MEDICAL PHYSICS LAB
LECTURE 7 – DIGITAL IMAGES

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Digital Images

- Part I – Sampling
- Part II – Aliasing
- Source: Ian A. Cunningham, Chapter 2 in Handbook of medical imaging.
  Volume 1, Physics and psychophysics.
  Richard Van Metter, Jacob Beutel, Harold Kundel, editors.

Part I – Sampling

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Digital images

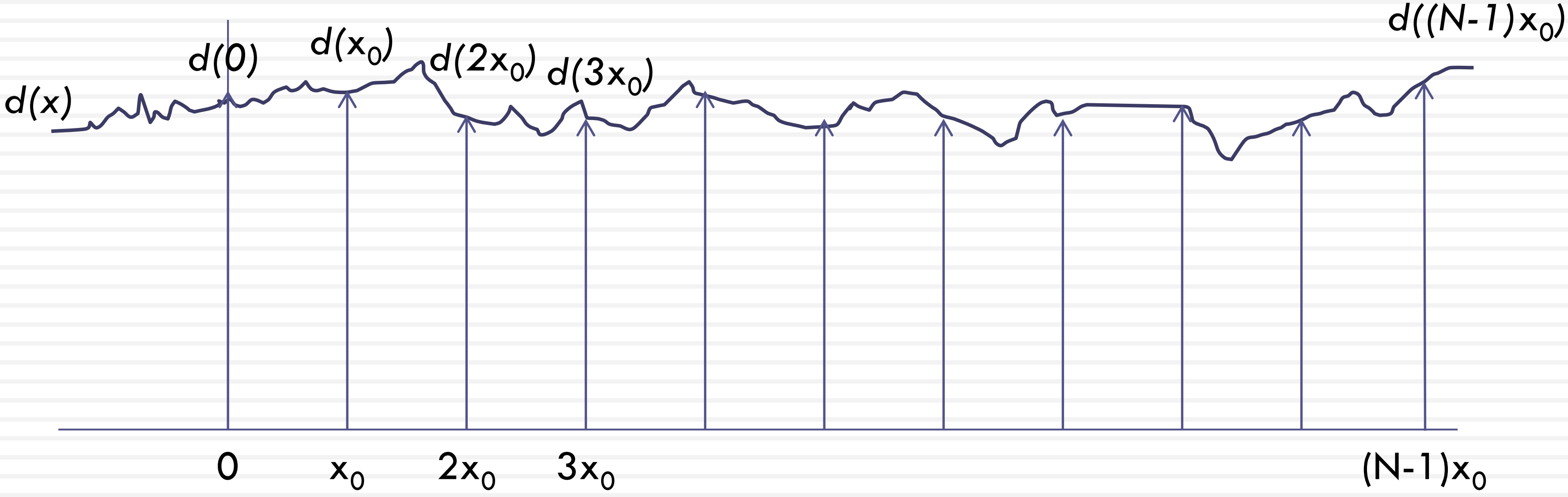
- Usually we have considered (and we will consider) images as analytical functions, e.g.
  - In the space domain: $d(x) = S\{h(x)\} = h(x) * irf(x)$
  - In the spatial frequency domain: $D(u) = H(u)T(u)$
- However, many imaging systems produce digital images, in which image brightness is represented as a sequence of numbers, e.g.
  - $d_n \quad 0 \leq n \leq N-1$

    where N is the number of pixels in the image
- Today we will investigate the relationship between a function and its discrete (sampled) representation.

Sampling

- The function $d(x)$ can be represented numerically with the N discrete values $d_n \;\; 0 \le n \le N-1$, where N is the number of pixels in the image and

