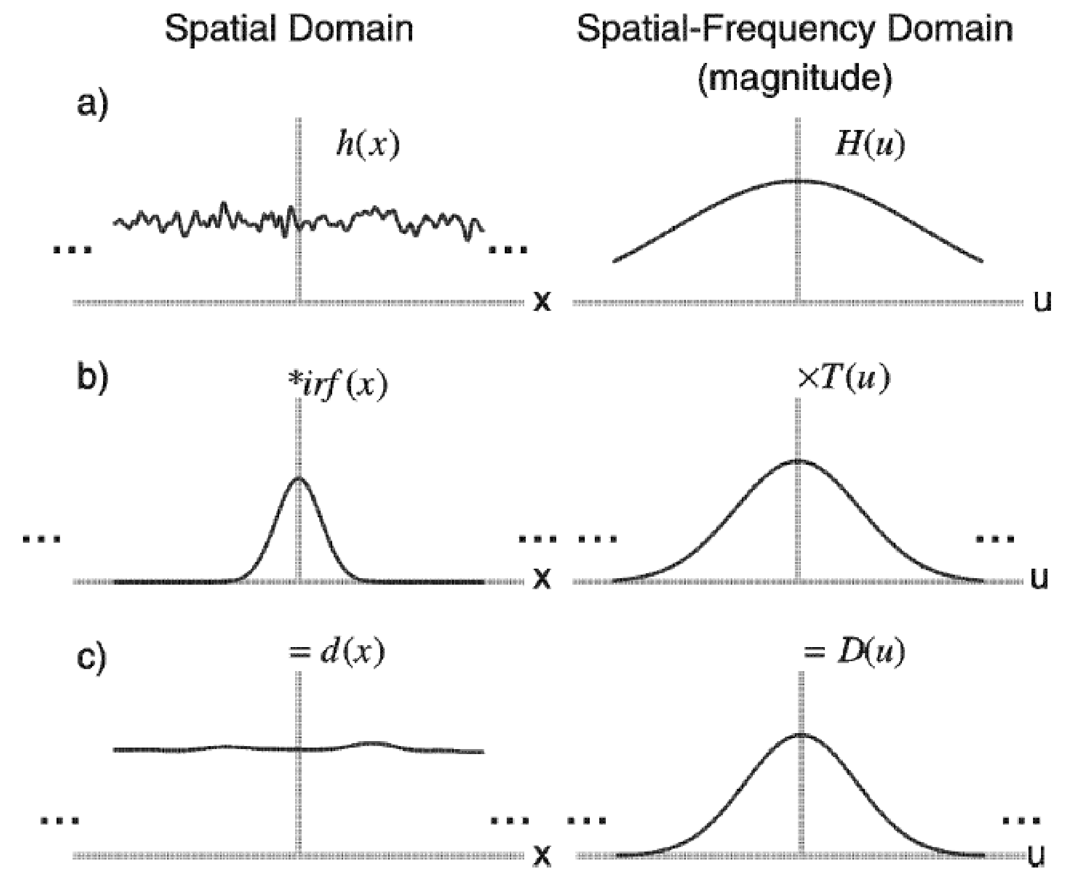
This is a very interesting result because it shows that the Fourier components $H(u)$ of the input are passed unchanged through the system other than a scaling by $\mathrm{T}(u)$. Thus, the signal-transfer characteristics of an LSI system can be expressed either as convolution with irf(x) in the spatial domain, or equivalently as multiplication with $\mathrm{T}(u)$ in the spatial-frequency domain. This relationship is illustrated graphically

