Copyright © 2004 WILEY-VCH Verlag GmbH & Co. KGaA

2

BASIC THEORY OF RADIATION FIELDS

2.1 REVIEW OF MAXWELL'S EQUATIONS

We open our study of electromagnetic phenomena by a review of the theory applied to nonrelativistic particles. Gaussian units are used throughout.

The operational definitions of the electric field $\mathbf{E}(\mathbf{r},t)$ and the magnetic field $\mathbf{B}(\mathbf{r},t)$ are made through observations on a particle of charge q at point $\mathbf{r}$ with velocity $\mathbf{v}$, and by means of the formula for the *Lorentz force*:

$$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right). \tag{2.1}$$

The rate of work done by the fields on a particle is

$$\mathbf{v} \cdot \mathbf{F} = q\mathbf{v} \cdot \mathbf{E}, \tag{2.2a}$$

because $\mathbf{v} \cdot (\mathbf{v} \times \mathbf{B}) = 0$. Since $\mathbf{F} = m d\mathbf{v}/dt$ for nonrelativistic particles, we have

$$q\mathbf{v} \cdot \mathbf{E} = \frac{d}{dt}\left(\tfrac{1}{2} m\mathbf{v}^2\right). \tag{2.2b}$$

These results may be generalized to total force on a volume element containing many charges. The force per unit volume is

$$\mathbf{f}=\rho\mathbf{E}+\frac{1}{c}\mathbf{j}\times\mathbf{B}, \tag{2.3}$$

where

$$\rho=\lim_{\Delta V\to 0}\frac{1}{\Delta V}\sum_i q_i, \tag{2.4a}$$

$$\mathbf{j}=\lim_{\Delta V\to 0}\frac{1}{\Delta V}\sum_i q_i\mathbf{v}_i, \tag{2.4b}$$

and ΔV is the volume element. ρ and $\mathbf{j}$ are charge and current densities, respectively. In Eqs. (2.3) and (2.4) ΔV must be chosen much smaller than characteristic scales but much larger than the volume containing a single particle.

The rate of work done by the field per unit volume is then

$$\frac{1}{\Delta V}\sum_i q_i\mathbf{v}_i\cdot\mathbf{E}=\mathbf{j}\cdot\mathbf{E}.$$

From Eq. (2.2b) this is also the rate of change of mechanical energy per unit volume due to the fields:

$$\frac{dU_{\text{mech}}}{dt}=\mathbf{j}\cdot\mathbf{E}. \tag{2.5}$$

Maxwell's equations relate $\mathbf{E}$, $\mathbf{B}$, ρ, and $\mathbf{j}$. In Gaussian units, they are

$$\begin{aligned}\nabla\cdot\mathbf{D}&=4\pi\rho & \nabla\cdot\mathbf{B}&=0\\ \nabla\times\mathbf{E}&=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} & \nabla\times\mathbf{H}&=\frac{4\pi}{c}\mathbf{j}+\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}.\end{aligned} \tag{2.6}$$

Here the fields $\mathbf{D}$ and $\mathbf{H}$ can often be related to $\mathbf{E}$ and $\mathbf{B}$ by the linear relations

$$\mathbf{D}=\epsilon\mathbf{E}, \tag{2.7a}$$

$$\mathbf{B}=\mu\mathbf{H}, \tag{2.7b}$$

where ϵ and μ are the dielectric constant and magnetic permeability of the medium, respectively. In the absence of dielectric or permeable media, $\epsilon=\mu=1$.

An immediate consequence of Maxwell's equation is *conservation of charge*: Taking $\nabla\cdot$ of the $\nabla\times\mathbf{H}$ equation gives

$$\nabla\cdot\mathbf{j}+\frac{\partial\rho}{\partial t}=0. \tag{2.8}$$

This expresses conservation of charge for a volume element.

We now give definitions of energy density and energy flux of the electromagnetic field. Consider the work done per unit volume on a particle distribution, [cf. Eq. (2.6)]:

$$\mathbf{j}\cdot\mathbf{E}=\frac{1}{4\pi}\left[c(\nabla\times\mathbf{H})\cdot\mathbf{E}-\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}\right], \tag{2.9}$$

where we have used Maxwell's equations. Now, use the vector identity:

$$\mathbf{E}\cdot(\nabla\times\mathbf{H})=\mathbf{H}\cdot(\nabla\times\mathbf{E})-\nabla\cdot(\mathbf{E}\times\mathbf{H}),$$

and again use Maxwell's equations to write Eq. (2.9) in the form

$$\mathbf{j}\cdot\mathbf{E}=\frac{1}{4\pi}\left[-\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}-c\nabla\cdot(\mathbf{E}\times\mathbf{H})-\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}\right]. \tag{2.10a}$$

Now, if ϵ and μ are independent of time, then the above relation may be written as [cf. Eq. (2.7)]

$$\mathbf{j}\cdot\mathbf{E}+\frac{1}{8\pi}\frac{\partial}{\partial t}\left(\epsilon E^2+\frac{B^2}{\mu}\right)=-\nabla\cdot\left(\frac{c}{4\pi}\mathbf{E}\times\mathbf{H}\right). \tag{2.10b}$$

Equation (2.10b) is *Poynting's theorem* in differential form and can be interpreted as saying that the rate of change of mechanical energy per unit volume plus the rate of change of field energy per unit volume equals minus the divergence of the field energy flux. Accordingly, we set the electromagnetic field energy per unit volume equal to

$$U_{\text{field}}=\frac{1}{8\pi}\left(\epsilon E^2+\frac{B^2}{\mu}\right)=U_E+U_B, \tag{2.11}$$

and the electromagnetic flux vector, or *Poynting vector*, equal to

$$\mathbf{S}=\frac{c}{4\pi}\mathbf{E}\times\mathbf{H}. \tag{2.12}$$

The above can also be understood by integrating over a volume element

and using the divergence theorem:

$$\int_V \mathbf{j}\cdot\mathbf{E}\,dV + \frac{d}{dt}\int_V \frac{\epsilon E^2 + B^2/\mu}{8\pi}\,dV = -\int_\Sigma \mathbf{S}\cdot d\mathbf{A},$$

or

$$\frac{d}{dt}(U_{\text{mech}} + U_{\text{field}}) = -\int_\Sigma \mathbf{S}\cdot d\mathbf{A}. \tag{2.13}$$

That is, the rate of change of total (mechanical plus field) energy within the volume V is equal to the net inward flow of energy through the bounding surface Σ.

Although U_{field} is called a field energy, it has contributions from the matter, because ϵ and μ are both macroscopic properties of matter. We are, in effect, putting the energy of the bound charges into the field. If we had treated all charges (free and bound) as part of the mechanical system, then we would use only the *microscopic* fields **E** and **B**. Then **j** would be replaced by the sum of the conduction current and induced molecular currents and $\mathbf{S}\to(c/4\pi)\mathbf{E}\times\mathbf{B}$. When both matter and fields are present, the allocation of energy into matter and field energies is somewhat arbitrary. What is not arbitrary is that the total energy is conserved.

