Corso di Laurea in Fisica - UNITS

ISTITUZIONI DI FISICA

PER IL SISTEMA TERRA

Wave Phenomena (superposition)

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Superposition of waves

When two waves meet in space their individual disturbances (represented by their wavefunctions) superimpose and add together.

The principle of superposition states:

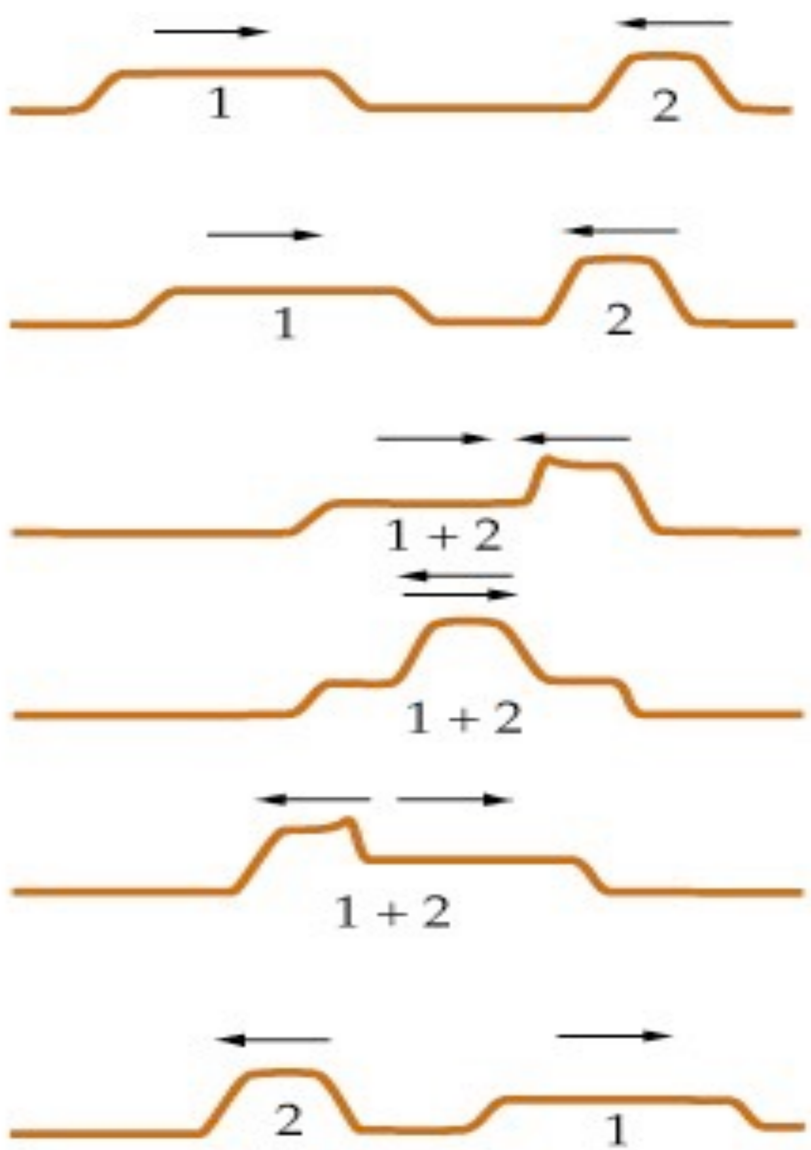
If two or more travelling waves are moving through a medium, the resultant wavefunction at any point is the algebraic sum of the wavefunctions of the individual waves

Waves that obey this principle are called **LINEAR WAVES**

Waves that do not are called **NONLINEAR WAVES**

Generally LW have small amplitudes, NLW have large amplitudes

One consequence of this is that two travelling waves can pass through each other without being altered or destroyed

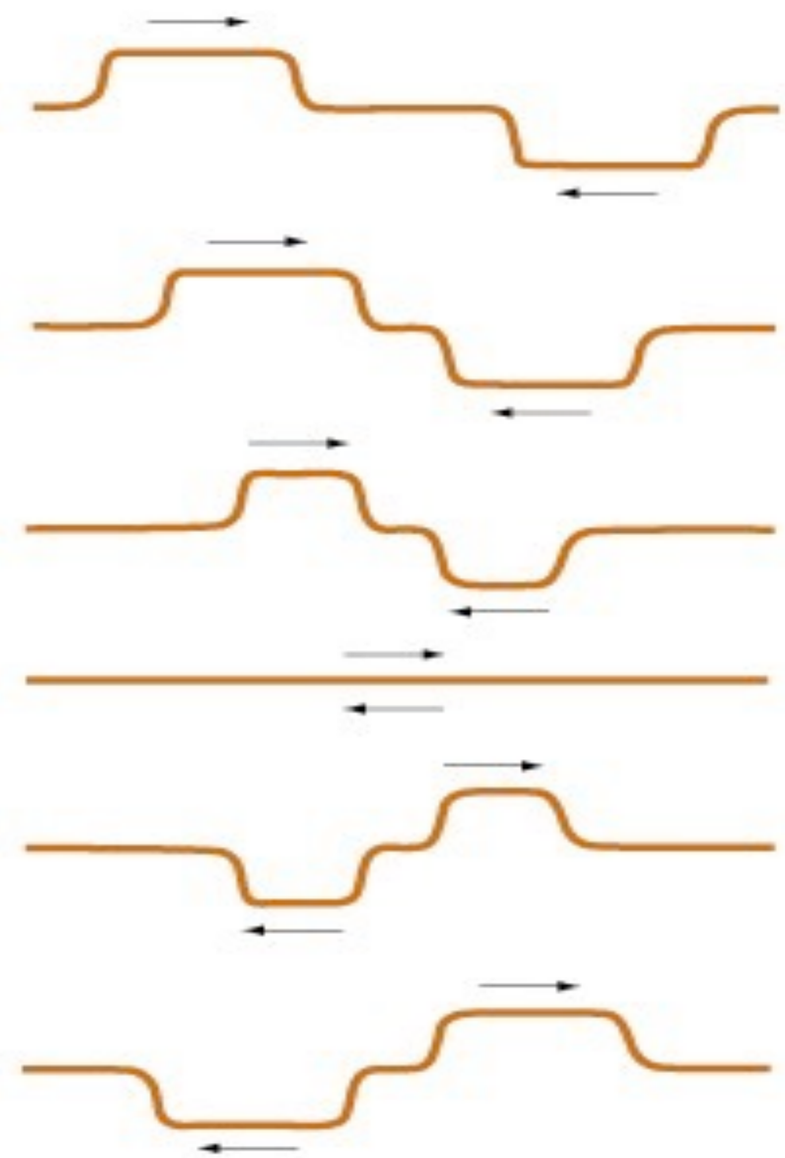


(left)

two +ve pulses

(right)

one +ve and one -ve pulse



Superposition and standing

Already looked at interference effects - the combination of two waves travelling simultaneously through a medium.

Now look at superposition of harmonic waves.

Beats

Standing waves

Modes of vibration

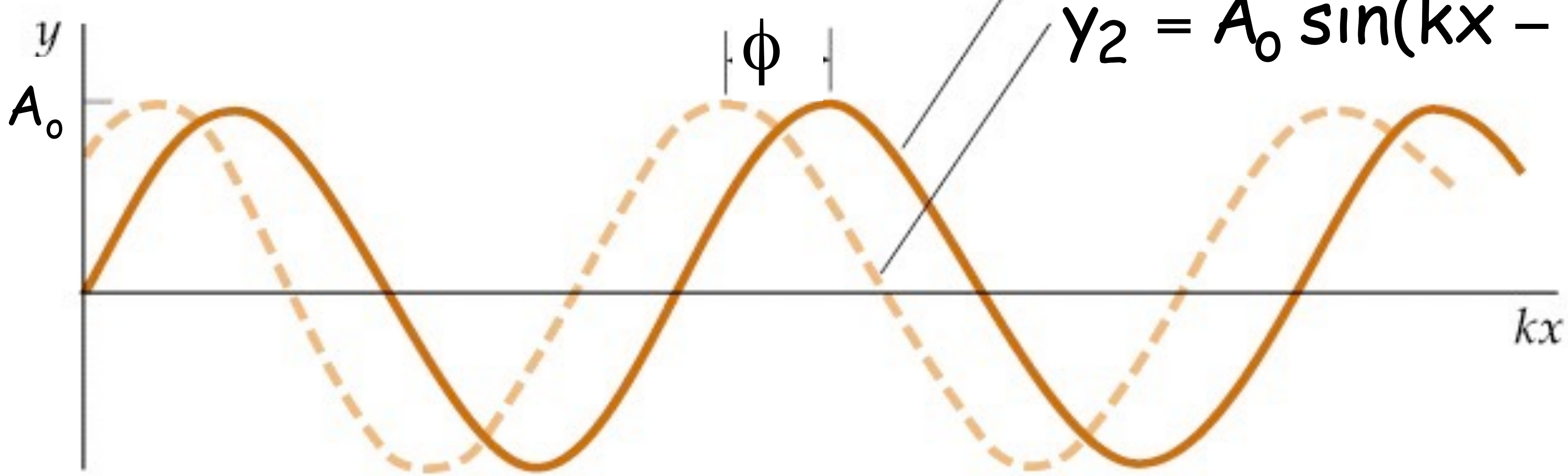
Superposition of Harmonic

Principle of superposition states that when two or more waves combine the net displacement of the medium is the algebraic sum of the two displacements.