The spatial-frequency domain

$$h(x) \supset H(u)$$

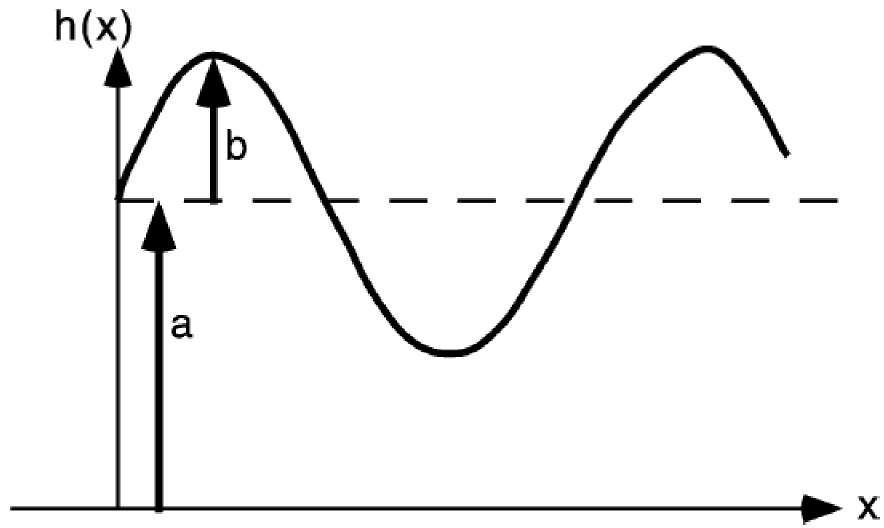
$$irf(x) \supset T(u)$$

$$d(x) \supset D(u)$$



$$d(x) = S\{h(x)\} = h(x) * irf(x) \qquad D(u) = H(u)T(u)$$

Another sinusoidal input



Consider the input $h(x)$ where

$$h(x) = a + be^{i2\pi ux}, \tag{2.36}$$

and where the real component of $h(x)$ corresponds to the real (measurable) input signal. Because of the sinusoidal nature of this input, it is more meaningful to characterize it in terms of its modulation than its contrast. The modulation of $h(x)$ in Figure 2.14 is given by

$$M_{in} = \frac{|h_{max}| - |h_{min}|}{|h_{max}| + |h_{min}|} = \frac{(a+b)-(a-b)}{(a+b)+(a-b)} = \frac{b}{a}. \tag{2.37}$$

The Modulation Transfer Function (MTF)

The output signal $d(x)$ is given by

$$d(x) = \mathrm{S}\{h(x)\} = \mathrm{S}\{a + be^{i2\pi ux}\} \tag{2.38}$$
$$= \mathrm{S}\{a\} + \mathrm{S}\{be^{i2\pi ux}\} \tag{2.39}$$
$$= a\mathrm{S}\{e^{i2\pi(u=0)x}\} + b\mathrm{S}\{e^{i2\pi ux}\} \tag{2.40}$$
$$= a\mathrm{T}(0) + b\mathrm{T}(u)e^{i2\pi ux}, \tag{2.41}$$

where $\mathrm{T}(u)$ is complex in general but $\mathrm{T}(0)$, which is equal to the area under the IRF, must be real only. The output modulation is therefore given by

$$M_{out} = \frac{|d_{max}| - |d_{min}|}{|d_{max}| + |d_{min}|} = \frac{b}{a}\frac{|\mathrm{T}(u)|}{\mathrm{T}(0)} = M_{in}\frac{|\mathrm{T}(u)|}{\mathrm{T}(0)}. \tag{2.42}$$

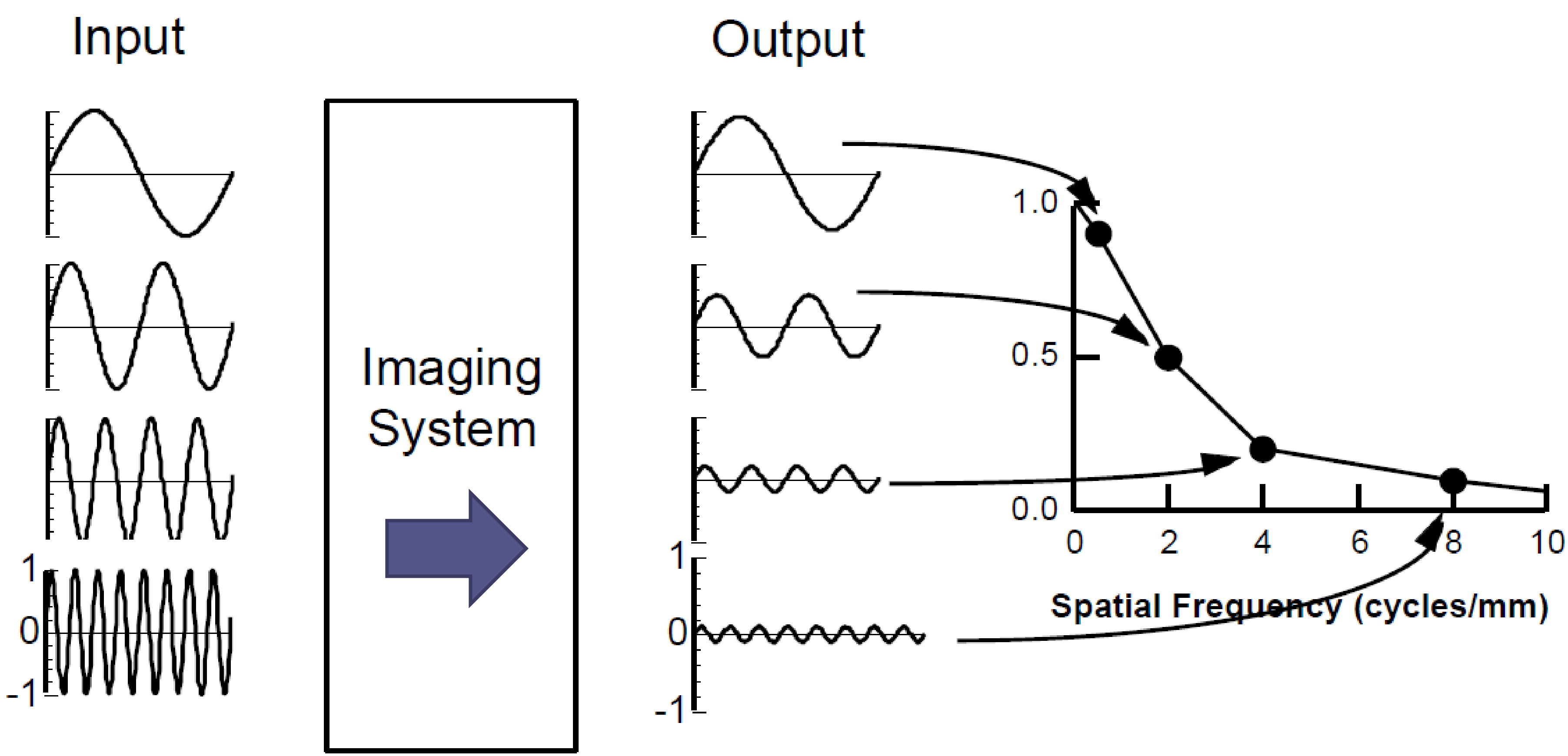
the ratio M_{out}/M_{in} is defined here as the *modulation* transfer function (MTF)