If we now consider either the microscopic energy flux in the field or the field in vacuum, and use Eq. (1.6) and the fact that $p = E/c$ for photons, then we can write the *momentum per unit volume in the field*, **g** as

$$\mathbf{g} = \frac{1}{4\pi c}\mathbf{E}\times\mathbf{B}. \tag{2.14}$$

The *angular momentum* carried by the field is given by $\mathfrak{L}$, the angular momentum density:

$$\mathfrak{L} = \mathbf{r}\times\mathbf{g}, \tag{2.15}$$

where **r** is the radius vector from the point about which the angular momentum is computed. We do not derive these results in general; however, this identification of momentum and angular momentum for electromagnetic radiation is verified in Problem 2.3.

Returning to the conservation of energy now, we can let the surface Σ approach infinity, and the question arises as to the limit of

$$\int_\Sigma \mathbf{S}\cdot d\mathbf{A}$$

In electrostatics and magnetostatics we recall that both **E** and **B** decrease like r^{-2} as $r\to\infty$. This implies that **S** decreases like r^{-4} in static problems. Thus the above integral goes to zero, since the surface area increases only as r^2. However, for time-varying fields we find that **E** and **B** may decrease only as r^{-1}. Therefore, the integral can contribute a finite amount to the rate of change of energy of the system. This finite energy flowing outward (or inward) at large distances is called *radiation*. Those parts of **E** and **B** that decrease as r^{-1} at large distances are said to constitute the *radiation field*.

2.2 PLANE ELECTROMAGNETIC WAVES

Maxwell's equations in vacuum become [cf. Eqs. (2.6)]

$$\begin{aligned} \nabla\cdot\mathbf{E}&=0 & \nabla\cdot\mathbf{B}&=0\\ \nabla\times\mathbf{E}&=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} & \nabla\times\mathbf{B}&=\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}. \end{aligned} \tag{2.16}$$

A basic feature of these equations is the existence of traveling wave solutions that carry energy. Taking the curl of the third equation and combining it with the fourth, we obtain

$$\nabla\times(\nabla\times\mathbf{E})=-\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}.$$

If we now use the vector identity

$$\nabla\times(\nabla\times\mathbf{E})=\nabla(\nabla\cdot\mathbf{E})-\nabla^2\mathbf{E}$$

(in Cartesian components) and the first equation, we obtain the *vector wave equation* for **E**:

$$\nabla^2\mathbf{E}-\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}=0. \tag{2.17}$$

An identical equation holds for **B**, since Eq. (2.16) is invariant under $\mathbf{E}\to\mathbf{B}$, $\mathbf{B}\to-\mathbf{E}$.

Let us now consider solutions of the form

$$\mathbf{E}=\hat{\mathbf{a}}_1E_0\,e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \tag{2.18a}$$

$$\mathbf{B}=\hat{\mathbf{a}}_2B_0\,e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \tag{2.18b}$$

where $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{a}}_2$ are unit vectors, E_0 and B_0 are complex constants, and $\mathbf{k}=k\mathbf{n}$ and ω are the "wave vector" and frequency, respectively. Clearly, such solutions represent waves traveling in the $\mathbf{n}$ direction, since surfaces of constant phase advance with time in the $\mathbf{n}$ direction. By superposing such solutions propagating in all directions and with all frequencies, we can construct the most general solution to the source-free Maxwell's equations. Substitution into Maxwell's equations yields:

$$\begin{aligned} i\mathbf{k}\cdot\hat{\mathbf{a}}_1 E_0 &= 0 & i\mathbf{k}\cdot\hat{\mathbf{a}}_2 B_0 &= 0 \\ i\mathbf{k}\times\hat{\mathbf{a}}_1 E_0 &= \frac{i\omega}{c}\hat{\mathbf{a}}_2 B_0 & i\mathbf{k}\times\hat{\mathbf{a}}_2 B_0 &= -\frac{i\omega}{c}\hat{\mathbf{a}}_1 E_0. \end{aligned} \tag{2.19}$$

The top two equations tell us that both $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{a}}_2$ are *transverse* (perpendicular) to the direction of propagation $\mathbf{k}$. With this information, the cross products in the bottom two equations can be done, and we see that $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{a}}_2$ are perpendicular to each other. The vectors $\hat{\mathbf{a}}_1$, $\hat{\mathbf{a}}_2$, and $\mathbf{k}$ form a right-hand triad of mutually perpendicular vectors. The values of E_0 and B_0 are related by

$$E_0=\frac{\omega}{kc}B_0, \qquad B_0=\frac{\omega}{kc}E_0,$$

so that

$$E_0=\left(\frac{\omega}{kc}\right)^2 E_0$$

and

$$\omega^2=c^2k^2. \tag{2.20a}$$

Taking k and ω positive, as implied by the above discussion, we have

$$\omega=ck. \tag{2.20b}$$

This in turn implies

$$E_0=B_0. \tag{2.21}$$

The waves propagate with a phase velocity that can be found from $v_{\text{ph}}=\omega/k$, so that

$$v_{\text{ph}}=c. \tag{2.22}$$

The waves, as expected, travel at the speed of light. (In a vacuum the group velocity, $v_g \equiv \partial\omega/\partial k$, equals c also.)

We can now compute the energy flux and energy density of these waves. Since **E** and **B** both vary sinusoidally in time, the Poynting vector and the energy density actually fluctuate; however, we take a *time average*, since this is in most cases what is measured.

Now, it can easily be shown (Problem 2.1) that if $A(t)$ and $B(t)$ are two complex quantities with the same sinusoidal time dependence, that is,

$$A(t) = \mathcal{A} e^{i\omega t} \qquad B(t) = \mathcal{B} e^{i\omega t},$$

then the time average of the product of their real parts is

$$\langle \mathrm{Re}\, A(t) \cdot \mathrm{Re}\, B(t) \rangle = \tfrac{1}{2}\, \mathrm{Re}(\mathcal{A}\mathcal{B}^*) = \tfrac{1}{2}\, \mathrm{Re}(\mathcal{A}^*\mathcal{B}). \tag{2.23}$$

We have used * to denote complex conjugation. Thus the time-averaged Poynting vector [cf. Eq. (2.12)] satisfies

$$\langle S \rangle = \frac{c}{8\pi} \mathrm{Re}(E_0 B_0^*). \tag{2.24a}$$

Since $E_0 = B_0$,

$$\langle S \rangle = \frac{c}{8\pi} |E_0|^2 = \frac{c}{8\pi} |B_0|^2. \tag{2.24b}$$

Similarly, the time-averaged energy density is [cf. Eq. (2.11)]

$$\langle U \rangle = \frac{1}{16\pi} \mathrm{Re}(E_0 E_0^* + B_0 B_0^*), \tag{2.25a}$$

or, with $E_0 = B_0$,

$$\langle U \rangle = \frac{1}{8\pi} |E_0|^2 = \frac{1}{8\pi} |B_0|^2. \tag{2.25b}$$

Therefore, the velocity of energy flow is $\langle S \rangle / \langle U \rangle = c$ also.