Consider two harmonic waves travelling in the same direction in a medium

$$y_1 = A_o \sin(kx - \omega t)$$

$$y_2 = A_o \sin(kx - \omega t - \phi)$$



Tipler

Fig 16-2

$$y_1 = A_o \sin(kx - \omega t) \qquad y_2 = A_o \sin(kx - \omega t - \phi)$$

The resultant wave function is given by

$$y = y_1 + y_2 = A_o \sin(kx - \omega t) + A_o \sin(kx - \omega t - \phi)$$
$$= A_o \left[\sin(kx - \omega t) + \sin(kx - \omega t - \phi)\right]$$

This can be simplified using

$$\sin A + \sin B = 2\cos\left(\frac{A - B}{2}\right)\sin\left(\frac{A + B}{2}\right)$$

with $A = (kx - \omega t)$ and $B = (kx - \omega t - \phi)$

$$y = A_o \left[\sin(kx - \omega t) + \sin(kx - \omega t - \phi)\right]$$

$$= 2A_o \cos\left(\frac{\phi}{2}\right)\sin\left(kx - \omega t - \frac{\phi}{2}\right)$$

The resulting wavefunction is harmonic and has the same frequency and wavelength as the original waves.

Amplitude of the resultant wave = $2A_o\cos(\phi/2)$

Phase of the resultant wave = $(\phi/2)$

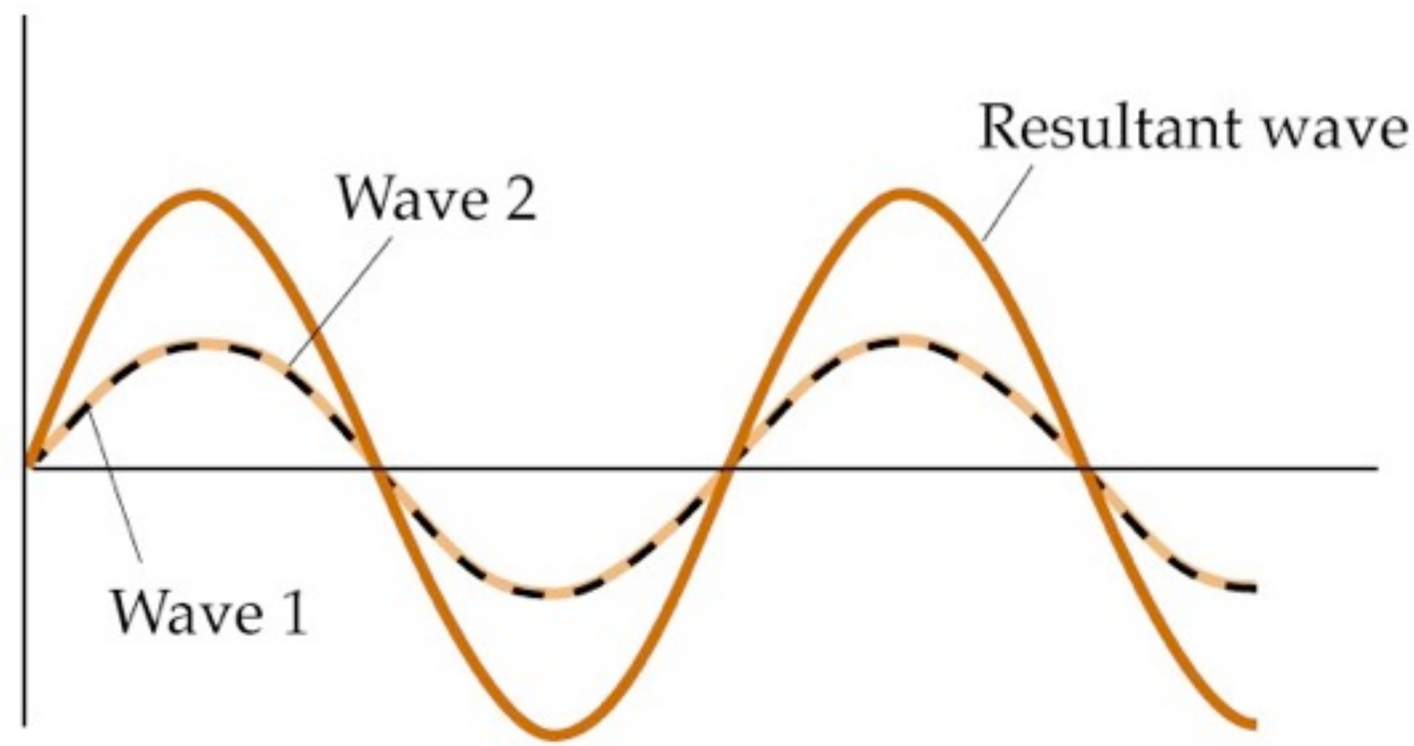
Constructive interference

$$y = 2A_0 \cos\left(\frac{\phi}{2}\right)\sin\left(kx - \omega t - \frac{\phi}{2}\right)$$

when $\phi = 0$ $\cos(\phi/2) = 1$

the amplitude of the resultant wave = $2A_o$

The waves are **in phase** and interfere **constructively.**



Tipler Fig 16-3

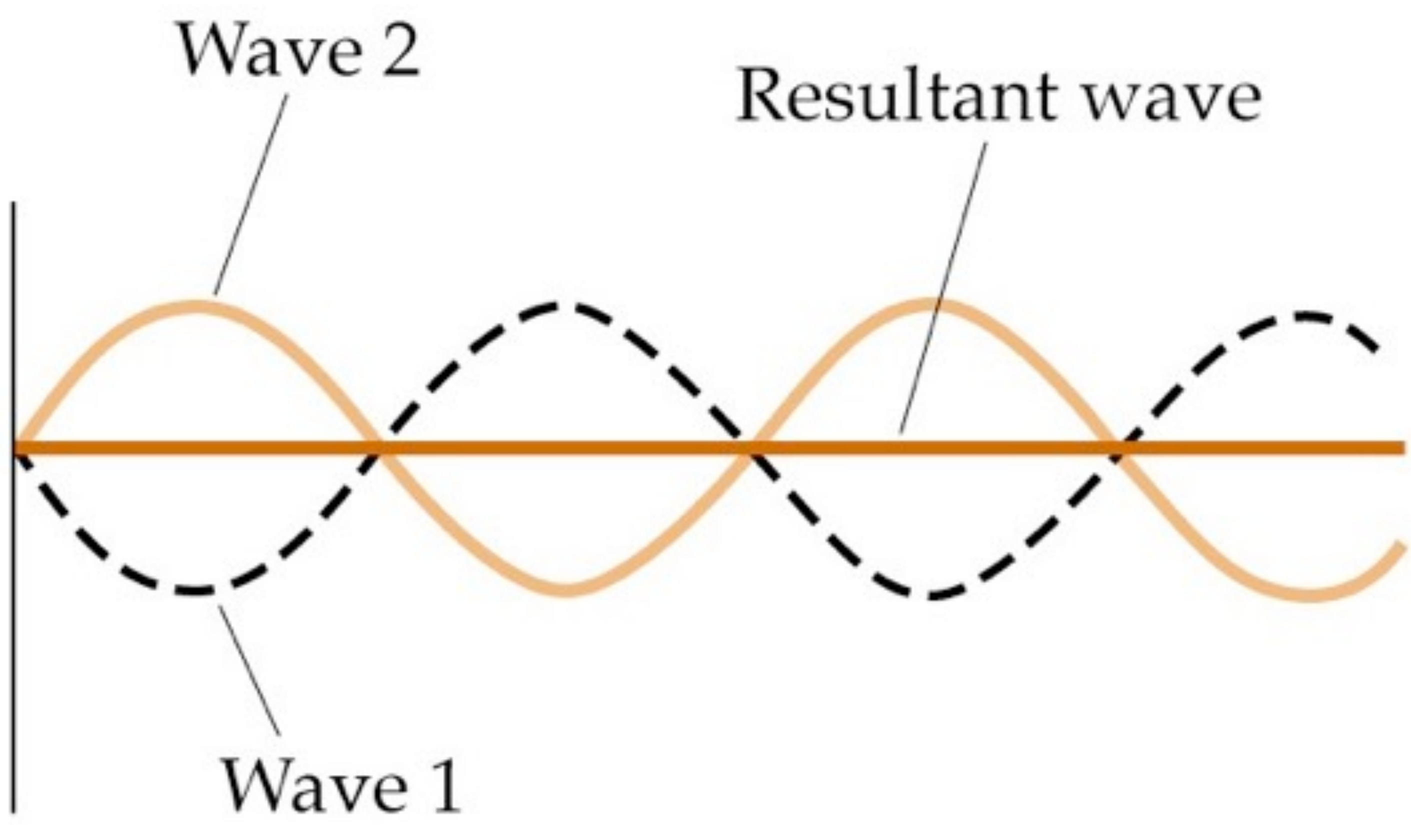
Destructive interference

$$y = 2A_0 \cos\left(\frac{\phi}{2}\right)\sin\left(kx - \omega t - \frac{\phi}{2}\right)$$