$$\boxed{\mathrm{MTF}(u) = \frac{|\mathrm{T}(u)|}{\mathrm{T}(0)}}$$

Part 3 – More on MTF

Source: Hasegawa, B. H. - The physics of medical X-ray imaging (or the photon and me: how I saw the light) - 1990

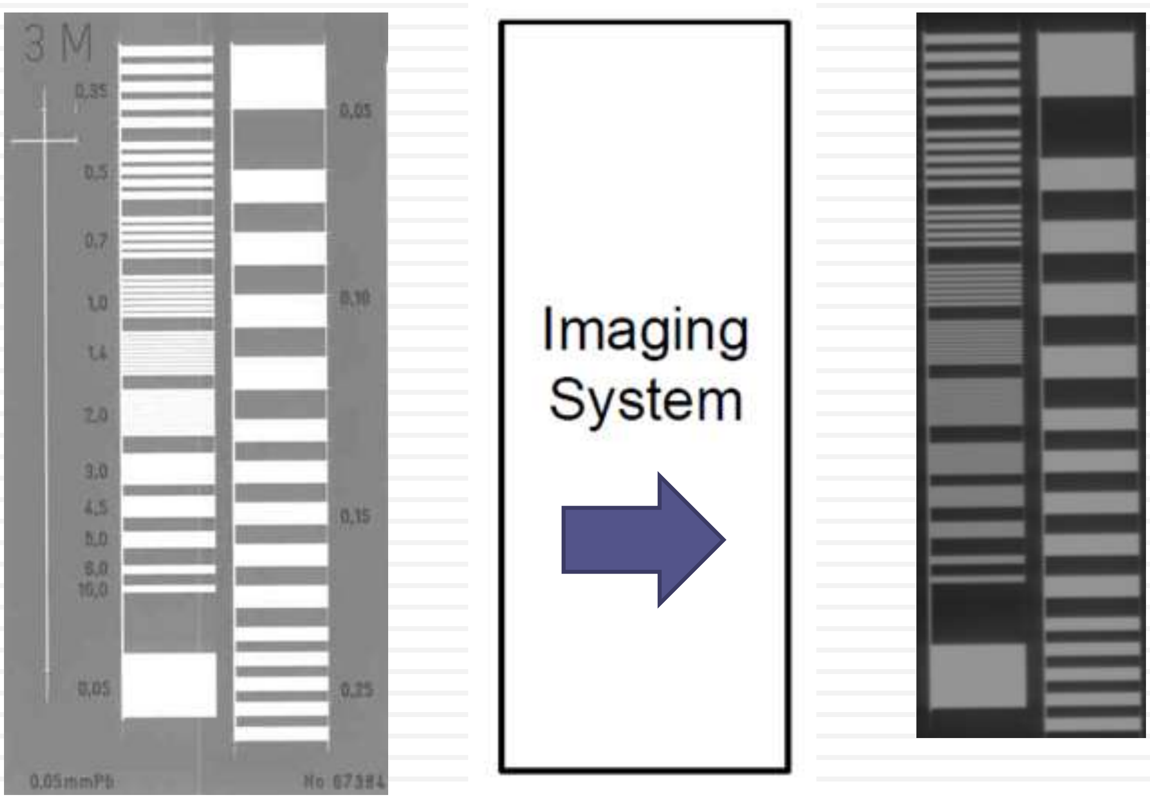
Measuring *MTF(u)* (conceptually)