The above results have all been for propagation in a vacuum. Similar results hold, at least formally, if we use a dielectric constant and permeability that are constants. However, in practice these quantities usually depend on frequency, so a more careful approach is required. Some effects of refraction and dispersion are treated in Chapter 8.

2.3 THE RADIATION SPECTRUM

The spectrum of radiation depends on the *time variation* of the electric field (we can ignore the magnetic field, since it mimics the electric field). A consequence is that one cannot give a meaning to the spectrum of radiation at a precise instant of time, knowing only the electric field at one point. Instead, one must talk about the spectrum of a train of waves, or of the radiation at a point during a sufficiently long time interval Δt. If we have such a time record of the radiation field of length Δt, we still can only define the spectrum to within a frequency resolution $\Delta\omega$ where

$$\Delta\omega\Delta t > 1. \tag{2.26}$$

This uncertainty relation is not necessarily quantum in nature (although it can be proved from the energy-time uncertainty relation), but is a property of any wave theory of light.

Let us assume, for mathematical simplicity, that the radiation is in the form of a finite pulse. (In practice, we only require that $\mathbf{E}(t)$ vanishes sufficiently rapidly for $t \to \pm\infty$.) Also, let us treat only one of the two independent components of the transverse electric field, say $E(t) \equiv \hat{\mathbf{a}} \cdot \mathbf{E}(t)$. With these assumptions we may express $E(t)$ in terms of a Fourier integral (Fourier transform):

$$\hat{E}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E(t) e^{i\omega t} dt. \tag{2.27}$$

The inverse of this is

$$E(t) = \int_{-\infty}^{\infty} \hat{E}(\omega) e^{-i\omega t} d\omega. \tag{2.28}$$

The function $\hat{E}(\omega)$ is complex; however, since $E(t)$ is real we can write

$$\hat{E}(-\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E(t) e^{-i\omega t} dt = \hat{E}^*(\omega),$$

so that the negative frequencies can be eliminated.

Contained in $\hat{E}(\omega)$ is all the information about the frequency behavior of $E(t)$. To convert this into frequency information about the energy we write the energy per unit time per unit area in terms of the Poynting vector:

$$\frac{dW}{dt\, dA} = \frac{c}{4\pi} E^2(t). \tag{2.29}$$

The total energy per unit area in the pulse is

$$\frac{dW}{dA} = \frac{c}{4\pi}\int_{-\infty}^{\infty} E^2(t)dt. \tag{2.30}$$

But from Parseval's theorem for Fourier transforms, we know that

$$\int_{-\infty}^{\infty} E^2(t)dt = 2\pi\int_{-\infty}^{\infty} |\hat{E}(\omega)|^2 d\omega. \tag{2.31}$$

By the above symmetry property of $\hat{E}(\omega)$ we have

$$|\hat{E}(\omega)|^2 = |\hat{E}(-\omega)|^2,$$

so that

$$\int_{-\infty}^{\infty} E^2(t)dt = 4\pi\int_{0}^{\infty} |\hat{E}(\omega)|^2 d\omega.$$

Thus we have the result

$$\frac{dW}{dA} = c\int_{0}^{\infty} |\hat{E}(\omega)|^2 d\omega, \tag{2.32}$$

and we may identify the energy per unit area per unit frequency:

$$\frac{dW}{dA\,d\omega} = c|\hat{E}(\omega)|^2 \tag{2.33}$$

It should be noted that this is the total energy per area per frequency range in the *entire pulse*; we have not written "per unit time." In fact, to write both dt and $d\omega$ would violate the uncertainty relation between ω and t. However, if the pulse repeats on an average time scale T, then we may *formally* write

$$\frac{dW}{dA\,d\omega\,dt} \equiv \frac{1}{T}\frac{dW}{dA\,d\omega} = \frac{c}{T}|\hat{E}(\omega)|^2. \tag{2.34}$$

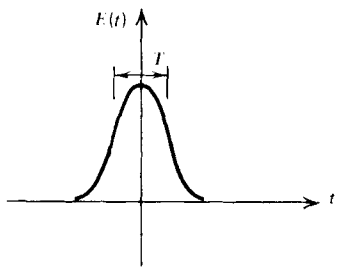
This formula also can be used to define the spectrum of a portion of length T of a much longer signal. If a very long signal has more or less the same properties over its entire length (property of *time stationarity*) then we expect that the result will be independent of T for large T, and we may write

$$\frac{dW}{dA\,d\omega\,dt} = c\lim_{T\to\infty}\frac{1}{T}|\hat{E}_T(\omega)|^2,$$

where we have written the subscript T on $\hat{E}_T(\omega)$ to emphasize that this is the transform of a portion of the function $E(t)$ of length T. In this way we can generalize our discussion to include infinitely long waves (such as sine waves) using formulas based on finite pulses.

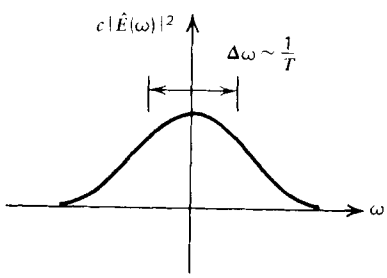
If the properties of $E(t)$ vary with time, then one expects that the spectrum as determined by analyzing a portion of length T will depend on just what portion is analyzed. In that case the whole efficacy of the concept of local spectrum depends on whether the changes of character of $E(t)$ occur on a time scale long enough that one can still define a length T in which a suitable frequency resolution $\Delta\omega \sim 1/T$ can be obtained. If this condition is not met, a local spectrum is not useful, and one must consider the spectrum of the entire pulse as the basic entity.

Let us consider now some typical pulse shapes and their corresponding spectra. (See Figs. 2.1, 2.2, and 2.3.) Study of these should give some



(*a*)

Figure 2.1a ***Electric field of a pulse of duration T.***



(*b*)

Figure 2.1b ***Power spectrum for a.***

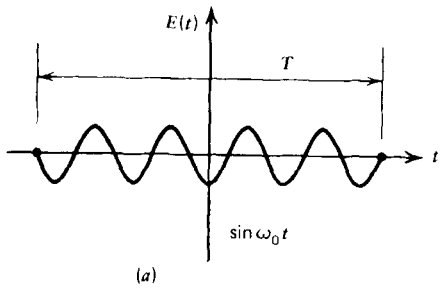


Figure 2.2a Electric field of a sinusoidal pulse of frequency ω_0 and duration T.

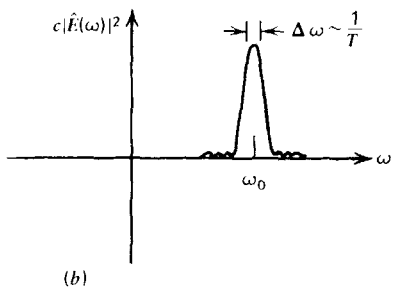


Figure 2.2b Power spectrum for a.