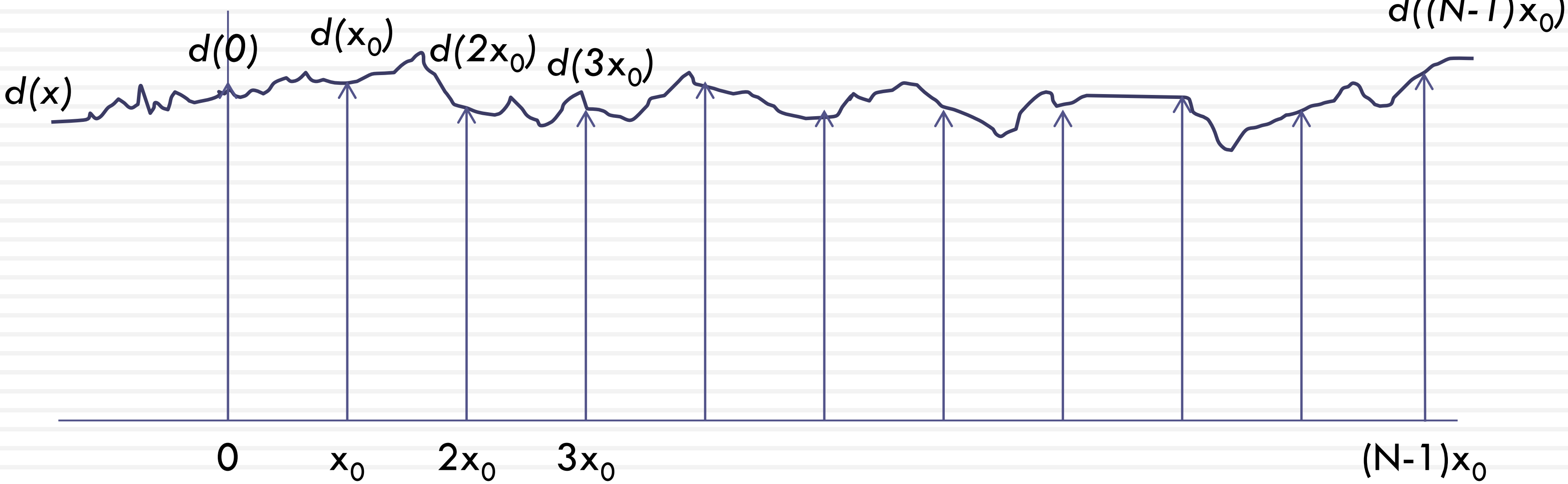
$$d_n \equiv d(nx_0) \equiv d(x)\Big|_{x=nx_0}$$



Sampling

- The process of evaluating a function at uniform spacing x_0 is called sampling
  - x_0 is the sampling interval.
  - $1/x_0$ is the sampling (spatial) frequency.

$$d_n \equiv d(nx_0) \equiv d(x)\big|_{x=nx_0}$$



Sampling

- One way to describe the sampling process is by making use of the sampling property of *III(x)* (see lect. 4).
- In particular, let us consider a function *d(x)* of infinite extent sampled with a sampling interval x_0, which gives an (infinite) sequence of sample values d_n.
- Ideally, this process can be represented as follows:

$$d^{\dagger}(x) \equiv \frac{1}{x_0} d(x) III(\frac{x}{x_0})$$

$$= d(x) \sum_{n=-\infty}^{+\infty} \delta(x - nx_0) = \sum_{n=-\infty}^{+\infty} d(nx_0)\delta(x - nx_0) \equiv \sum_{n=-\infty}^{+\infty} d_n \delta(x - nx_0)$$

- Thus, $d^{\dagger}(x)$ is a sequence of delta functions scaled by the sample values of the “presampling” function *d(x)*.

