BT1 4.2 THE JEANS EQUATION 1

Boltzmann eq. → taking moments → Jeans eq.

B. eq. 4.13c $\frac{\partial f}{\partial t} + \bar{v}\,\nabla f - \nabla\phi\,\frac{\partial f}{\partial \bar{v}} = 0$

4.13b $\underbrace{\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i}}_{\Sigma_i} - \underbrace{\frac{\partial \phi}{\partial x_i}\frac{\partial f}{\partial v_i}}_{\Sigma_i} = 0$

$\int 4.13b \; d^3\bar{v}$ $\bar{x}$ e $\bar{v}$ are indep.

$$\int \frac{\partial f}{\partial t} d^3\bar{v} + \int v_i \frac{\partial f}{\partial x_i} d^3\bar{v} - \int \frac{\partial \phi}{\partial x_i}\frac{\partial f}{\partial v_i} d^3\bar{v} = 0 \qquad 4.19$$

range of velocities are indep. of time

is indep. of $\bar{v}$

$$\frac{\partial}{\partial t}\underbrace{\int f\, d^3\bar{v}}_{\equiv \nu} + \frac{\partial}{\partial x_i}\underbrace{\int v_i f\, d^3\bar{v}}_{\nu \bar{v}_i} - \frac{\partial \phi}{\partial x_i}\underbrace{\cancel{\int \frac{\partial f}{\partial v_i} d^3\bar{v}}}_{\text{divergence theorem}} = 0$$

$\int_S f\, d^2S = 0$

S → surface on the space of velocities with a radius $v \to \infty$ on the surface no objects ⟹ $f = 0$

$$\frac{\partial \nu}{\partial t} + \frac{\partial(\nu \bar{v}_i)}{\partial x_i} = 0 \qquad \boxed{4.-21}$$

CONTINUITY EQ. (see Fluid IE-3)

$\nu \to$ density (3D)

NOTE

Divergence theor.

$$\int_V \bar{\nabla}\bar{F}\, d^3\bar{x} = \int_{x_{1a}}^{x_{1b}} dx_1 \int_{x_{2a}}^{x_{2b}} dx_2 \int_{x_{3a}}^{x_{3b}} dx_3 \left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}\right) =$$

$$= \int_{x_{2a}}^{x_{2b}} dx_2 \int_{x_{3a}}^{x_{3b}} \left[F_1(x_{1b}, x_2, x_3) - F_2(x_{1a}, x_2, x_3)\right] + \left[F_2(x_{1b}, x_2, x_3) - \ldots\right] + \left[F_3 \ldots\right]$$

$$= \int_S \vec{F}\hat{n}\, d^2S = \int_S \bar{F}\, d^2\bar{S}$$

EXTENDED DIV. THEO.

$$\int g \bar{\nabla}\bar{F} d^3\bar{x} = \int_S g \bar{F}\, d^2\bar{S} - \int \bar{F}(\bar{\nabla} g)\, d^3\bar{x}$$

int. x parti

4.13b $\times v_j \; d^3\bar{v}$

$$\int \frac{\partial f}{\partial t} v_j \, d^3\bar{v} + \int v_i v_j \frac{\partial f}{\partial x_i} d^3\bar{v} - \int \frac{\partial \phi}{\partial x_i} v_j \frac{\partial f}{\partial v_i} d^3\bar{v} = 0$$

$$\frac{\partial}{\partial t}\int f v_j \, d^3\bar{v} + \frac{\partial}{\partial x_i}\int v_i v_j \, d^3\bar{v} - \frac{\partial \phi}{\partial x_i} \underbrace{\int v_j \frac{\partial f}{\partial v_i} d^3\bar{v}}_{} = 0$$

$$\int v_j \frac{\partial f}{\partial v_i} d^3\bar{v} = \cancel{\int_S v_j f \, d^2 S} - \int \frac{\partial v_j}{\partial v_i} f \, d^3\bar{v} =$$

div. theo. (ext.)