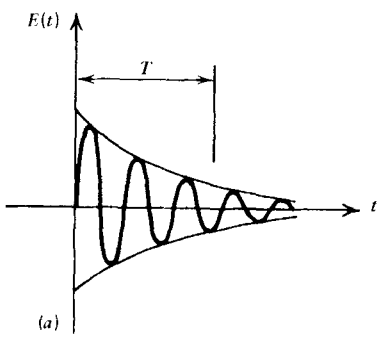


Figure 2.3a Electric field of a damped sinusoid of the form $\exp(-t/T)\sin\omega_0 t$.

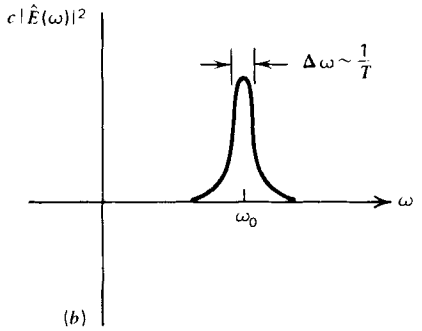


Figure 2.3b Power spectrum for a.

insight into the relationships that are useful in estimating spectra from particular processes. Note that the graphs of $c|\hat{E}(\omega)|^2$ are always symmetric about the origin—sometimes we have drawn the curves for both positive and negative ω for convenience, while in other cases we have only drawn them for positive ω. Only the values for positive ω need concern us.

Some general rules can be seen in these simple examples: First, the time extent of the pulse T determines the width of the finest features in the spectrum by means of $\Delta\omega \sim 1/T$. Second, the existence of a sinusoidal time dependence within the pulse shape causes the spectrum to be concentrated near $\omega \sim \omega_0$.

2.4 POLARIZATION AND STOKES PARAMETERS

Monochromatic Waves

The monochromatic plane waves described in Eq. (2.18) are *linearly polarized*; that is, the electric vector simply oscillates in the direction $\hat{\mathbf{a}}_1$, which, with the propagation direction, defines the *plane of polarization*. By superposing solutions corresponding to two such oscillations in perpendicular directions, we can construct the most general state of polarization for a wave of given $\mathbf{k}$ and ω. We need consider only the electric vector $\mathbf{E}$; the magnetic vector simply stays perpendicular to and has the same magnitude as $\mathbf{E}$. Let us examine the electric vector at an arbitrary point (say, $\mathbf{r}=0$) and choose axes x and y with corresponding unit vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ (see Fig. 2.4). The direction of the wave is out of the page, toward the observer. Then the electric vector is the real part of

$$\mathbf{E} = (\hat{\mathbf{x}}E_1 + \hat{\mathbf{y}}E_2)e^{-i\omega t} \equiv \mathbf{E}_0 e^{-i\omega t}. \tag{2.35}$$

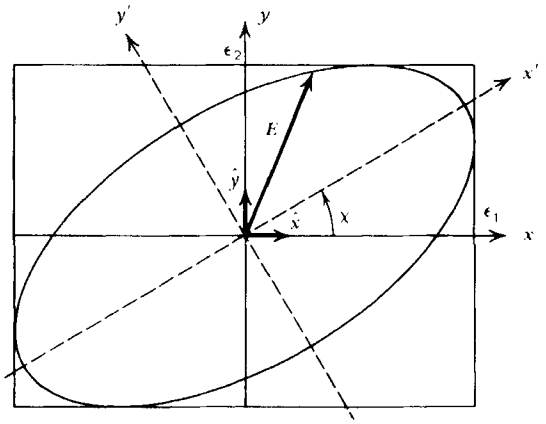


Figure 2.4 Rotation of x and y electric field components through angle χ to coincide with principal axes of the polarization ellipse.

This generalization of Eq. (2.18) can be characterized as having replaced $\hat{\mathbf{a}}_1 E_1$ by the general complex vector $\mathbf{E}_0$. The complex amplitudes E_1 and E_2 can be expressed as

$$E_1 = \mathcal{E}_1 e^{i\phi_1}, \qquad E_2 = \mathcal{E}_2 e^{i\phi_2}. \tag{2.36}$$

Taking the real part of **E**, we find the physical components of the electric field along $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ to be

$$E_x = \mathcal{E}_1 \cos(\omega t - \phi_1), \qquad E_y = \mathcal{E}_2 \cos(\omega t - \phi_2). \tag{2.37}$$

These equations describe the tip of the electric field vector in the x-y plane.

We now show that the figure traced out is an ellipse, and hence the general wave is said to be *elliptically polarized*. First of all, note that the equations for a general ellipse relative to its principal axes x' and y', which are tilted at an angle χ to the x- and y-axes (see Fig. 2.4), can be written

$$E'_x = \mathcal{E}_0 \cos\beta \cos\omega t, \qquad E'_y = -\mathcal{E}_0 \sin\beta \sin\omega t, \tag{2.38}$$

where $-\pi/2 \leqslant \beta \leqslant \pi/2$. The magnitudes of the principal axes are clearly $\mathcal{E}_0 |\cos\beta|$ and $\mathcal{E}_0 |\sin\beta|$, since $(E'_x/\mathcal{E}_0 \cos\beta)^2 + (E'_y/\mathcal{E}_0 \sin\beta)^2 = 1$.The ellipse will be traced out in a clockwise sense for $0 < \beta < \pi/2$ and counterclockwise sense for $-\pi/2 < \beta < 0$, as viewed by an observer toward whom the wave is propagating. These possibilities are called, respectively, *right-* and *left-handed* elliptical polarization. Other terms are, respectively, *negative* and *positive helicity*.

Two degenerate cases of elliptical polarization can occur: When $\beta = \pm \pi/4$ the ellipse becomes a circle, and the wave is said to be *circularly polarized*. When $\beta = 0$ or $\pm \pi/2$, the ellipse narrows to a straight line, and the wave is said to be *linearly polarized*. In this latter case the wave is neither right-handed nor left-handed.