[From Robert M. Nishikawa]

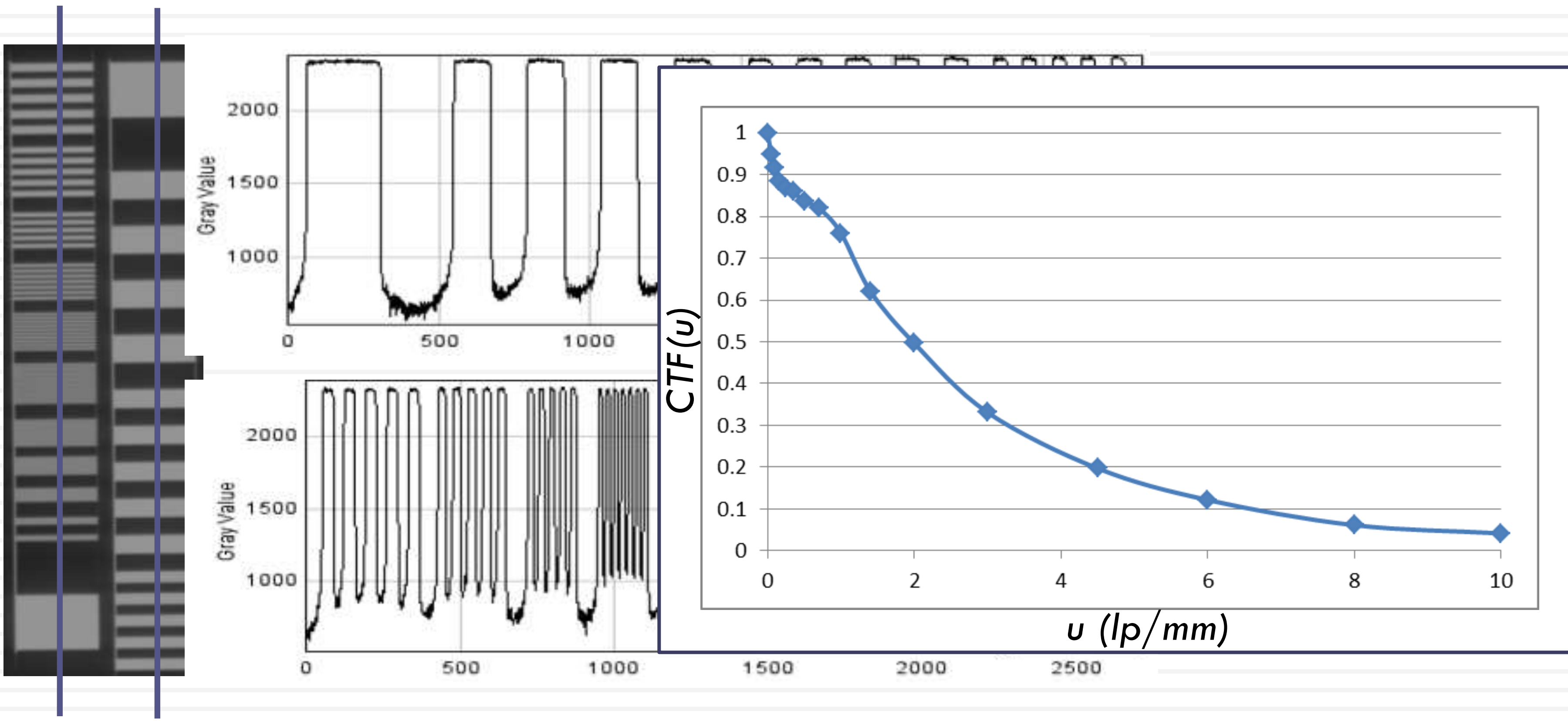
Measuring *MTF(u)* (a simple method)

- A very simple method to measure the MTF of a system is by means of a bar pattern, which provides an input object with several square waves of different spatial frequencies



Measuring *MTF(u)* (a simple method)

- The modulation of each square wave of the bar pattern is then calculated from the image and the result is plotted as a function of the spatial frequency to yield the *CTF(u)*