$f = 0$ on the surface

$$= - \int \delta_{ij} f \, d^3\bar{v} = - \delta_{ij} \nu$$

$$\frac{\partial(\nu \bar{v}_j)}{\partial t} + \frac{\partial(\nu \overline{v_i v_j})}{\partial x_i} + \nu \frac{\partial \phi}{\partial x_j} = 0 \quad \boxed{4.24e}$$

used for the virial theorem

4.24e − 4.21 $\times \bar{v}_j$

$$-\left[\bar{v}_j \frac{\partial \nu}{\partial t} + \frac{\partial(\nu \bar{v}_i)}{\partial x_i} \bar{v}_j\right] = 0$$

$$\nu \frac{\partial \bar{v}_j}{\partial t} - \bar{v}_j \frac{\partial(\nu \bar{v}_i)}{\partial x_i} + \frac{\partial(\nu \overline{v_i v_j})}{\partial x_i} = -\nu \frac{\partial \phi}{\partial x_j} \qquad 4.25$$

$$\frac{\partial(\nu \overline{v_i v_j})}{\partial x_i} - \bar{v}_j \frac{\partial(\nu \bar{v}_i)}{\partial x_i} - \bar{v}_i \frac{\partial \bar{v}_j}{\partial x_i} + \nu \bar{v}_i \frac{\partial \bar{v}_j}{\partial x_i} =$$

$$= \frac{\partial(\nu \overline{v_i v_j})}{\partial x_i} - \frac{\partial(\nu \bar{v}_i \bar{v}_j)}{\partial x_i} + \nu \bar{v}_i \frac{\partial \bar{v}_j}{\partial x_i} = \frac{\partial(\nu \sigma_{ij}^2)}{\partial x_i} + \nu \bar{v}_i \frac{\partial \bar{v}_j}{\partial x_i}$$

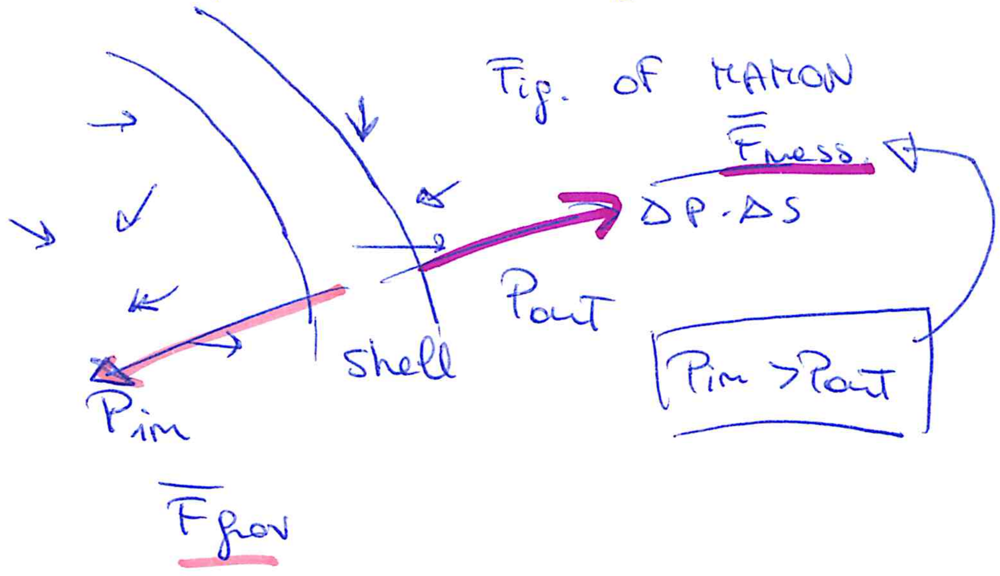
since $\sigma_{ij}^2 \equiv \overline{(v_i - \bar{v}_i)(v_j - \bar{v}_j)}$
$= \overline{v_i v_j} - \bar{v}_i \bar{v}_j$

$$\nu \frac{\partial \bar{v}_j}{\partial t} + \nu \bar{v}_i \frac{\partial \bar{v}_j}{\partial x_i} = -\nu \frac{\partial \phi}{\partial x_j} - \frac{\partial(\nu \sigma_{ij}^2)}{\partial x_i} \qquad 4.27$$

EQ. OF JEANS

EQ. OF JEANS is the equation of the local equilibrium (at each radius, e.g.)