Let us now make the connections between the quantities that appear in Eq. (2.37) and those defining the principal axes of the ellipse. To do this we transform the electric field components in Eq. (2.38) to the x- and y-axes by rotating through the angle χ (see Fig. 2.4). This yields

$$E_x = \mathcal{E}_0(\cos\beta\cos\chi\cos\omega t + \sin\beta\sin\chi\sin\omega t)$$
$$E_y = \mathcal{E}_0(\cos\beta\sin\chi\cos\omega t - \sin\beta\cos\chi\sin\omega t)$$

These are identical with Eq. (2-37) if we take

$$\mathcal{E}_1\cos\phi_1 = \mathcal{E}_0\cos\beta\cos\chi, \tag{2.39a}$$
$$\mathcal{E}_1\sin\phi_1 = \mathcal{E}_0\sin\beta\sin\chi, \tag{2.39b}$$
$$\mathcal{E}_2\cos\phi_2 = \mathcal{E}_0\cos\beta\sin\chi, \tag{2.39c}$$
$$\mathcal{E}_2\sin\phi_2 = -\mathcal{E}_0\sin\beta\cos\chi. \tag{2.39d}$$

Given $\mathcal{E}_1$, ϕ_1, $\mathcal{E}_2$, ϕ_2 these equations can be solved for $\mathcal{E}_0$, β, and χ. A convenient way of doing this is by means of the *Stokes parameters for monochromatic waves*, which are defined by the equations:

$$I \equiv \mathcal{E}_1^2 + \mathcal{E}_2^2 = \mathcal{E}_0^2 \tag{2.40a}$$
$$Q \equiv \mathcal{E}_1^2 - \mathcal{E}_2^2 = \mathcal{E}_0^2\cos 2\beta\cos 2\chi \tag{2.40b}$$
$$U \equiv 2\mathcal{E}_1\mathcal{E}_2\cos(\phi_1 - \phi_2) = \mathcal{E}_0^2\cos 2\beta\sin 2\chi \tag{2.40c}$$
$$V \equiv 2\mathcal{E}_1\mathcal{E}_2\sin(\phi_1 - \phi_2) = \mathcal{E}_0^2\sin 2\beta. \tag{2.40d}$$

The alternate forms follow from manipulations of Eqs. (2.39). Thus we have

$$\mathcal{E}_0 = \sqrt{I} \tag{2.41a}$$
$$\sin 2\beta = \frac{V}{I} \tag{2.41b}$$
$$\tan 2\chi = \frac{U}{Q}. \tag{2.41c}$$

Pure elliptical polarization is determined solely by three parameters: $\mathcal{E}_0$, β, and χ. Therefore, one expects a relation to exist between the four Stokes parameters in this case; in fact, we have

$$I^2 = Q^2 + U^2 + V^2 \tag{2.42}$$

for a monochromatic wave (pure elliptical polarization.)

The meanings of the Stokes parameters are as follows: I is nonnegative and is proportional to the total energy flux or intensity of the wave. In practice, it is customary to choose a single proportionality factor in all of the definitions of (2.40) so that I is precisely the flux or intensity, but we shall omit it here. V is the *circularity* parameter that measures the ratio of principal axes of the ellipse. The wave has right- or left-handed polarization when V is positive or negative, respectively; $V=0$ is the condition for linear polarization. There is only one remaining independent parameter, Q or U, which measures the orientation of the ellipse relative to the x-axis; $Q=U=0$ is the condition for circular polarization.

Quasi-monochromatic Waves

The monochromatic waves just treated are said to be *completely* or *100%* polarized, since the electric vector displays a simple, nonrandom directional behavior in time. However, in practice we never see a single monochromatic component but rather a superposition of many components, each with its own polarization. An important case of interest occurs when the amplitudes and phases of the wave possess a relatively slow time variation, so that instead of Eq. (2.36) we have

$$E_1(t) = \mathcal{E}_1(t)e^{i\phi_1(t)}, \qquad E_2(t) = \mathcal{E}_2(t)e^{i\phi_2(t)} \tag{2.43}$$

To be precise, we assume that over short times, of order $1/\omega$, the wave looks completely polarized with a definite state of elliptical polarization, but over much longer times, $\Delta t \gg 1/\omega$, characterizing the times over which $\mathcal{E}_1$, $\mathcal{E}_2$, ϕ_1 and ϕ_2 change substantially, this state of polarization can change completely. Such a wave is no longer monochromatic; by the uncertainty relation its frequency spread $\Delta\omega$ about the value ω can be estimated as $\Delta\omega > 1/\Delta t$ so that $\Delta\omega \ll \omega$. For this reason the wave is called *quasi-monochromatic*. The frequency spread $\Delta\omega$ is called the *bandwidth* of the wave, and the time Δt is called the *coherence time*.

The quantitative characterization of quasi-monochromatic waves depends on what kind of measurements can be made. In principle, for strong

waves the precise time variations of the quantities $\mathcal{E}_1$, $\mathcal{E}_2$, ϕ_1, and ϕ_2 could be measured; this would be the most detailed characterization possible. On the other hand, most measurements are not so detailed and usually involve some apparatus in which the radiation eventually falls on a detector that measures the time-averaged square of the electric field, for example, the energy flux (2.24b). Before falling on the detector the radiation may pass through a variety of devices that have the effect of forming a linear combination of the two independent electric field components with arbitrary weights and phases. For radio waves such devices include dipole antennas and electric delay lines; the optical equivalents are found in polarizing filters and quarter-wave plates.

If we suppose that any time delays involved are short compared to the coherence time of the wave, then we can show that the outcome of a measurement with such a device depends on simple extensions of the Stokes parameters previously introduced.

We first note that the most general linear transformation of field components by devices of the type described above can be written

$$\begin{aligned} E_1' &= \lambda_{11}E_1 + \lambda_{12}E_2 \\ E_2' &= \lambda_{21}E_1 + \lambda_{22}E_2, \end{aligned} \tag{2.44}$$

where λ_{ij}, $(i,j=1,2)$, are complex constants describing the measuring apparatus. What is measured is the average sum of the squares of the x' and y' components of electric field. The average of the square of the x' component is

$$\begin{aligned} 2\langle[\operatorname{Re}E_1'e^{-i\omega t}]^2\rangle = {} & |\lambda_{11}|^2\langle E_1E_1^*\rangle + \lambda_{11}\lambda_{12}^*\langle E_1E_2^*\rangle \\ & + \lambda_{12}\lambda_{11}^*\langle E_2E_1^*\rangle \\ & + |\lambda_{12}|^2\langle E_2E_2^*\rangle. \end{aligned} \tag{2.45}$$

Eq. (2.23) has been used to average over the "fast" variations in the field described by the $e^{-i\omega t}$ term. The brackets $\langle\ \rangle$ on the right-hand side then refer only to time averaging of the slowly varying combinations of $E_1(t)$ and $E_2(t)$. For example,

$$\langle E_1E_2^*\rangle = \frac{1}{T}\int_0^T E_1(t)E_2^*(t)dt, \tag{2.46}$$

where 0 to T is the time interval over which the measurement is made. The average square of the y component yields a result analogous to Eq. (2.45) with λ_{21} and λ_{22} replacing λ_{11} and λ_{12}, respectively.