Measuring the PSF

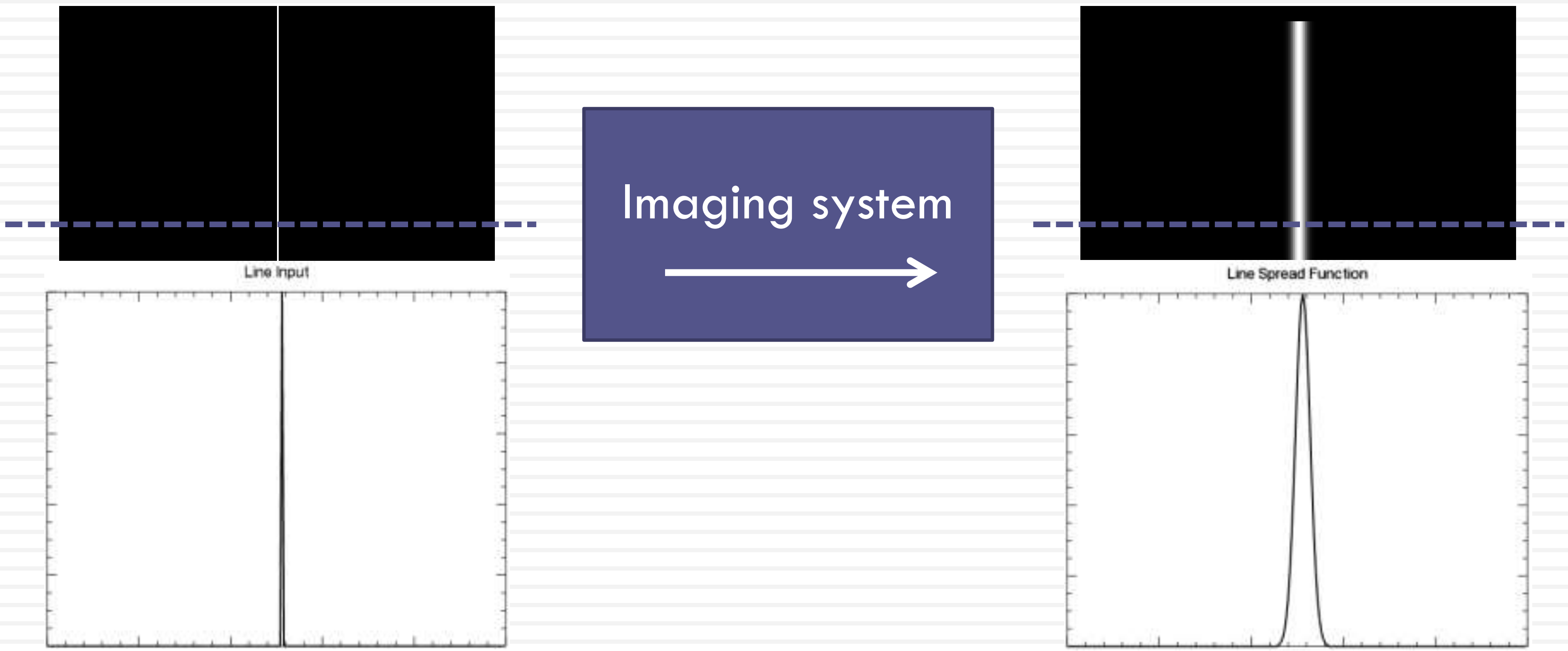
- Alternatively, the PSF could be measured instead
- Note: following Hasegawa, in this section we will assume the *irf(x,y)* is normalized, i.e.
  - *irf(x,y) =PSF(x,y)*
  - *T(0,0)=1* and *T(u,v)=OTF(u,v)*
- Thus the PSF(x,y) is defined as the action of the system S[ ] on a point-like object $\delta(x,y)$

$$PSF(x,y) = S[\delta(x,y)]$$

- In practice, however, utilizing a point-like input $\delta(x,y)$ can be impractical
  - it can be technically challenging to realize it
  - the input can be weak and the output dominated by noise

PSF and LSF

- An alternative approach is to consider a line input, e.g. a bright line corresponding to the y axis in the image plane
- The action of the imaging system on this line input defines the Line Spread Function (LSF)



PSF and LSF

- Formally the line input can be written as:

$$line(x) = \delta(x) = \int \delta(x, y)dy$$

- We define the Line Spread Function as follows:

$$LSF(x) = S[line(x)] = S[\int \delta(x, y)dy] = \int S[\delta(x, y)]dy = \int PSF(x, y)dy$$

- As a consequence

$$\Im(LSF(x)) = \int LSF(x)e^{-2\pi iux}dx = \int\int PSF(x, y)dye^{-2\pi iux}dx$$