Fig. of MAMON



MAMON → spherical coordinates

Steady state $\quad \dfrac{\partial -}{\partial t} = 0 \rightarrow \dfrac{\partial}{\partial r} = \dfrac{d}{dr}$

$$\frac{\partial(\nu \bar{v}_r)}{\partial t} + \frac{\partial(\nu \overline{v_r^2})}{\partial r} + \frac{\nu}{r}\left[2\overline{v_r^2} - (\overline{v_\theta^2} + \overline{v_\phi^2})\right] = -\nu \frac{\partial \phi}{\partial r}$$

$$\frac{d(\nu \overline{v_r^2})}{dr} + \frac{\nu}{r}\left[2\overline{v_r^2} - (\overline{v_\theta^2} + \overline{v_\phi^2})\right] = -\nu \frac{d\phi}{dr}$$

$\bar{v}_\theta = \bar{v}_\phi = 0$ $\quad$ $\boxed{\text{no rotations}}$ $\quad$ but still $\bar{v}_r \neq 0$ radial motions are possible!

$\overline{v_\theta^2} \rightarrow \sigma_\theta^2$ and by symmetry

$\sigma_\theta^2 = \sigma_\phi^2$ $\quad \left(\text{tangential } \sigma_t^2 = \dfrac{\sigma_\theta^2 + \sigma_\phi^2}{2}\right)$

$\bar{v}_r = 0$ $\quad$ moto medio $= \phi$

$\Rightarrow \overline{v_r^2} \rightarrow \sigma_r^2$

$\left(\beta = 1 - \dfrac{\sigma_t^2}{\sigma_r^2} \text{ anisotropy parameter}\right)$

$\beta =$
- 1 if $\sigma_t = 0$ i.e. radial orbits
- 0 $\sigma_t = \sigma_r$ isotropy
- $\rightarrow -\infty$ if $\sigma_r = 0$ circular orbits

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$$\frac{d(\nu \sigma_r^2)}{dr} + 2\beta \frac{\nu \sigma_r^2}{r} = -\nu \frac{d\phi}{dr}$$

simpliest version of 4.54e Jeans equation

Notice $\nu = \nu(r)$

$\sigma_r^2 = \sigma_r^2(r)$

$\beta = \beta(r)$

$\phi = \phi(r)$

Several models of β are used, e.g.

$$\beta(r) = \frac{r^2}{r^2 + r_a^2}$$ Osipkov - Merritt 1985

$$\beta(r) = \frac{1}{2} \frac{r}{r + r_a}$$ Mamon - Lokas 2005

For galaxy clusters: gals with isotropic orbits at the center ($r \ll r_a$) and radial externally ($r \gg r_a$)

if isotropic velocities (i.e. $\beta = 0$)

$$\frac{d(\nu \sigma_r^2)}{dr} = -\nu \frac{d\phi}{dr}$$

(cf. with eq. of hydrostatic equilibrium for ICM (hot gas) in clusters

$$\frac{1}{\nu}\left\{\frac{d(\nu \sigma_r^2)}{dr} + 2\frac{\beta \sigma_r^2}{dr}\right\} = -\frac{d\phi}{dr} = -\frac{GM}{r^2} \quad \boxed{4.54 \text{ BT1}}$$

used for β spec ($\nu \equiv n$) $\longrightarrow$

$$\frac{1}{\nu}\left(\nu\frac{d^2\sigma_r^2}{dr}+\sigma_r^2\frac{d\nu}{dr}\right)+2\beta\frac{\sigma_r^2}{r}=-\frac{GM}{r^2}$$

$$\frac{GM}{r}=-\sigma_r^2\left(\frac{r}{\nu}\frac{d\nu}{dr}+\frac{r}{\sigma_r^2}\frac{d\sigma_r^2}{dr}+2\beta\right)=$$

$$=-\sigma_r^2\left(\frac{d\ln\nu}{d\ln r}+\frac{d\ln\sigma_r^2}{d\ln r}+2\beta\right)$$

$$M=-\frac{r\sigma_r^2}{G}\left(\frac{d\ln\nu}{d\ln r}+\frac{d\ln\sigma_r^2}{d\ln r}+2\beta\right) \qquad 4.56$$

PRO

1) close to observables ($\int d^3v$ is better!)
2) test particles ~~⇏~~ ϕ or M given by other
   → gals, ↓ DM ← e.g.
3) different types of particles