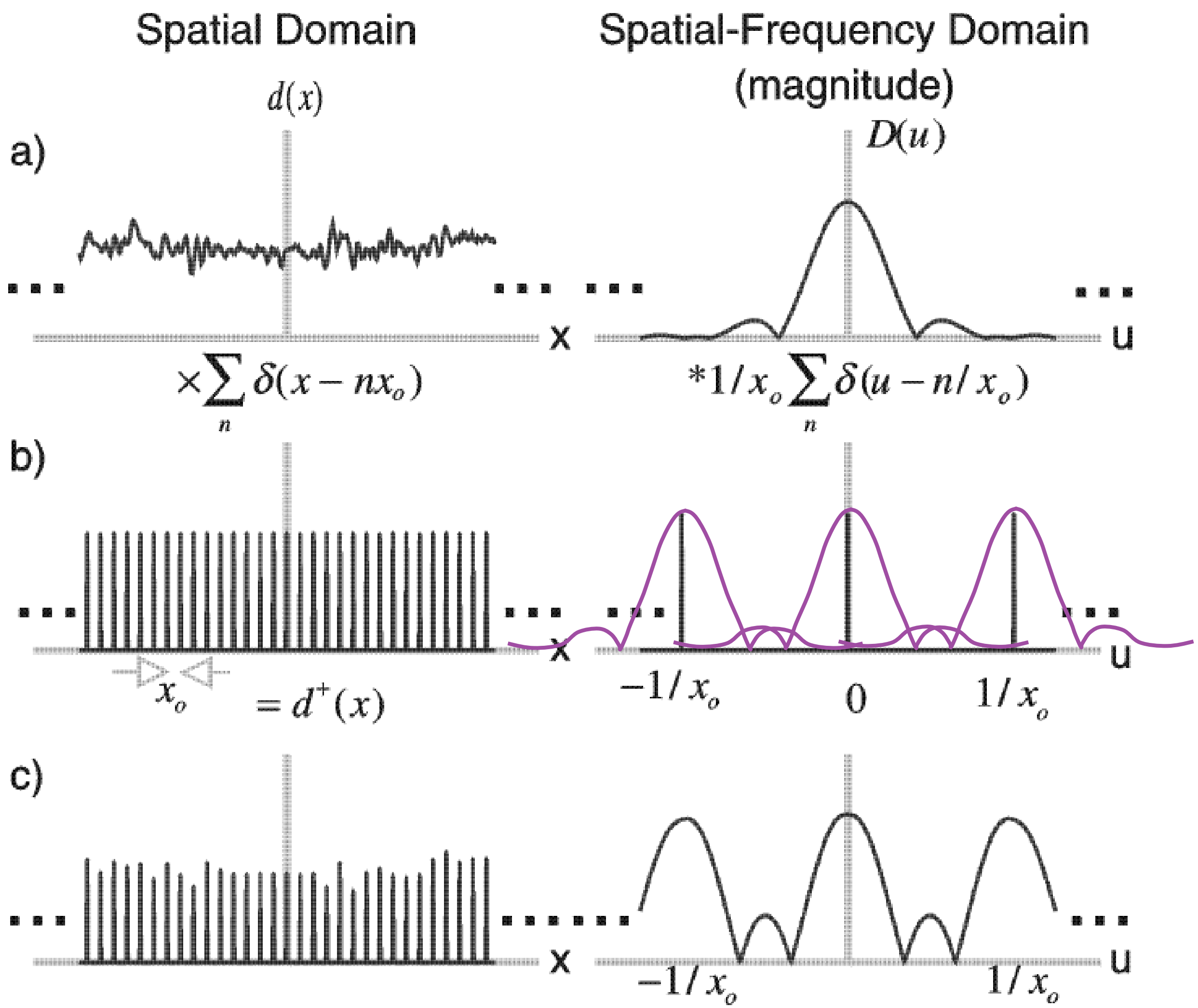
Sampling in the Fourier space

- The formula
$$d^{\dagger}(x) \equiv \frac{1}{x_0} d(x) III(\frac{x}{x_0})$$
provides an analytical representation of the sampling process
- In particular, by means of this expression we can study the sampling process in the Fourier space
- Let us assume $d(x) \supset D(u)$ and $d^{\dagger}(x) \supset D^{\dagger}(u)$
  - by virtue of the similarity theorem: $III(\frac{x}{x_0}) \supset x_0 III(x_0 u)$
  - and by virtue of the convolution theorem

we have

$$d^{\dagger}(x) \equiv \frac{1}{x_0} d(x) III(\frac{x}{x_0}) \supset D(u) * III(x_0 u) \equiv D^{\dagger}(u)$$

$$d(x) \qquad \times \frac{1}{x_0} III\left(\frac{x}{x_0}\right) \qquad = d^{\dagger}(x)$$



$$D(u) \qquad *III(x_0 u) \qquad = D^{\dagger}(u)$$

Figure 2.16: Sampling the function $d(x)$ is represented as $d^{\dagger}(x) = d(x)\sum_{n=-\infty}^{\infty}\delta(x - nx_0)$ and consists of a sequence of δ functions scaled by the discrete values d_n where n is an integer over $-\infty \leqslant n \leqslant \infty$. Spectral aliasing occurs when the aliases overlap in the spatial-frequency domain. Only the magnitude is shown in the frequency domain.

Part II – Aliasing

- Ian A. Cunningham, Chapter 2 in Handbook of medical imaging.
  Volume 1, Physics and psychophysics.
  Richard Van Metter, Jacob Beutel, Harold Kundel, editors.

Sampling and aliasing

- Thus, in general, $D^{\dagger}(u)$ consists of an infinite number of replicas (aliases) of $D(u)$ (scaled by $1/x_0$) and centered at frequencies $u = n/x_0$.
- It is important to notice that if $D(u)$ extends beyond the "cutoff frequency" $u = \pm 1/2x_0$ then the aliases will overlap and $D(u)$ cannot be simply obtained from $D^{\dagger}(u)$ (aliasing).
- This is equivalent to saying that the original function $d(x)$ cannot be simply obtained from the sampled function $d^{\dagger}(x)$ once aliasing has occurred.