when $\phi = \pi$ or any odd multiple of π $\cos(\phi/2) = 0$

the amplitude of the resultant wave = 0

The waves are **out of phase** and interfere **destructively.**

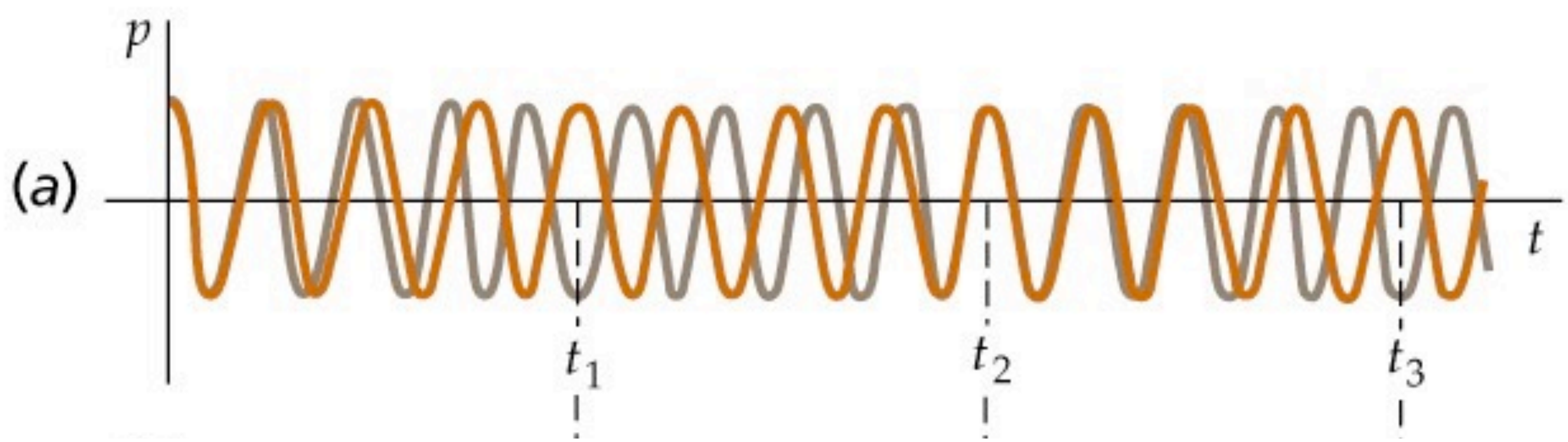


Tipler Fig 16-4

Beats

What happens if the wave have different frequencies ?

Consider two waves travelling in the same direction but with slightly different frequencies



Tipler Fig 16-5a

$$y_1 = A_0 \cos(2\pi f_1 t) \qquad y_2 = A_0 \cos(2\pi f_2 t)$$

Using the principle of superposition we can say

$$y = y_1 + y_2 = A_0\left[\cos(2\pi f_1 t) + \cos(2\pi f_2 t)\right]$$

Beats

$$y = y_1 + y_2 = A_0\left[\cos(2\pi f_1 t) + \cos(2\pi f_2 t)\right]$$

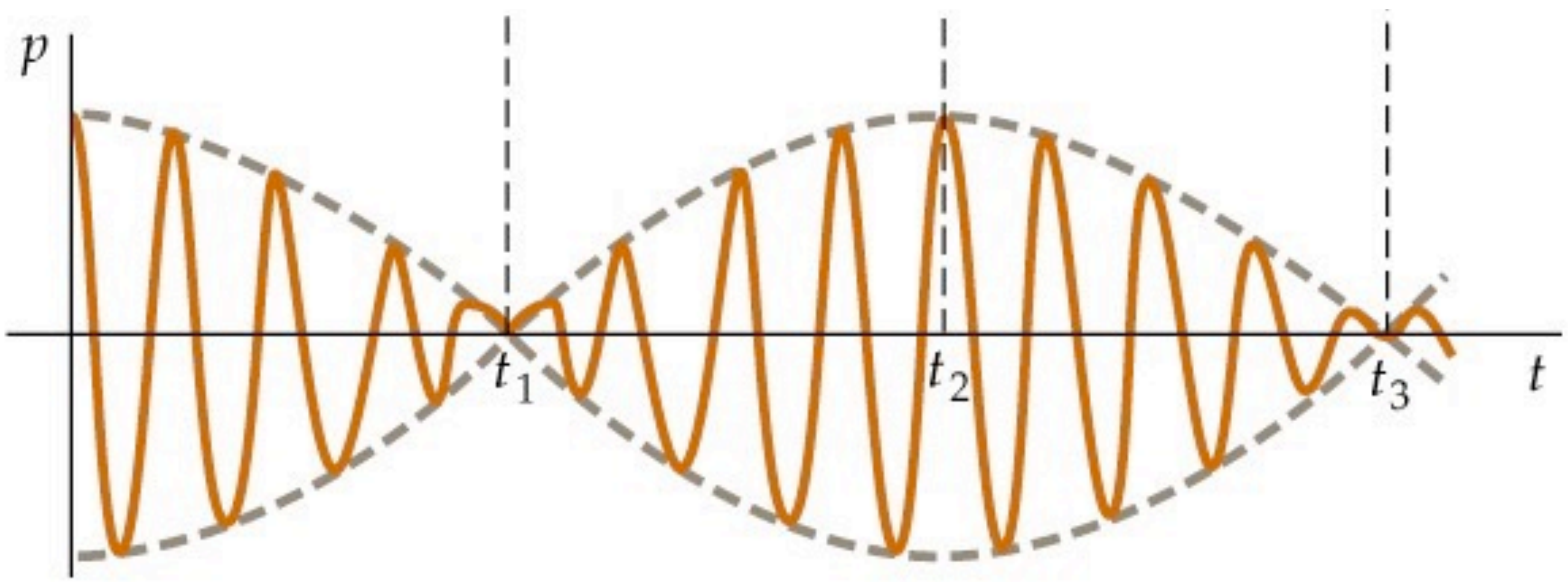
This can be simplified using

$$\cos A + \cos B = 2\cos\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

with $A = (2\pi f_1 t)$

and $B = (2\pi f_2 t)$

$$\therefore \quad y = 2A_0 \cos\left[2\pi t\left(\frac{f_1 - f_2}{2}\right)\right] \cos\left[2\pi t\left(\frac{f_1 + f_2}{2}\right)\right]$$



Tipler Fig 16-5b

Beats

$$y = 2A_0 \cos\left[2\pi t\left(\frac{f_1 - f_2}{2}\right)\right] \cos\left[2\pi t\left(\frac{f_1 + f_2}{2}\right)\right]$$

Compare this to the individual wavefunctions:

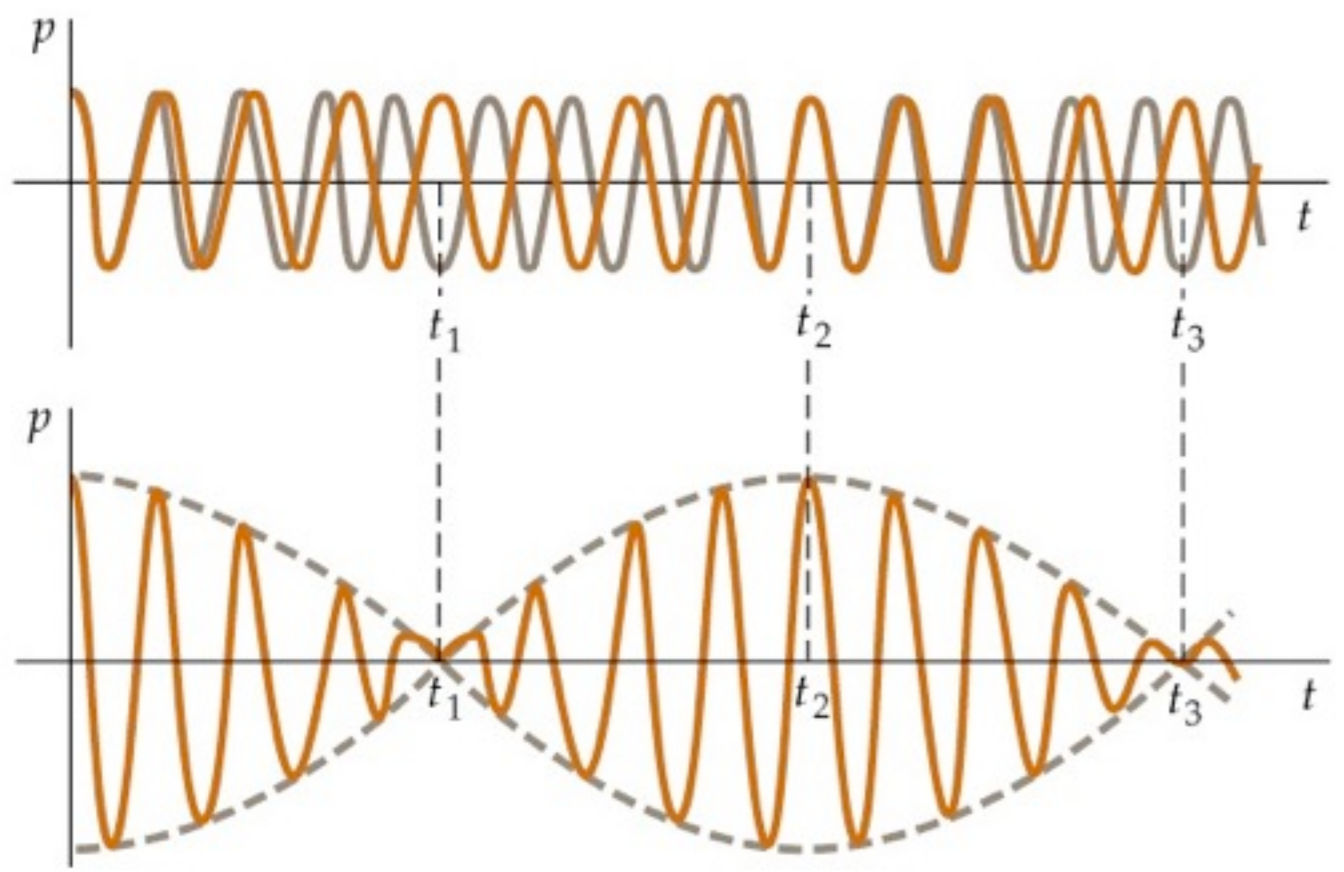
$$y_1 = A_0 \cos(2\pi f_1 t) \qquad y_2 = A_0 \cos(2\pi f_2 t)$$