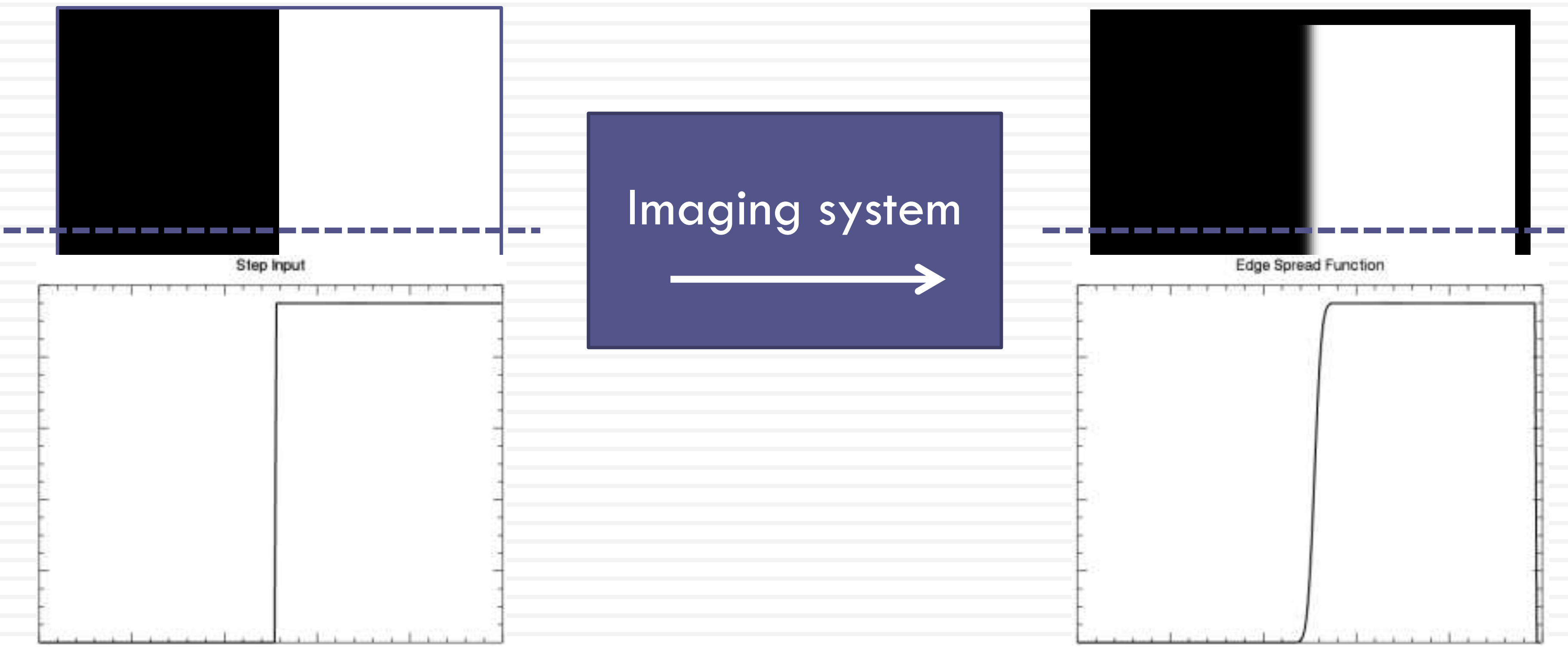
$$= \int\int PSF(x, y)e^{-2\pi i(ux+vy)}dydx\Big|_{v=0} = \Im[PSF(x, y)]\Big|_{v=0}$$

$$= OTF(u, 0)$$

- Often, the OTF is characterized by some symmetry properties (e.g. circular symmetry) and thus it is sufficient to evaluate it along one direction in the spatial-frequency plane

LSF and ESF

- An even more practical approach is by considering a step input, that can easily be obtained placing an opaque edge across the field of view
- We thus define an "edge spread function" (ESF):



LSF and ESF

- Formally, the step input can be written as:

$$step(x,y) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \qquad step(x,y) = \int_{-\infty}^{x} \delta(x')dx' = \int_{-\infty}^{x} line(x')dx'$$

- We thus define an "edge spread function":

$$ESF(x) = S[step(x,y)] = S\left[\int_{-\infty}^{x} line(x')dx'\right] = \int_{-\infty}^{x} S[line(x')]dx' = \int_{-\infty}^{x} LSF(x')dx'$$

- The LSF can then be obtained by differentiating the previous equation:

$$\boxed{LSF(x) = \frac{d}{dx} ESF(x)}$$

- The *OTF(u,0)* can then be obtained calculating the FT of the *LSF(x)*

technicalities

The Optical Transfer Function (OTF)