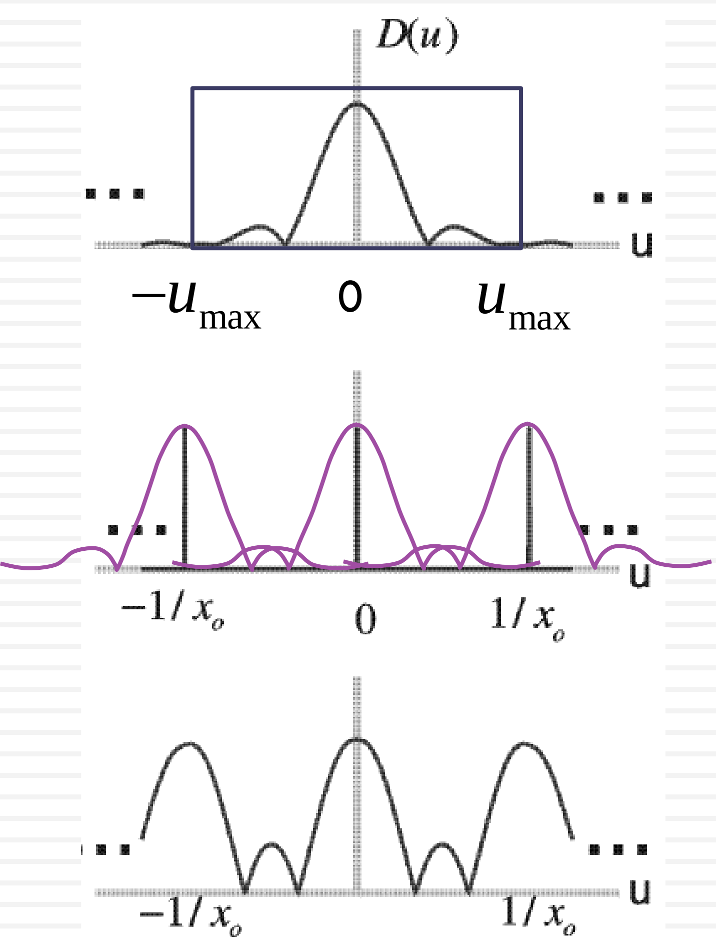
Sampling and aliasing

- Let us consider a function *d(x)* of infinite extent sampled with a sampling interval x_0, which gives an (infinite) sequence of sample values d_n.
- Let us further suppose that *d(x)* is <u>band-limited</u>, i.e. its FT *D(u)* has zero value for every $|u| \geq u_{\max}$
- Then, the infinite number of replicas (aliases) of $D(u)$ will not overlap if they are separated by more than $2u_{\max}$ which is called the Nyquist sampling frequency $u_{Ny} \equiv 2u_{\max}$
- This requirement is equivalent to saying that the original function must be sampled with a (spatial) frequency

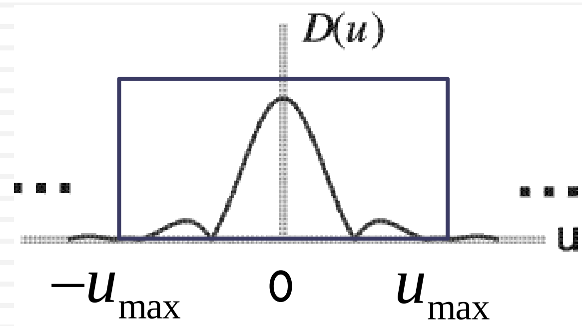
$$\frac{1}{x_0} \geq u_{Ny} \equiv 2u_{\max}$$