Resultant vibration has an effective frequency $(f_1 + f_2)/2$ and an amplitude given by

$$2A_0 \cos\left[2\pi t\left(\frac{f_1 - f_2}{2}\right)\right]$$

The amplitude varies with time with a frequency $(f_1 - f_2)/2$

Beats



Tipler fig 16-5

A beat is detected when $\cos\left[2\pi t\left(\frac{f_1 - f_2}{2}\right)\right] = \pm 1$

There are two beats per cycle

Beat frequency $= 2\,(f_1 - f_2)/2 = f_1 - f_2$

What will we hear ?

Consider two tuning forks vibrating at frequencies of 438 and 442Hz.

The resultant sound wave would have a frequency of
(438+442) / 2 = 440Hz (A on piano)

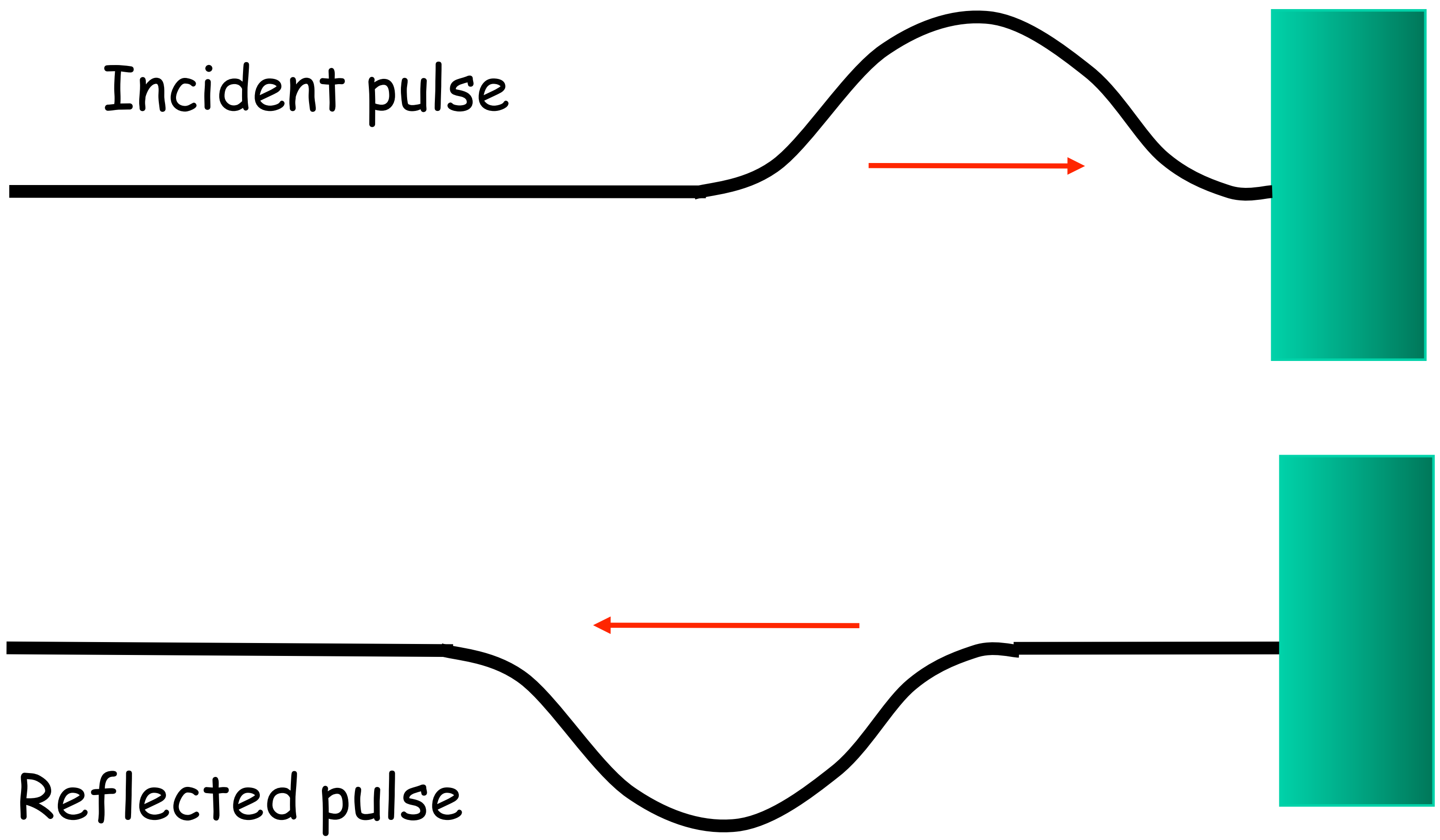
and a beat frequency of 442 - 438 = 4Hz

The listener would hear the 440Hz sound wave go through an intensity maximum four times per second