- More generally, the Optical Transfer Function of an imaging system is defined as

$$\boxed{\mathrm{OTF}(u) = \frac{\mathrm{T}(u)}{\mathrm{T}(0)}}$$

- In general, while the MTF is always real, the OTF has complex values. Thus, it can be written in the polar form

$$OTF(u) = MTF(u)e^{iPTF(u)}$$

- where we have introduced the
  - Modulation Transfer Function $MTF(u) = |OTF(u)|$
  - Phase Transfer Function $PTF(u)$

The Point Spread Function (PSF)

- The OTF and the MTF are normalized (by definition):

$$OTF(0) = 1$$
$$MTF(0) = 1$$

- The normalized impulse response function is said ***point spread function***

$$psf(x) = \frac{irf(x)}{\int_{-\infty}^{+\infty} irf(x)dx} = \frac{irf(x)}{T(0)}$$

- Thus

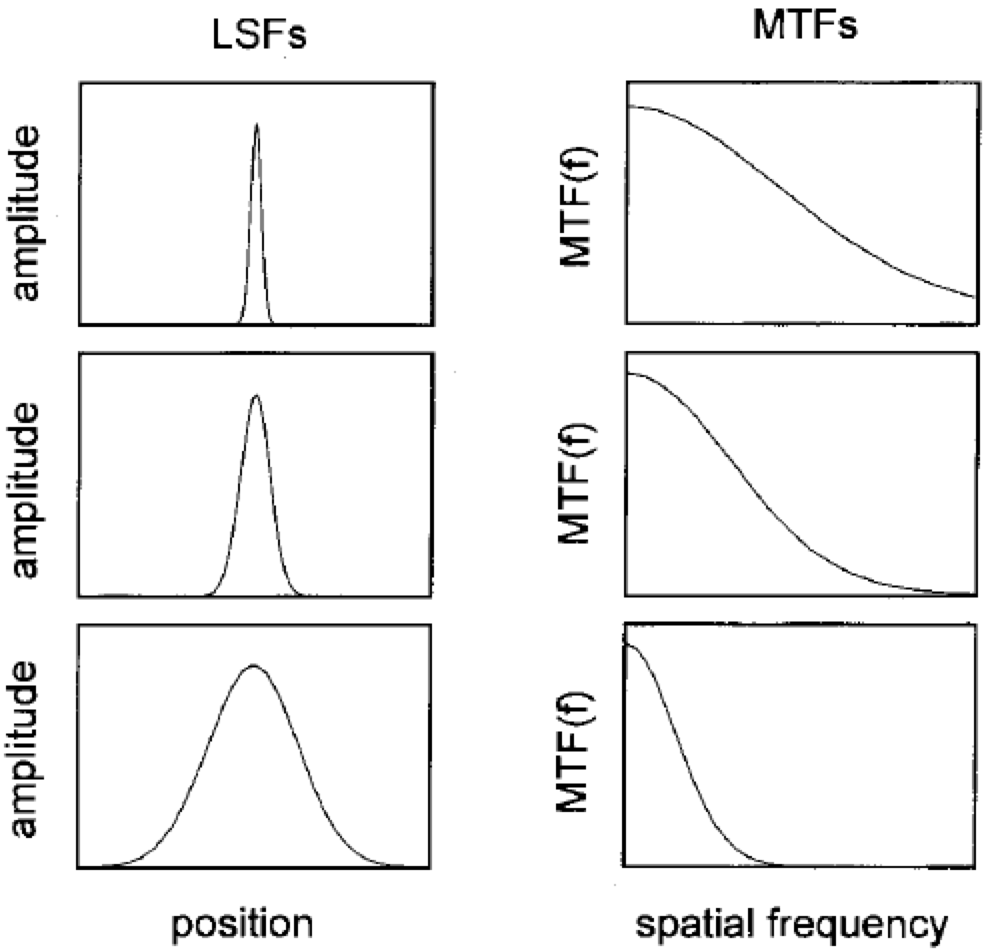
$$psf(x) \supset \frac{T(u)}{T(0)} = OTF(u)$$

irf(x), psf(x), T(u), MTF(u), OTF(u)

- In the previous slides we gave formal definitions of irf(x), psf(x), T(u), MTF(u), and OTF(u).
- However, most often in practical cases some property applies so that a simplification is possible
- For instance, if the impulse response function is normalized: $\int_{-\infty}^{+\infty} irf(x)dx = T(0) = 1$

  then $psf(x) = irf(x)$ and $OTF(u) = T(u)$
- Moreover, if the impulse response function is normalized, real and even:
  then $psf(x) = irf(x)$ and $MTF(u) = OTF(u) = T(u)$
- Some textbooks just assume these conditions apply and do no even introduce irf(x), T(u) and OTF(u), but they simply use psf(x) and MTF(u) - or OTF(u)