$$\frac{1}{x_0} < u_{Ny} \equiv 2u_{max}$$

$$\frac{1}{x_0} \ge u_{Ny} \equiv 2u_{max}$$

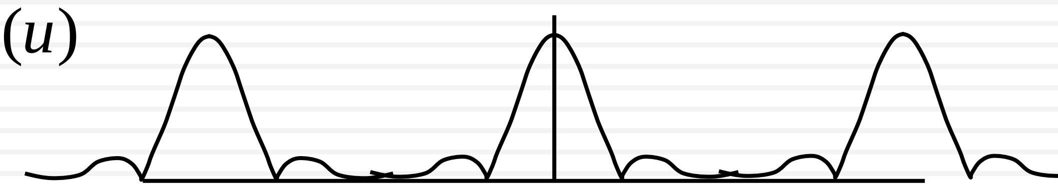


$D(u)$



$* III(x_0 u)$

$= D^{\dagger}(u)$



aliasing

NO aliasing

The sampling theorem

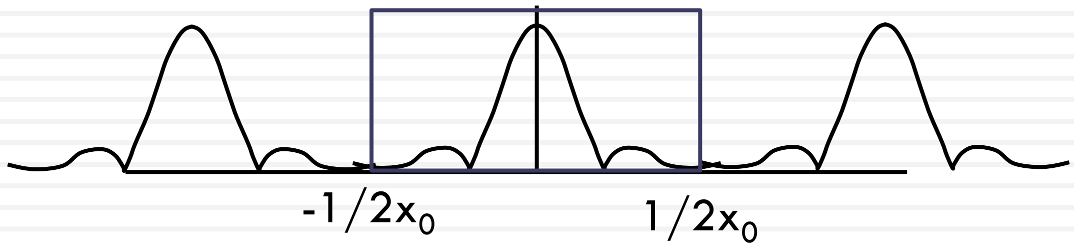
The question should be asked whether the original presampling function $d(x)$ can be recovered exactly from the sample values d_n. In the conjugate domain this question is equivalent to asking whether $D(u)$ can be recovered exactly from the aliased spectrum $\mathrm{F}\{d^{\dagger}(x)\}$ in the lower part of the previous slide

Nyquist-Shannon theorem

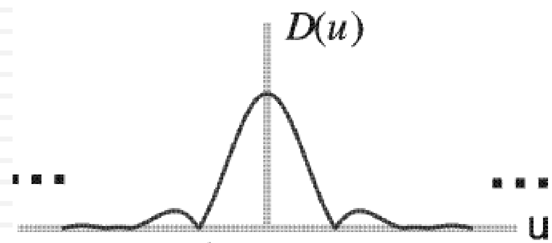
Any band-limited function having infinite extent and no component frequencies at frequencies greater than $u = u_{max}$ can be fully determined from an infinite set of discrete samples if sampled at a frequency greater than $u_{Ny} = 2u_{max}$ where u_{Ny} is called the Nyquist sampling frequency.

Recovering a continuous function from sample values

- In the absence of aliasing:



$$D(u) = D^{\dagger}(u) \cdot x_0 \Pi(x_0 u)$$



- Since $x_0 \Pi(x_0 u) \subset \text{sinc}(\frac{x}{x_0})$
- This is equivalent to saying that the original function *d*(*x*) can be recovered from the sample values:

$$d(x) = d^{\dagger}(x) * \text{sinc}(\frac{x}{x_0})$$

Recovering a continuous function from sample values

- Recalling: $d^{\dagger}(x) = \frac{1}{x_0} d(x) III\left(\frac{x}{x_0}\right) = \sum_{n=-\infty}^{+\infty} d_n \delta(x - nx_0)$

  we can write the previous result more explicitly as:

  $$\begin{aligned} d(x) &= d^{\dagger}(x) * sinc\left(\frac{\mathrm{x}}{\mathrm{x}_0}\right) \\ &= \int_{-\infty}^{\infty} d^{\dagger}(x') sinc\left(\frac{x - x'}{x_0}\right) dx' \\ &= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{+\infty} d_n \delta(x' - nx_0)\, sinc\left(\frac{x - x'}{x_0}\right) dx' \\ &= \sum_{n=-\infty}^{+\infty} d_n \int_{-\infty}^{\infty} \delta(x' - nx_0) sinc\left(\frac{x - x'}{x_0}\right) dx' = \sum_{n=-\infty}^{+\infty} d_n sinc\left(\frac{x - nx_0}{x_0}\right) \end{aligned}$$

- Thus, *d(x)* can be obtained as the *superposition* of (infinite) scaled sinc functions, one for each sampled point