Musicians use beats to tune an instrument

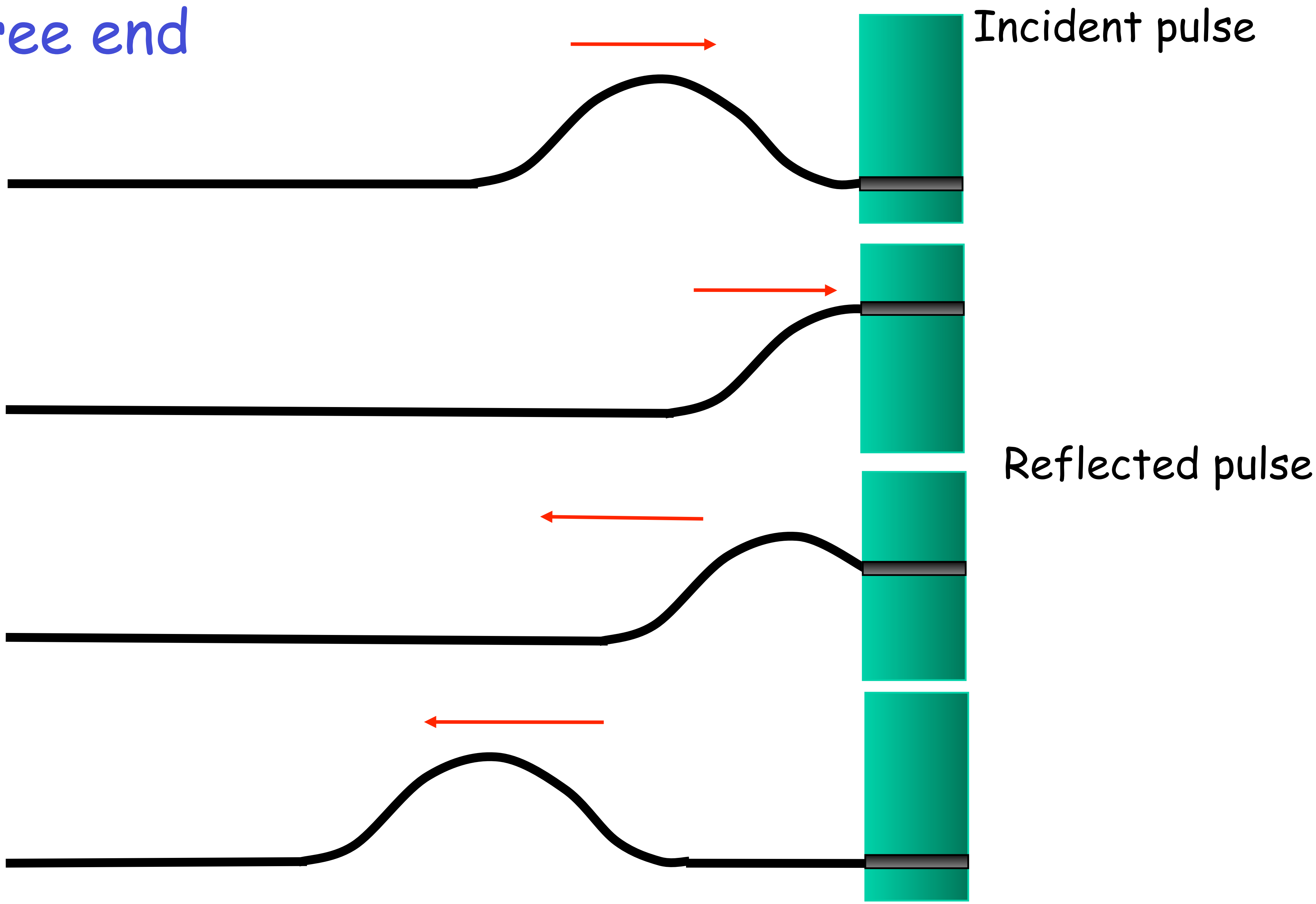
Reflection of travelling waves

A pulse travelling a string fixed at one end

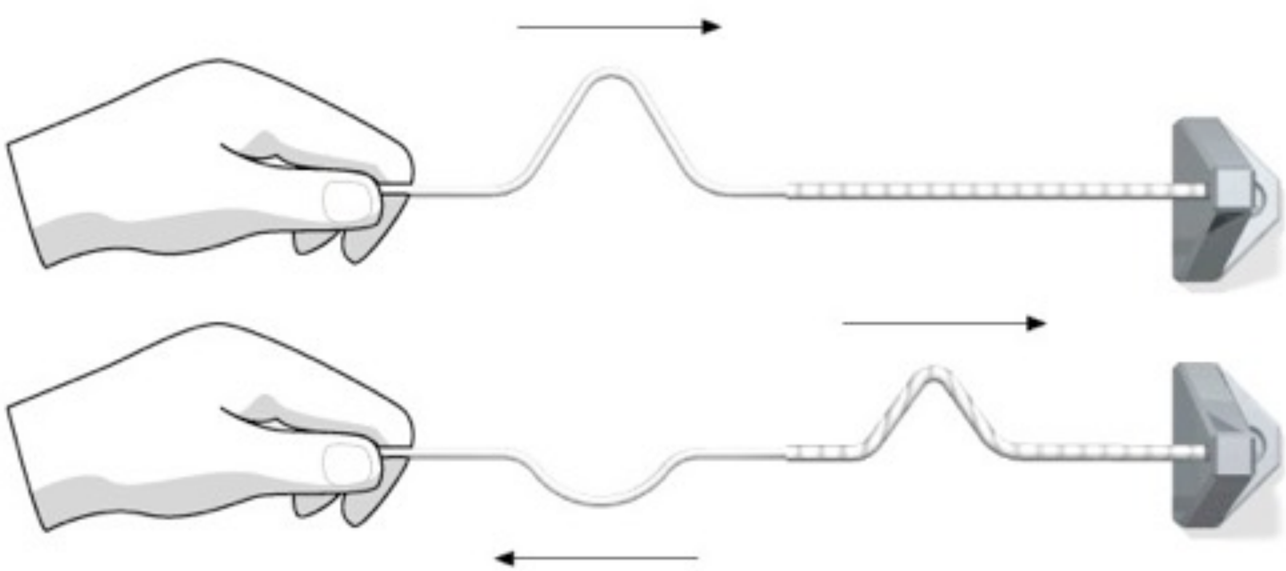


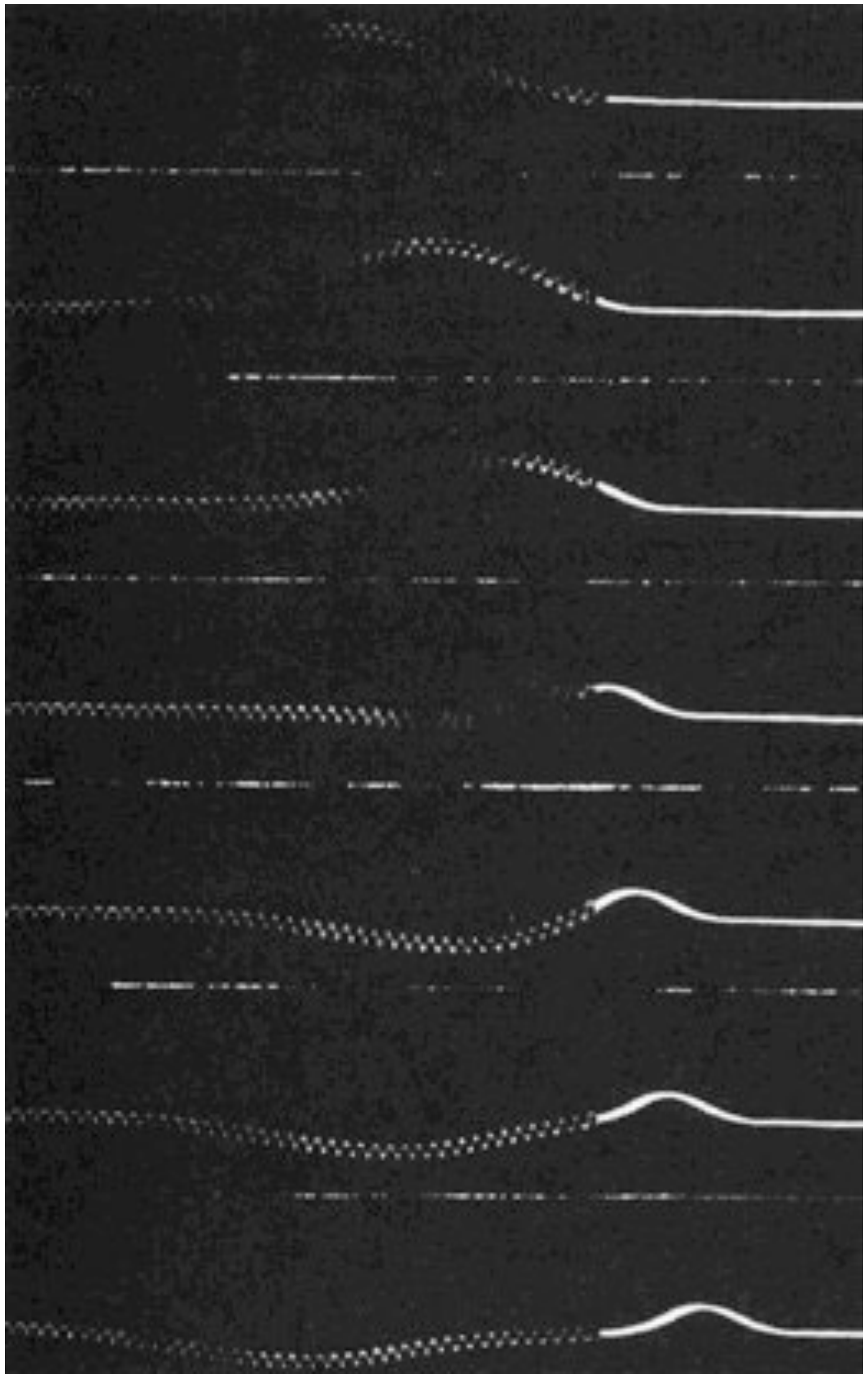
NB. we assume that the wall is rigid and the wave does not transmit any part of the disturbance to the wall