It is clear from the above that the measurement depends on the radiation field only through the four complex quantities $\langle E_i(t)E_j^*(t)\rangle$, where i, $j=1, 2$. These in turn are equivalent to four real quantities, since $\langle E_1E_1^*\rangle$ and $\langle E_2E_2^*\rangle$ are real and $\langle E_1E_2^*\rangle$ and $\langle E_2E_1^*\rangle$ are complex conjugates. A common and convenient set of four real quantities used to express $\langle E_iE_j^*\rangle$ are the *Stokes parameter for quasi-monochromatic waves*,

$$I\equiv\langle E_1E_1^*\rangle+\langle E_2E_2^*\rangle=\langle\mathcal{E}_1^2+\mathcal{E}_2^2\rangle \tag{2.47a}$$

$$Q\equiv\langle E_1E_1^*\rangle-\langle E_2E_2^*\rangle=\langle\mathcal{E}_1^2-\mathcal{E}_2^2\rangle \tag{2.47b}$$

$$U\equiv\langle E_1E_2^*\rangle+\langle E_2E_1^*\rangle=\langle 2\mathcal{E}_1\mathcal{E}_2\cos(\phi_1-\phi_2)\rangle \tag{2.47c}$$

$$V\equiv\frac{1}{i}(\langle E_1E_2^*\rangle-\langle E_2E_1^*\rangle)=\langle 2\mathcal{E}_1\mathcal{E}_2\sin(\phi_1-\phi_2)\rangle, \tag{2.47d}$$

using Eqs. (2.36). We see that these definitions are generalizations of Eqs. (2.40), to which they reduce when $\mathcal{E}_1$, $\mathcal{E}_2$, ϕ_1, and ϕ_2 are time independent. The Stokes parameters are the most complete description of the radiation field, in the sense that two waves having the same parameters cannot be distinguished by any measurements using an apparatus of the type described above.

Equation (2.42) will not hold for arbitrary quasi-monochromatic waves. It is easy to show from the Schwartz inequality, that

$$\langle E_1E_1^*\rangle\langle E_2E_2^*\rangle\geqslant\langle E_1E_2^*\rangle\langle E_2E_1^*\rangle, \tag{2.48}$$

the equality sign holding only when the ratio of $E_1(t)$ to $E_2(t)$ is a complex constant, independent of time. This latter condition implies that the electric vector traces out an ellipse of fixed shape and fixed orientation and only its overall size changes slowly with time. Such a wave is completely equivalent to a pure elliptically polarized (a monochromatic) wave because their Stokes parameters are the same. Summarizing, we have from Eqs. (2.47) and (2.48) that

$$I^2\geqslant Q^2+U^2+V^2, \tag{2.49}$$

the equality holding for a completely elliptically polarized wave.

At the other extreme there is the completely *unpolarized* wave, where the phases between E_1 and E_2 maintain no permanent relation and where there is no preferred orientation in the x-y plane, so that $\langle\mathcal{E}_1^2\rangle=\langle\mathcal{E}_2^2\rangle$. In this case

$$Q=U=V=0, \tag{2.50a}$$

or

$$Q^2+U^2+V^2=0. \tag{2.50b}$$

An important property of the Stokes parameters is that they are additive for a superposition of independent waves. By independent we mean that there are no permanent phase relations between the various waves, and that over the relevant time scales the relative phases can be assumed to be randomly and uniformly distributed from 0 to 2π. For a superposition of different waves, each having its own $E_1^{(k)}$ and $E_2^{(k)}$, $k=1, 2, 3\cdots$, we have

$$E_1=\sum_k E_1^{(k)}, \qquad E_2=\sum_l E_2^{(l)} \tag{2.51}$$

so that

$$\langle E_i E_j^* \rangle = \sum_k \sum_l \langle E_1^{(k)} E_2^{(l)*} \rangle = \sum_k \langle E_1^{(k)} E_2^{(k)*} \rangle. \tag{2.52}$$

Because of the random phases only terms with $k=l$ survive the averaging, as indicated. It follows that

$$I=\sum I^{(k)} \tag{2.53a}$$

$$Q=\sum Q^{(k)} \tag{2.53b}$$

$$U=\sum U^{(k)} \tag{2.53c}$$

$$V=\sum V^{(k)}, \tag{2.53d}$$

proving the additivity.

By the superposition principle, an arbitrary set of Stokes parameters can be represented as

$$\begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix} = \begin{bmatrix} I-\sqrt{Q^2+U^2+V^2} \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \sqrt{Q^2+U^2+V^2} \\ Q \\ U \\ V \end{bmatrix}. \tag{2.54}$$

The first term on the right represents the Stokes parameters of a completely unpolarized wave of intensity $I-\sqrt{Q^2+U^2+V^2}$ and the second represents the Stokes parameters of a completely (elliptically) polarized wave of intensity $\sqrt{Q^2+U^2+V^2}$, since it satisfies Eq. (2.42). Therefore an arbitrary wave can be regarded as the independent superposition of a

completely polarized and a completely unpolarized wave. With this decomposition the meaning of the Stokes parameters for a quasi-monochromatic wave can be reduced to the meanings previously given for the completely polarized part plus that for the unpolarized part. Such a wave is therefore said to be *partially polarized.* The *degree of polarization* is defined in terms of this representation as the ratio of the intensity of the polarized part to the total intensity:

$$\Pi \equiv \frac{I_{\text{pol}}}{I} = \frac{\sqrt{Q^2+U^2+V^2}}{I}. \tag{2.55}$$

This is often given in terms of percentages.

A special case that appears frequently in applications is partial linear polarization, where $V=0$. Such radiation can be analyzed using a single linear polarizing filter (or dipole antenna), which picks out the component of the electric field in one direction. The measurement consists of rotating the filter until the maximum values of intensity are found. The maximum value I_{max} will occur when the filter is aligned with the plane of polarization (the x'-axis), and the minimum value will occur along in the direction perpendicular to it (the y'-axis). The unpolarized intensity only contributes one-half of its intensity to any given measurement, since the total is shared between any two perpendicular directions. Therefore, the maximum and minimum values of intensity are

$$I_{\text{max}} = \tfrac{1}{2} I_{\text{unpol}} + I_{\text{pol}}, \tag{2.56a}$$

$$I_{\text{min}} = \tfrac{1}{2} I_{\text{unpol}}, \tag{2.56b}$$

where $I_{\text{unpol}} = I - \sqrt{Q^2+U^2}$ and $I_{\text{pol}} = \sqrt{Q^2+U^2}$. From Eq. (2.55) we have, finally,

$$\Pi = \frac{I_{\text{max}} - I_{\text{min}}}{I_{\text{max}} + I_{\text{min}}}. \tag{2.57}$$

One should be cautioned that this formula applies only in cases in which the polarization is known to be of plane type. It will underestimate the true degree of polarization if circular or elliptical polarization is present.

2.5 ELECTROMAGNETIC POTENTIALS

Because of the form of Maxwell's equations, [cf. Eqs. (2.6)], especially the "internal equations," it is found that the **E** and **B** fields may be expressed

completely in terms of a *scalar potential* $\phi(r,t)$ and a *vector potential* $\mathbf{A}(r,t)$. There are several reasons for wanting to do this: One scalar plus one vector is simpler than two vectors. Also, the equations determining ϕ and $\mathbf{A}$ are quite a bit simpler than Maxwell's equations for $\mathbf{E}$ and $\mathbf{B}$. Finally, the relativistic formulation of electromagnetic theory is simpler in terms of the potentials than in terms of the electric and magnetic fields.