Recovering a continuous function from sample values

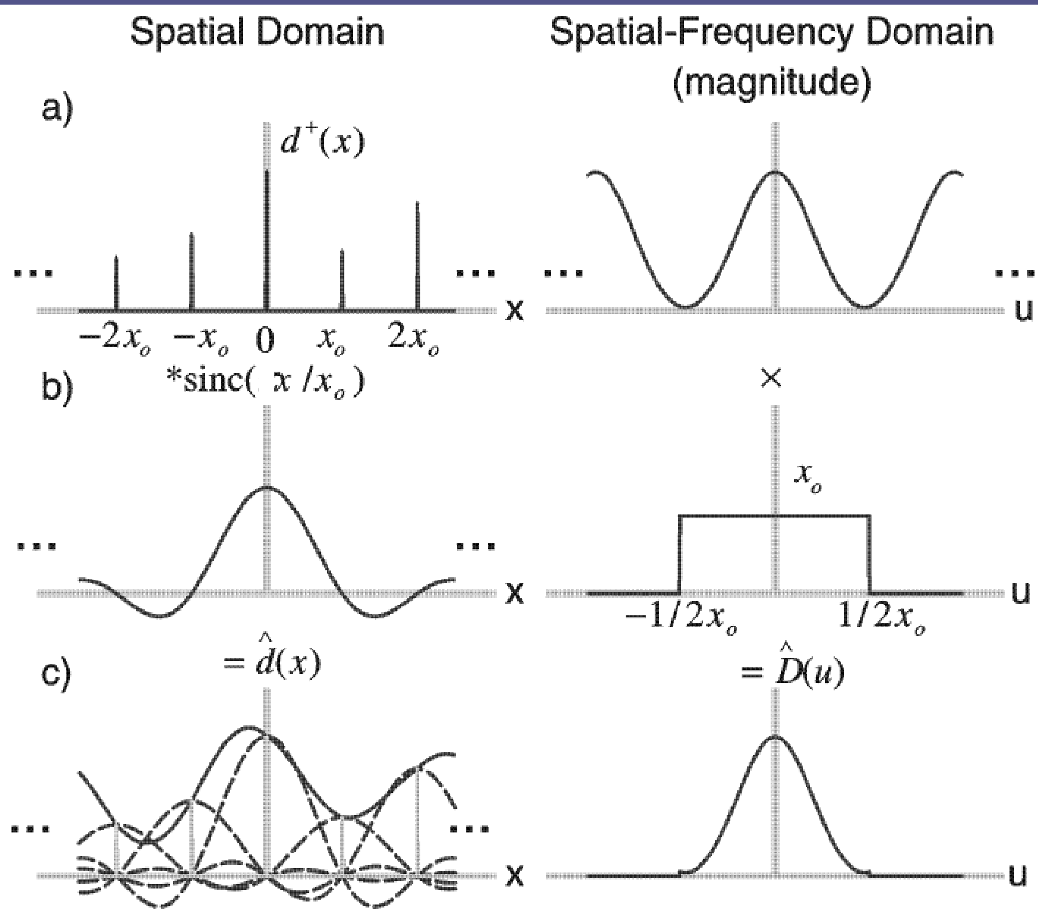
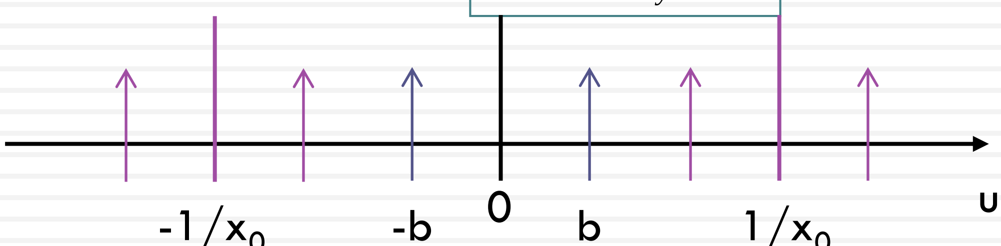


Figure 2.17: The recovered function $\hat{d}(x)$ is obtained by convolving $d^\dagger(x)$ with $\mathrm{sinc}(\ x/x_0)$, resulting in the superposition of a scaled sinc function for each sampled point as indicated by the dashed lines in c). Only the magnitude is shown in the frequency domain.