A pulse travelling on a string with a free end

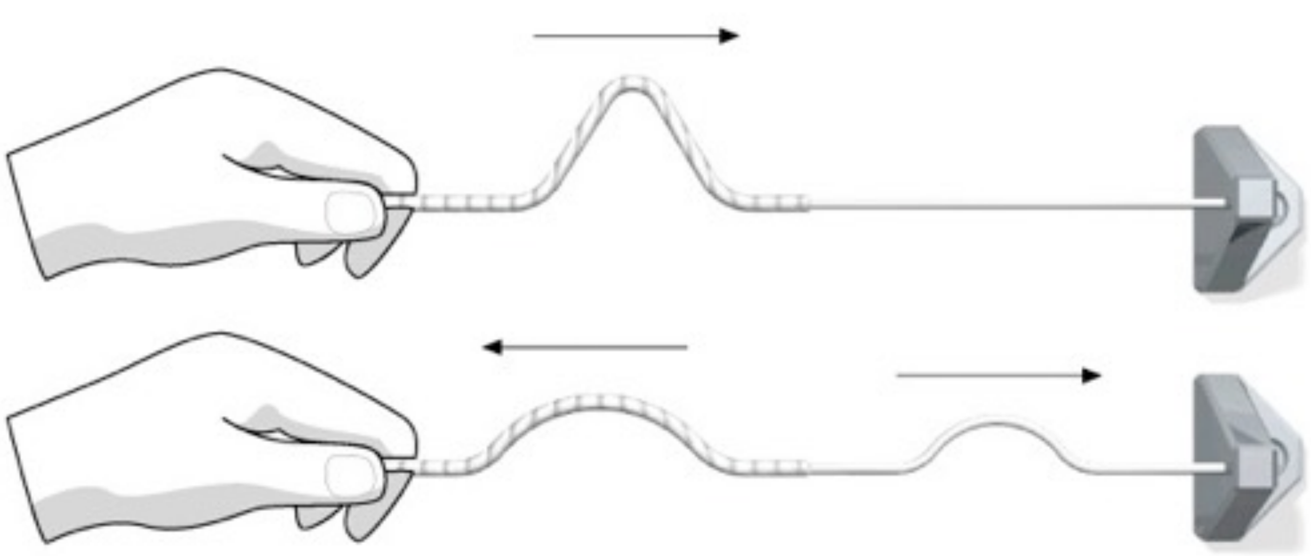


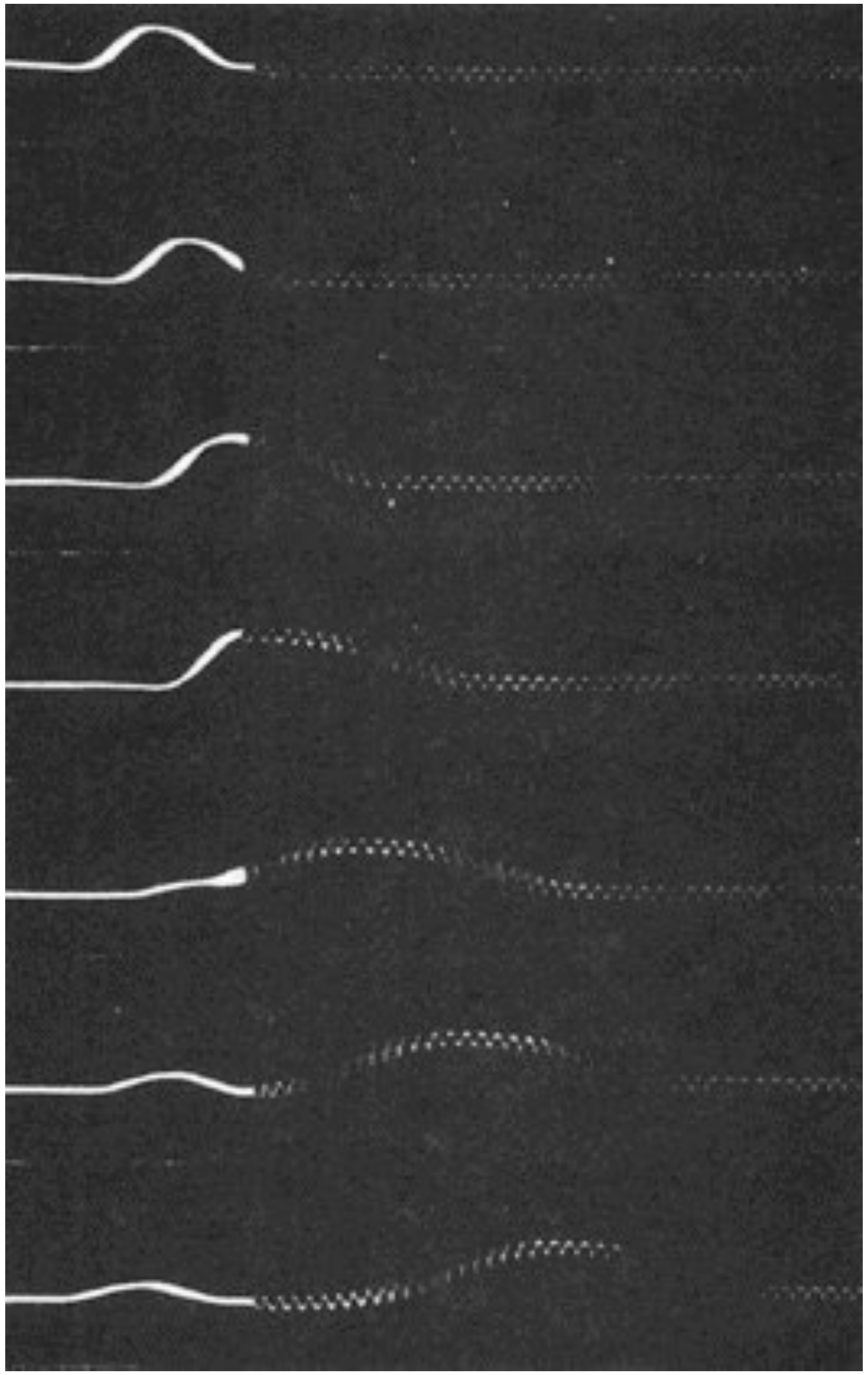
A pulse travelling on a light string attached to a heavier string





A pulse travelling on a heavy string attached to a lighter string





Impedance

- Any medium through which waves propagates will present an impedance to those waves.
- If medium is lossless or possesses no dissipative mechanism, the impedance is real and can be determined by the energy storing parameters, inertia and elasticity.
- Presence of loss mechanism will introduce a complex term.
- Impedance presented by a string to a traveling wave propagating on it is called transverse impedance.

Impedance

$$\text{transverse impedance} = \frac{\text{transverse force}}{\text{transverse velocity}}$$

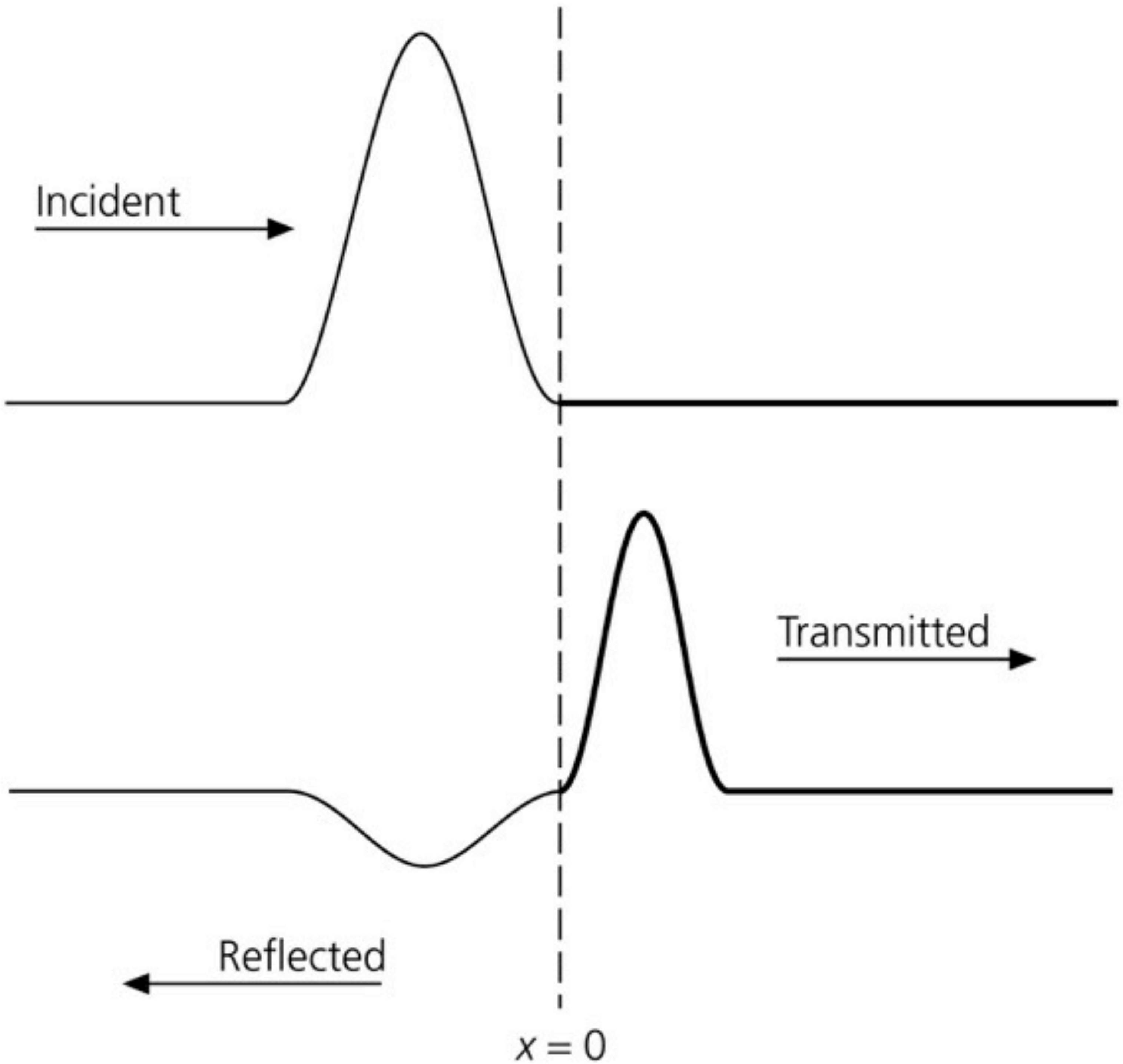
$$F_T \approx -T\tan\theta = -T\left(\frac{\partial y}{\partial x}\right) = T\frac{\omega}{v}y \quad v_T \approx \omega y$$

$$Z = \frac{T}{v} = \sqrt{T\rho} = \rho v$$

$$\text{acoustic impedance} = \frac{\text{pressure}}{\text{sound flux}}$$

Heterogeneous string

Figure 2.2-5: Transmitted and reflected wave pulses.