From Maxwell's equation $\nabla\cdot\mathbf{B}=0$ it follows that $\mathbf{B}$ may be expressed as the curl of some vector field $\mathbf{A}$:

$$\mathbf{B}=\nabla\times\mathbf{A}. \tag{2.58}$$

The $\nabla\times\mathbf{E}$ equation can be written

$$\nabla\times\left(\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right)=0. \tag{2.59}$$

It follows that $\mathbf{E}+\frac{1}{c}\partial\mathbf{A}/\partial t$ may be expressed as the gradient of some scalar field $-\phi$:

$$\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}=-\nabla\phi,$$

$$\mathbf{E}=-\nabla\phi-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}. \tag{2.60}$$

Two of Maxwell's equations have already been satisfied identically by virtue of the definitions of the potentials. The $\nabla\cdot\mathbf{E}$ equation can be written

$$\nabla^2\phi+\frac{1}{c}\frac{\partial}{\partial t}(\nabla\cdot\mathbf{A})=-4\pi\rho, \tag{2.61}$$

where we have used the *microscopic* form of Maxwell's equations ($\rho=\rho_{\text{free}}+\rho_{\text{bound}}$). Equation (2.61) may also be written in the form

$$\nabla^2\phi-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}+\frac{1}{c}\frac{\partial}{\partial t}\left(\nabla\cdot\mathbf{A}+\frac{1}{c}\frac{\partial\phi}{\partial t}\right)=-4\pi\rho. \tag{2.62}$$

The $\nabla\times\mathbf{H}$ equation can be written

$$\nabla\times(\nabla\times\mathbf{A})-\frac{1}{c}\frac{\partial}{\partial t}\left(-\nabla\phi-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right)=\frac{4\pi}{c}\mathbf{j} \tag{2.63}$$

With the vector identity $\nabla\times(\nabla\times\mathbf{A})=-\nabla^2\mathbf{A}+\nabla(\nabla\cdot\mathbf{A})$ this becomes

$$\nabla^2\mathbf{A}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}-\nabla\left(\nabla\cdot\mathbf{A}+\frac{1}{c}\frac{\partial\phi}{\partial t}\right)=-\frac{4\pi}{c}\mathbf{j}. \tag{2.64}$$

The potentials are not uniquely determined by the conditions imposed above. For example, the addition to **A** of the gradient of an arbitrary scalar function ψ will leave **B** unchanged:

$$\mathbf{A}\rightarrow\mathbf{A}+\nabla\psi, \qquad \mathbf{B}\rightarrow\mathbf{B}.$$

The electric field will also be unchanged if at the same time ϕ is changed by

$$\phi\rightarrow\phi-\frac{1}{c}\frac{\partial\psi}{\partial t}, \qquad \mathbf{E}\rightarrow\mathbf{E}.$$

These alterations of **A** and ϕ are called *Gauge transformations*. Their value for our purposes lies in the possibility of choosing potentials in such a way to simplify the above equations. Note that since we have one free function, we can satisfy one scalar constraint equation. The most important choice made is a gauge for which the *Lorentz condition* is satisfied

$$\nabla\cdot\mathbf{A}+\frac{1}{c}\frac{\partial\phi}{\partial t}=0. \tag{2.65}$$

The gauge corresponding to Eq. (2.65) is called the *Lorentz gauge*. With this gauge Eqs. (2.62) and (2.64) now become the following inhomogeneous equations:

$$\nabla^2\phi-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}=-4\pi\rho, \tag{2.66a}$$

$$\nabla^2\mathbf{A}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=-\frac{4\pi}{c}\mathbf{j}. \tag{2.66b}$$

The solutions to Eqs. (2.66) may be written (see, e.g., Jackson 1975) as integrals over the sources:

$$\phi(\mathbf{r},t)=\int\frac{[\rho]d^3\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}, \tag{2.67a}$$

$$\mathbf{A}(\mathbf{r},t)=\frac{1}{c}\int\frac{[\mathbf{j}]d^3\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}. \tag{2.67b}$$

Equations (2.67) are the *retarded potentials*. The notation $[Q]$ means that Q is to be evaluated at the *retarded time*

$$[Q]\equiv Q\left(\mathbf{r}',t-\frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right).$$

The retarded time refers to conditions at the point $\mathbf{r}'$ that existed at a time earlier than t by just the time required for light to travel between $\mathbf{r}$ and $\mathbf{r}'$. The interpretation is that information at point $\mathbf{r}'$ propagates at the speed of light, so that the potentials at point $\mathbf{r}$ can only be affected by conditions at point $\mathbf{r}'$ at such a retarded time. (A similar set of solutions with the advanced time $t+c^{-1}|\mathbf{r}-\mathbf{r}'|$ are also possible mathematically, but are ordinarily excluded on the physical grounds of causality.)

We now have a rather straightforward way of finding the electric and magnetic fields due to a given charge and current density: first, find the retarded potentials by means of the above integrals, and then determine $\mathbf{E}$ and $\mathbf{B}$ by their expressions in terms of the potentials. In the next chapter we determine the retarded potentials for a point charge in this way.

2.6 APPLICABILITY OF TRANSFER THEORY AND THE GEOMETRICAL OPTICS LIMIT

Following our discussion of waves, it is now possible to discuss more quantitatively the applicability of geometrical optics. In standard discussions of the propagation, or transfer, of radiation through matter, the specific intensity, with its associated concept of rays, is used as a fundamental variable. However, there are certain limitations imposed on transfer theory by the wave or quantum nature of light. For example, we defined specific intensity by the relation

$$dE = I_\nu \, dA \, d\Omega \, d\nu \, dt,$$

where dA, $d\Omega$, $d\nu$, and dt were presumed to be infinitesimal. However, dA and $d\Omega$ cannot both be made arbitrarily small because of the uncertainty principle for photons:

$$dx \, dp_x \, dy \, dp_y = p^2 \, dA \, d\Omega \gtrsim h^2,$$

$$dA \, d\Omega \gtrsim \lambda^2. \tag{2.68}$$

As soon as the size of dA is of order of the square of the wavelength, the direction cannot be defined with any precision and the concept of rays breaks down.

There is another limitation on the sizes of dt and $d\nu$ because of the energy uncertainty principle

$$dE \, dt \gtrsim h,$$

$$d\nu \, dt \gtrsim 1. \tag{2.69}$$

For these reasons, when the wavelength of light is larger than atomic dimensions, as in the optical, we cannot describe the interaction of light on the atomic scale in terms of specific intensity. However, we may still regard transfer theory as a valid macroscopic theory, provided the absorption and emission properties are correctly calculated from electromagnetic theory or quantum theory.