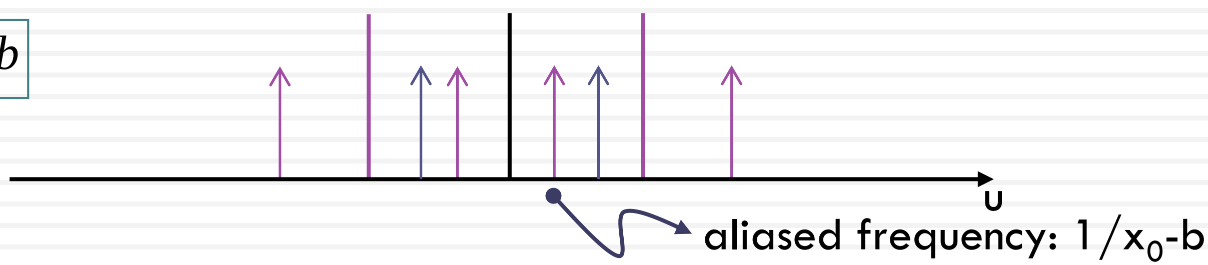
Artifacts from aliasing: a simple approach

- Let us consider a simple function *d(x)* such a cosine wave of infinite extent sampled with a sampling interval x_0, which gives an (infinite) sequence of sample values d_n.
  - $\mathrm{d(x)} = \cos(2\pi b x) \supset \frac{1}{2}[\delta(u-b) + \delta(u+b)]$
- *d(x)* is obviously <u>band-limited</u>, with $u_{\max} = b$
- To satisfy the Nyquist crterion $1/x_0 \geq u_{Ny} = 2b$



- If the Nyquist criterion is not satisfied, this may lead to artifacts, since I may take the aliased frequency for the original one (<u>undersampling</u>).

$1/x_0 < u_{Ny} = 2b$



Artifacts from aliasing: a simple approach

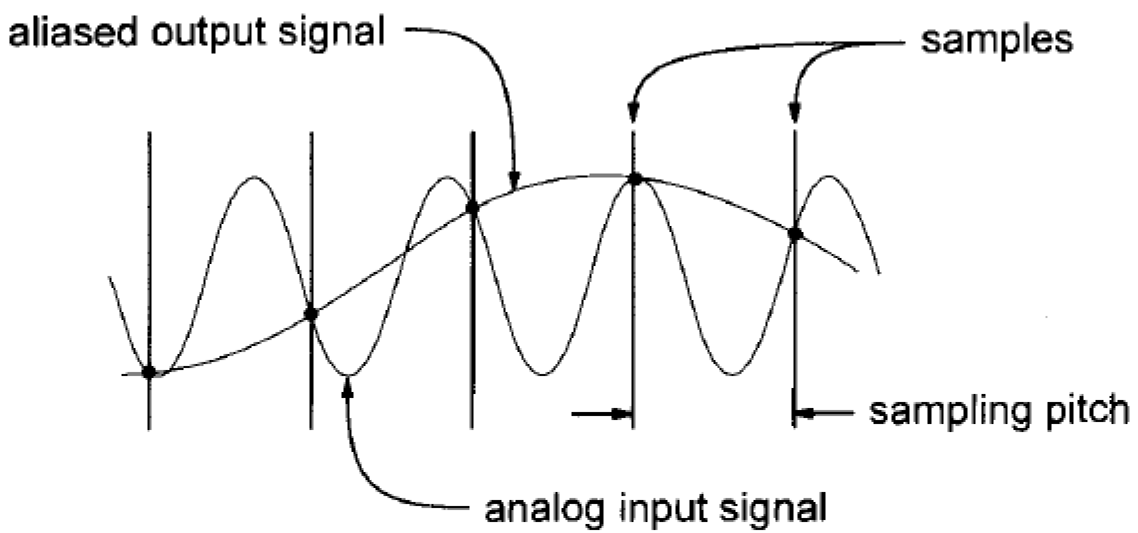


FIGURE 10-34. The geometric basis of aliasing. An analog input signal of a certain frequency F is sampled with a periodicity shown by the *vertical lines* ("samples"). The interval between samples is wider than that required by the Nyquist criterion, and thus the input signal is *undersampled*. The sampled points (*solid circles*) are consistent with a much lower frequency signal ("aliased output signal"), illustrating how undersampling causes high input frequencies to be recorded as aliased, lower measured frequencies.