PSFs and MTFs



note: in this slide the PSF is called LSF (which we will introduce later) and the spatial frequency is indicated as *f* (instead of *u*)

FIGURE 10-21. The MTF is typically calculated from a measurement of the line spread function (LSF). As the line spread function gets broader (*left column, top to bottom*), the corresponding MTFs plummet to lower MTF values at the same spatial frequency, and the cutoff frequency (where the MTF curve meets the x-axis) is also reduced. The best LSF-MTF pair is at the *top*, and the worst LSF-MTF pair is at the *bottom*.

Measuring *MTF(u)* (a simple method)

- The basic idea was to measure the modulation of the images obtained with a bar-pattern test-object: the input is a square wave (rather than a sine wave)
- However, since we measure the response of the system to a square wave the result is <u>not exactly</u> the *MTF(u)*: it's a different function which sometimes is called <u>Contrast Transfer Function</u> *CTF(u)*

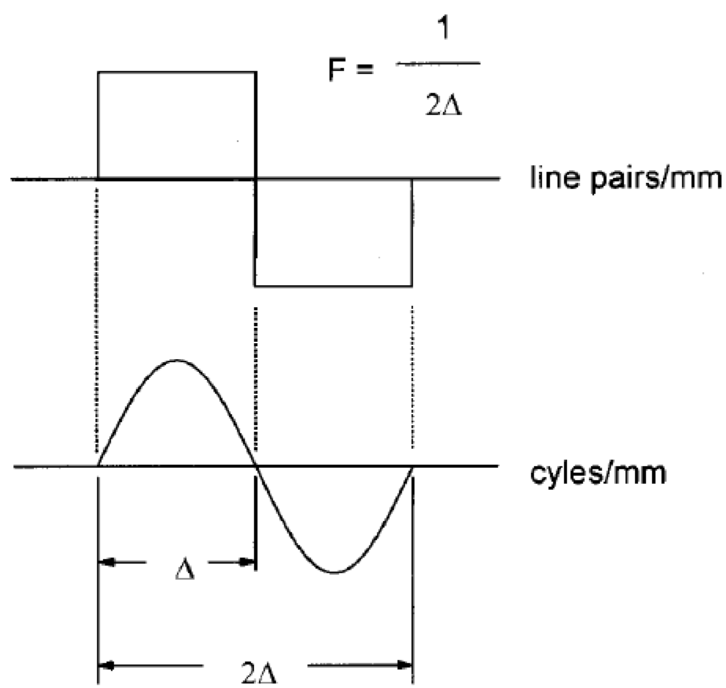


FIGURE 10-17. The concept of *spatial frequency*. A single sine wave (*bottom*) with the width of one-half of the sine wave, which is equal to a distance Δ. The complete width of the sine wave (2Δ) corresponds to one *cycle*. With Δ measured in millimeters, the corresponding spatial frequency is $F = \frac{1}{2\Delta}$. Smaller objects (small Δ) correspond to higher spatial frequencies, and larger objects (large Δ) correspond to lower spatial frequencies. The square wave (top) is a simplification of the sine wave, and the square wave shown corresponds to a single line pair.

Measuring $MTF(u)$ (a simple method)

- Then, a more accurate estimate for the $MTF(u)$ can be obtained form the values of $CTF(u)$ according to the Coltman formula [J.W. Coltman JOSA **44** 468-469, 1954] :

Given the CTF, the Coltman formula to determine the MTF, is

$$M(f) = \frac{\pi}{4}\left[C(f) + \frac{C(3f)}{3} - \frac{C(5f)}{5} + \frac{C(7f)}{7} + \frac{C(11f)}{11} - \frac{C(13f)}{13} - \frac{C(15f)}{15} - \frac{C(17f)}{17} + \frac{C(19f)}{19} \ldots\right]$$

and given the MTF, the Coltman formula to determine the CTF, is

$$C(f) = \frac{4}{\pi}\left[M(f) - \frac{M(3f)}{3} + \frac{M(5f)}{5} - \frac{M(7f)}{7} + \frac{M(9f)}{9} - \frac{M(11f)}{11} + \frac{M(13f)}{13} - \frac{M(15f)}{15} + \frac{M(17f)}{17} - \frac{M(19f)}{19} \ldots\right]$$

where, M(f) = sine wave MTF
C(f) = bar target CTF
f = spatial frequency