A more precise, classical treatment of the validity of rays is known as the *eikonal approximation*. The essential features of this approach can be seen if we treat a scalar field rather than the vector electromagnetic fields. Rays are curves whose tangents at each point lie along the direction of propagation of the wave. Clearly, these rays are well defined only if the amplitude and direction of the wave is practically constant over a distance of a wavelength λ. This limit is called the *geometrical optics limit*. Let the wave be represented by a function $g(\mathbf{r},t)$ of the form

$$g(\mathbf{r},t)=a(\mathbf{r},t)e^{i\psi(\mathbf{r},t)}, \tag{2.70}$$

where $a(\mathbf{r},t)$ is the slowly varying *amplitude* and $\psi(\mathbf{r},t)$ is the rapidly varying phase. If a were strictly constant, then the local direction of propagation $\mathbf{k}$ of the wave (normal to the surfaces of constant phase ψ), is given by

$$\mathbf{k}=\nabla\psi, \tag{2.71a}$$

and the local frequency, ω, is given by

$$\omega=-\frac{\partial\psi}{\partial t}. \tag{2.71b}$$

The exact behavior of a and ψ is constrained by the *wave* equation for $g(\mathbf{r},t)$,

$$\nabla^2 g(\mathbf{r},t)-\frac{1}{c^2}\frac{\partial^2 g}{\partial t^2}=0,$$

or, substituting in Eq. (2.70) for $g(\mathbf{r},t)$,

$$\nabla^2 a-\frac{1}{c^2}\frac{\partial^2 a}{\partial t^2}+ia\left(\nabla^2\psi-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}\right)+2i\left(\nabla a\cdot\nabla\psi-\frac{1}{c^2}\frac{\partial\psi}{\partial t}\frac{\partial a}{\partial t}\right)$$
$$-a(\nabla\psi)^2+\frac{a}{c^2}\left(\frac{\partial\psi}{\partial t}\right)^2=0. \tag{2.72}$$

The geometrical optics limit can now be made precise. If

$$\frac{1}{a}|\nabla a| \ll |\nabla \psi|, \qquad \frac{1}{a}\left|\frac{\partial a}{\partial t}\right| \ll \left|\frac{\partial \psi}{\partial t}\right|,$$

$$|\nabla^2 \psi| \ll |\nabla \psi|^2, \qquad \left|\frac{\partial^2 \psi}{\partial t^2}\right| \ll \left|\frac{\partial \psi}{\partial t}\right|^2,$$

$$\frac{1}{a}|\nabla^2 a| \ll |\nabla \psi|^2,$$

then the above equation reduces to

$$(\nabla \psi)^2 - \frac{1}{c^2}\left(\frac{\partial \psi}{\partial t}\right)^2 = 0, \tag{2.73}$$

which is the eikonal equation. If Eqs. (2.71) are substituted for the gradients of ψ, we obtain

$$|\mathbf{k}|^2 - \frac{\omega^2}{c^2} = 0,$$

which will be recognized [cf. Eq. (2.20a)] as the relationship between wave number and frequency of a plane wave.

PROBLEMS

2.1—Two oscillating quantities $A(t)$ and $B(t)$ are represented as the real parts of the complex quantities $\mathscr{A}e^{-i\omega t}$ and $\mathscr{B}e^{-i\omega t}$. Show that the time average of AB is given by

$$\langle AB \rangle = \tfrac{1}{2}\,\mathrm{Re}(\mathscr{A}^*\mathscr{B}) = \tfrac{1}{2}\,\mathrm{Re}(\mathscr{A}\mathscr{B}^*).$$

2.2—In certain cases the process of absorption of radiation can be treated by means of the macroscopic Maxwell equations. For example, suppose we have a conducting medium, so that the current density $\mathbf{j}$ is related to the electric field $\mathbf{E}$ by *Ohm's law*:

$$\mathbf{j} = \sigma \mathbf{E},$$

where σ is the conductivity (cgs unit = sec^{-1}). Investigate the propagation

of electromagnetic waves in such a medium and show that:

a. The wave vector $\mathbf{k}$ is complex

$$\mathbf{k}^2 = \frac{\omega^2 m^2}{c^2},$$

where m is the *complex index of refraction*, defined by

$$m^2 = \mu\epsilon\left(1 + \frac{4\pi i\sigma}{\omega\epsilon}\right).$$

b. The waves are attenuated as they propagate, corresponding to an absorption coefficient

$$\alpha_\nu = \frac{2\omega}{c} Im(m).$$

(Note: In some literature, minus signs appear in these formulas. This is because the wave is often taken to be $\exp(-i\mathbf{k}\cdot\mathbf{r} + i\omega t)$ rather than the $\exp(i\mathbf{k}\cdot\mathbf{r} - i\omega t)$ chosen here.)

2.3—This problem is meant to deduce the momentum and angular momentum properties of radiation and does not necessarily represent any real physical system of interest. Consider a charge Q in a viscous medium where the viscous force is proportional to velocity: $\mathbf{F}_{visc} = -\beta\mathbf{v}$. Suppose a *circularly polarized* wave passes through the medium. The equation of motion of the change is

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F}_{visc} + \mathbf{F}_{Lorentz}.$$

We assume that the terms on the right dominate the inertial term on the left, so that approximately

$$0 = \mathbf{F}_{visc} + \mathbf{F}_{Lorentz}.$$

Let the frequency of the wave be ω and the strength of the electric field be E.

a. Show that to lowest order (neglecting the magnetic force) the charge moves on a circle in a plane normal to the direction of propagation of the wave with speed QE/β and with radius $QE/\beta\omega$.

b. Show that the power transmitted to the fluid by the wave is Q^2E^2/β.

c. By considering the small magnetic force acting on the particle show that the momentum per unit time (force) given to the fluid by the wave is in the direction of propagation and has the magnitude $Q^2E^2/\beta c$.

d. Show that the angular momentum per unit time (torque) given to the fluid by the wave is in the direction of propagation and has magnitude $\pm Q^2E^2/\beta\omega$, where $(\pm)$ is for $\begin{pmatrix}\text{left}\\ \text{right}\end{pmatrix}$ circular polarization.

e. Show that the absorption cross section of the charge is $4\pi Q^2/\beta c$.

f. If we now regard the radiation to be composed of circularly polarized photons of energy $E_\gamma = \hbar\omega$, show that these results imply that the photon has momentum $p = \hbar k = h/\lambda = E_\gamma/c$ and has angular momentum $J = \pm\hbar$ along the direction of propagation.

g. Repeat this problem with appropriate modifications for a linearly polarized wave.

2.4—Show that Maxwell's equations before Maxwell, that is, without the "displacement current" term $c^{-1}\partial\mathbf{D}/\partial t$, unacceptably constrained the sources of the field and also did not permit the existence of waves.

REFERENCES

Born, M. and Wolf, E., 1964, *Principles of Optics* (Macmillan, New York).
Chandrasekhar, S., 1960, *Radiative Transfer*, (Dover, New York).
Jackson, J. D., 1975, *Classical Electrodynamics*, (J. Wiley, New York).