Left side:

$$u_1(x, t) = Ae^{i(\omega t - k_1 x)} + Be^{i(\omega t + k_1 x)}$$

Right side:

$$u_2(x, t) = Ce^{i(\omega t - k_2 x)}$$

Displacement continuity

Left side:

$$u_1(x, t) = Ae^{i(\omega t - k_1 x)} + Be^{i(\omega t + k_1 x)}$$

Right side:

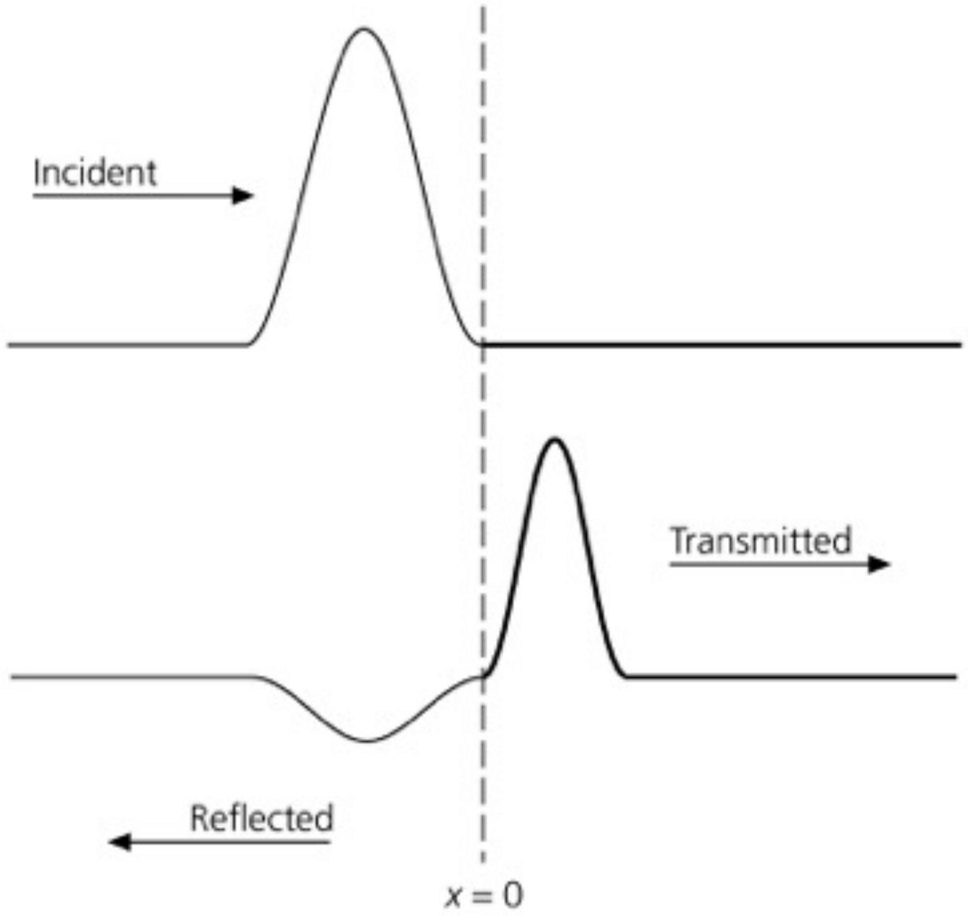
$$u_2(x, t) = Ce^{i(\omega t - k_2 x)}$$

$$\boxed{u_1(0, t) = u_2(0, t)}$$

$$Ae^{i\omega t} + Be^{i\omega t} = Ce^{i\omega t}$$

$$A + B = C$$

Figure 2.2-5: Transmitted and reflected wave pulses.



Force continuity

Left side:

$$u_1(x, t) = Ae^{i(\omega t - k_1 x)} + Be^{i(\omega t + k_1 x)}$$

Right side:

$$u_2(x, t) = Ce^{i(\omega t - k_2 x)}$$

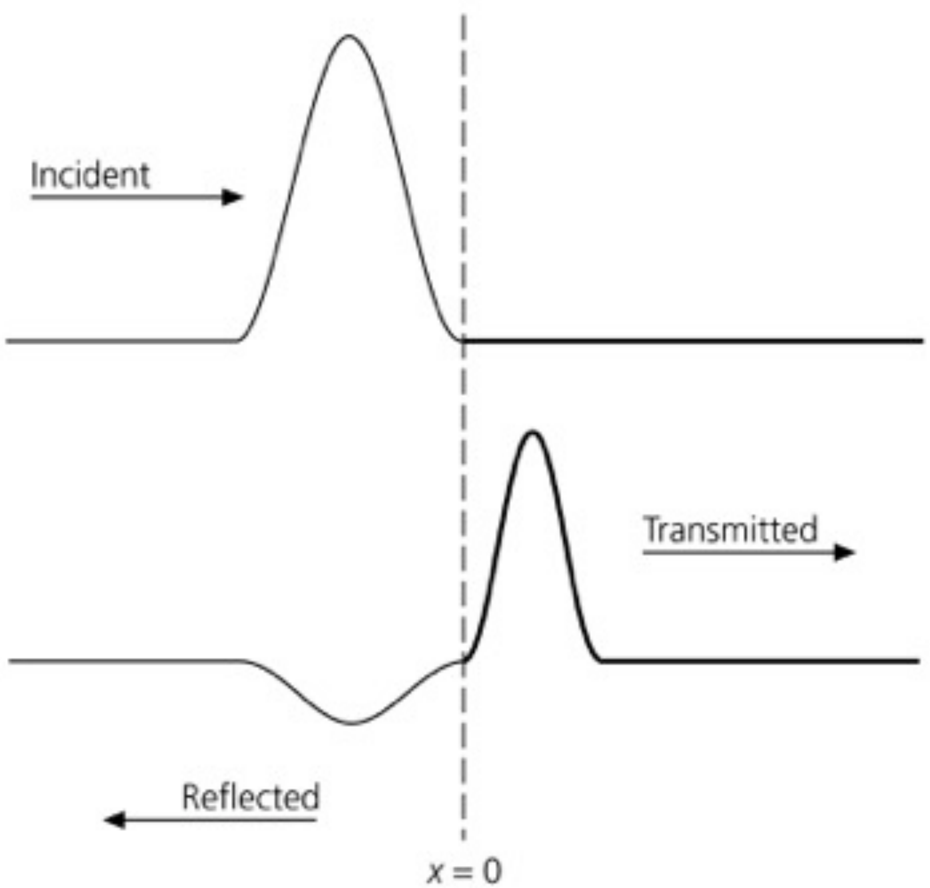
$$\boxed{\tau \frac{\partial u_1(0, t)}{\partial x} = \tau \frac{\partial u_2(0, t)}{\partial x}}$$

$$\tau k_1 (A - B) = \tau k_2 C$$

Because the velocities on the two sides are $v_i = (\tau/\rho_i)^{1/2}$ and $k_i = \omega/v_i$,

$$\rho_1 v_1 \, (A - B) = \rho_2 v_2 C$$

Figure 2.2-5: Transmitted and reflected wave pulses.



R&T coefficients

$$A + B = C$$

$$\rho_1 v_1 (A - B) = \rho_2 v_2 C$$

Reflection coefficient:

$$R_{12} = \frac{B}{A} = \frac{\rho_1 v_1 - \rho_2 v_2}{\rho_1 v_1 + \rho_2 v_2}$$

Transmission coefficient:

$$T_{12} = \frac{C}{A} = \frac{2\rho_1 v_1}{\rho_1 v_1 + \rho_2 v_2}$$

$$R_{12} = -R_{21} \qquad T_{12} + T_{21} = 2$$

$$R_{12} = \frac{B}{A} = \frac{\rho_1 v_1 - \rho_2 v_2}{\rho_1 v_1 + \rho_2 v_2}$$

Fixed end?