Artifacts from aliasing: a simple approach

$1/x_0 = 10$ cycles/mm

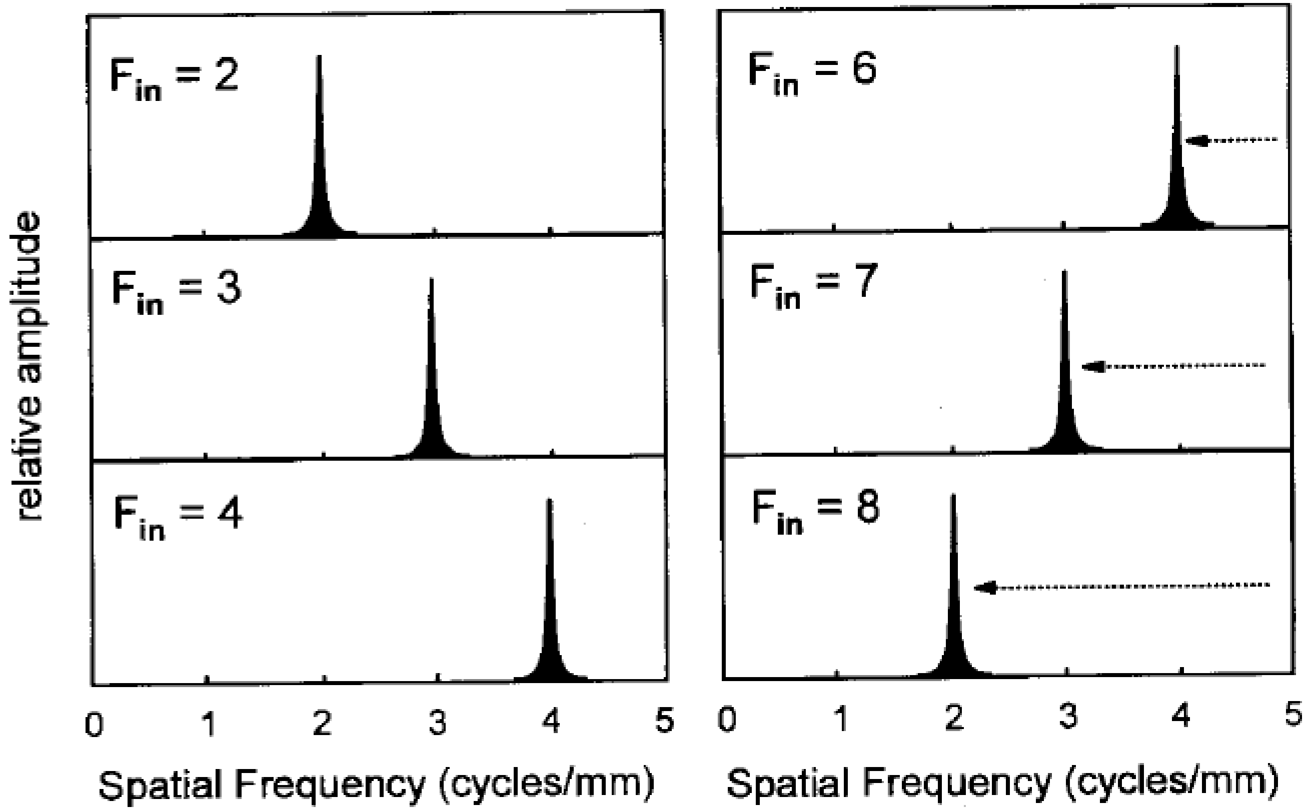


FIGURE 10-35. Each of the six panels illustrates the Fourier transform of a sine wave with input frequency F_{IN}, after it has been detected by an imaging system with a Nyquist frequency, F_N, of 5 cycles/mm (i.e., sampling pitch = 0.10 mm). **Left panels:** Input frequencies that obey the Nyquist criterion ($F_{IN} < F_N$), and the Fourier transform shows the recorded frequencies to be equal to the input frequencies. **Right panels:** Input frequencies that do not obey the Nyquist criterion ($F_{IN} > F_N$), and the Fourier transform of the signal indicates recorded frequencies quite different from the input frequencies, demonstrating the effect of *aliasing*.

Appendix: The Discrete Fourier Transform (DFT)

One commonly used form for the DFT of a sequence of N values d_n for $0 \leqslant n \leqslant N-1$ is given by

$$D_m = \mathrm{DFT}\{d_n\} = \sum_{n=0}^{N-1} d_n e^{-i2\pi nm/N}, \tag{2.63}$$

which consists of a sequence of the N complex values D_m for $0 \leqslant m \leqslant N-1$. The inverse DFT is given by

$$d_n = \mathrm{DFT}^{-1}\{D_m\} = \frac{1}{N}\sum_{m=0}^{N-1} D_m e^{i2\pi nm/N}. \tag{2.64}$$

Other forms of the DFT exist, differing primarily by a scaler constant of N or $\sqrt{N}$. The dimensions of d_n and D_m must necessarily be the same, and they are often dimensionless.