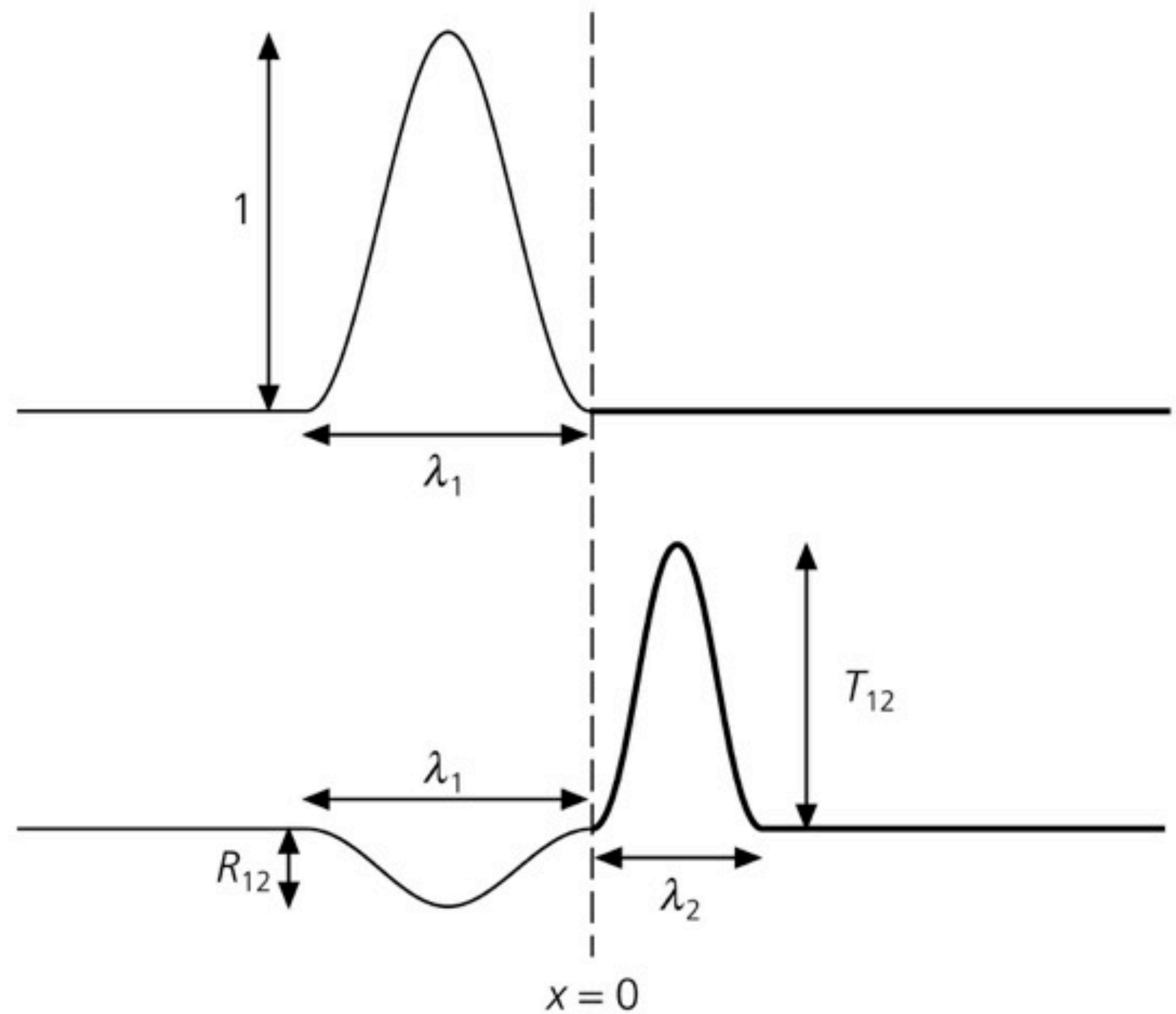
$$R_{fixed} = \frac{\rho_1 v_1 - \infty}{\rho_1 v_1 + \infty} = -1$$

Free end?

$$R_{fixed} = \frac{\rho_1 v_1 - 0}{\rho_1 v_1 + 0} = 1$$

Polarity?

Figure 2.2-7: Reflected and transmitted amplitudes.



$$\omega = v_1 k_1 = v_2 k_2 = v_1 2\pi/\lambda_1 = v_2 2\pi/\lambda_2$$

Kinetic energy

Kinetic energy:

$$KE = \frac{\rho}{2}\left(\frac{\partial u}{\partial t}\right)^2 dx$$

because the mass of the spring is $m = \rho dx$

Averaged over one wavelength, with $u(x, t) = A\cos(\omega t - kx)$:

$$KE = \frac{\rho}{2\lambda}\int_0^\lambda \left(\frac{\partial u}{\partial t}\right)^2 dx = \frac{\rho\, A^2\, \omega^2}{2\lambda}\int_0^\lambda \sin^2(\omega t - kx)dx$$

Identity:

$$\int_0^\lambda \sin^2(\omega t - kx)dx = \lambda/2$$

$$KE = A^2\omega^2\rho/4$$

Potential energy

Potential energy:

strain:

$$e = \frac{(dx^2 + du^2)^{1/2} - dx}{dx} = \left[1 + \left(\frac{du}{dx}\right)^2\right]^{1/2} - 1 = \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2$$

(using the Taylor series approximation $(1 + a^2)^{1/2} \approx 1 + a^2/2$ for small a)

$$PE = \int_0^L e\tau\, dx = \frac{\tau}{2}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx$$

$$PE = \frac{\tau}{2\lambda}\int_0^\lambda \left(\frac{\partial u}{\partial x}\right)^2 dx = \frac{\tau\ A^2\ k^2}{2\lambda}\int_0^\lambda \sin^2(\omega t - kx)dx$$

$$PE = \tau A^2 k^2/4 = A^2\omega^2\rho/4$$

Total energy

$$KE = A^2 \omega^2 \rho/4$$

$$PE = A^2 \omega^2 \rho/4$$

Total energy:

$$E = PE + KE = A^2 \omega^2 \rho/2$$

Energy flux:

$$\dot{\mathbf{E}} = A^2 \omega^2 \rho \mathrm{v}/2$$

$$\dot{\mathbf{E}}_R + \dot{\mathbf{E}}_T = R_{12}^2 \omega^2 \rho_1 \mathrm{v}_1/2 + T_{12}^2 \omega^2 \rho_2 \mathrm{v}_2/2$$

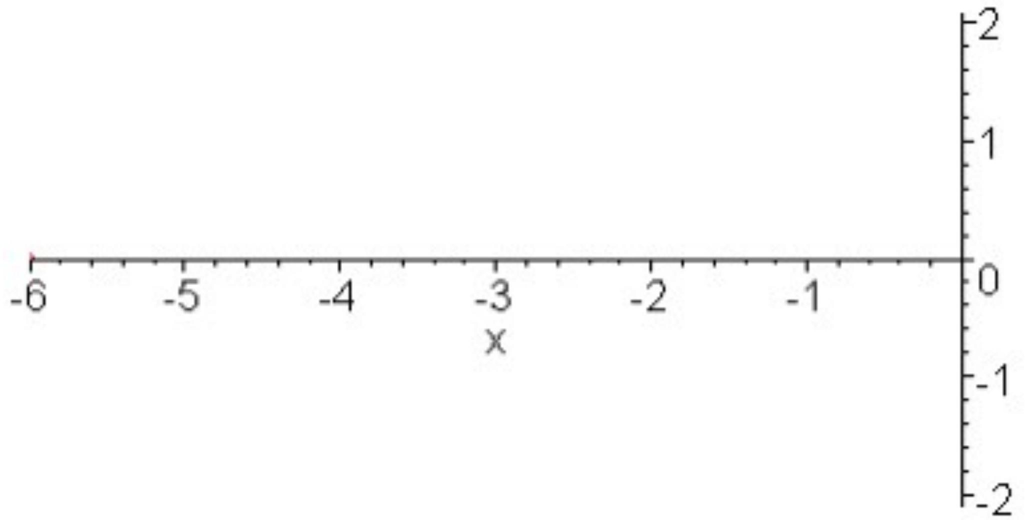
$$= (\omega^2/2)\,[R_{12}^2 \mathrm{v}_1 \rho_1 + T_{12}^2 \mathrm{v}_2 \rho_2] = \omega^2 \rho_1 \mathrm{v}_1/2 = \dot{\mathbf{E}}_I$$

Standing waves

If a string is clamped at both ends, waves will be reflected from the fixed ends and a standing wave will be set up.

The incident and reflected waves will combine according to the principle of superposition

Essential in music and quantum theory !



Wavefunction for a standing wave

Consider two sinusoidal waves in the same medium with the same amplitude, frequency and wavelength but travelling in opposite directions

$$y_1 = A_0 \sin(kx - \omega t) \qquad \longrightarrow$$

$$y_2 = A_0 \sin(kx + \omega t) \qquad \longleftarrow$$

$$y = A_0 \left[\sin(kx - \omega t) + \sin(kx + \omega t)\right]$$

Using the identity $\sin A + \sin B = 2\cos\left(\frac{A-B}{2}\right)\sin\left(\frac{A+B}{2}\right)$

$$\boxed{y = 2A_0 \sin(kx)\cos(\omega t)}$$

This is the wavefunction of a standing wave

Standing waves

$$y = 2A_0 \sin(kx) \cos(\omega t)$$

Amplitude = $2A_0 \sin(kx)$

Angular frequency = ω

Every particle on the string vibrates in SHM with the same frequency.

The amplitude of a given particle depends on x

Compare this to travelling harmonic wave where all particles oscillate with the same amplitude and at the same frequency

Antinodes

$$y = 2A_o \sin(kx)\cos(\omega t)$$

At any x maximum amplitude ($2A_o$) occurs when sin(kx) = 1

or when $$kx = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \ldots\ldots$$

but $k = 2\pi / \lambda$ and positions of maximum amplitude occur at

$$x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4} \ldots\ldots = \frac{n\lambda}{4} \quad \text{with } n = 1, 3, 5, \ldots.$$

Positions of maximum amplitude are **ANTINODES** and are separated by a distance of $\lambda/2$.

Nodes

$$y = 2A_0 \sin(kx)\cos(\omega t)$$

Similarly zero amplitude occurs when sin(kx) = 0

or when $kx = \pi,\ 2\pi,\ 3\pi.......$

$$x = \frac{\lambda}{2}, \frac{2\lambda}{2}, \frac{3\lambda}{2}....... = \frac{n\lambda}{2} \quad \text{with } n = 1, 2, 3, ...$$

Positions of zero amplitude are **NODES** and are also separated by a distance of $\lambda/2$.

The distance between a node and an antinode is $\lambda/4$