CONTRA

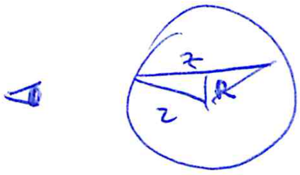
1) solution of J. eq. might be not solution of B. eq.
2) in J. eq. there are (we need) profiles of $\sigma_r(r)$ and $\nu(r)$
   difficult to compute (many data!)
   virial theorem (has not problem, it uses global velocity dispersion!)
3) we can observe only projected quantities!
4) 2 eqs. but 3 variables
→

2 eq.s Jeans + eq. of projected profile

$$M(r) = -\frac{r\,\sigma_r^2}{G}\left(\frac{d\ln\nu}{d\ln r} + \frac{d\ln\sigma_r^2}{d\ln r} + 2\beta\right)$$

variables M, ν, β, σ_r → 3 variables

ν → if spherical symmetry recovered via Abel integral



$z = \sqrt{r^2 - R^2}$

$$\Sigma(R) = \int_{-\infty}^{\infty} \cancel{\frac{\nu(r)\,dr}{\sqrt{r^2+R^2}}}\ \nu(r)\,dz =$$

Σ ↓ projected density

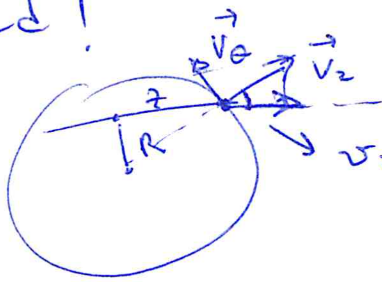
R ↓ projected radius

$$= 2\int_0^{\infty} \nu(r)\,dz = 2\int_R^{\infty} \frac{\nu\, r\, dr}{\sqrt{r^2-R^2}}$$

$$\nu(r) = -\frac{1}{\pi}\int_r^{\infty} \frac{d\Sigma}{2R\,dR}\,\frac{2R\,dR}{\sqrt{R^2-r^2}} - \lim_{R\to\infty}\frac{\Sigma(R)}{\sqrt{R^2-r^2}}$$

→ can be recovered!

Fig 4.4 BT1



observed

$v_r \cos\alpha - v_\theta \operatorname{sen}\alpha = v_{LOS}$

eq. of projected profile

$$\Sigma(R)\,\sigma_{LOS}^2(R) = 2\int_R^{\infty}\left(1 - \beta(r)\frac{R^2}{r^2}\right)\frac{\nu(r)\,\sigma_r^2(r)}{\sqrt{r^2-R^2}}\, r\, dr$$

2eq.s but 3 variables

OR $\beta = 0$ better for stars in ellipticals

OR $\nu(r) \propto \rho(r)$ better for clusters

ρ → mass density

ν → number density of galaxies in clusters

MASS FOLLOWS LIGHT ASSUMPTION (Good see. ep. Slides)

$$\Sigma(R) = \int_{-\infty} \nu(z)\,dz$$

$$= 2\int_R^\infty \frac{\nu(z)\, z\, dz}{\sqrt{z^2-R^2}}$$

$$\Sigma(R) = 2\int_R^\infty \frac{\nu z\, dz}{\sqrt{z^2-R^2}}$$

int. di Abel

$0<\alpha<1$

$$f(x) = \int_0^x \frac{g(t)}{(x-t)^\alpha}\,dt$$

$$g(t) = \frac{\operatorname{sen}\pi\alpha}{\pi}\frac{d}{dt}\int_0^t \frac{f(x)}{(t-x)^{1-\alpha}}\,dx$$

$$= \frac{\operatorname{sen}\pi\alpha}{\pi}\left[\int_0^t \frac{df}{dx}\frac{dx}{(t-x)^{1-\alpha}} + \frac{f(0)}{t^{1-\alpha}}\right]$$

$$f(x) = \int_x^\infty \frac{g(t)\,dt}{(t-x)^\alpha} \qquad 0<\alpha<1$$

$$g(t) = -\frac{\operatorname{sen}\pi\alpha}{\pi}\frac{d}{dt}\int_t^\infty \frac{f(x)}{(x-t)^{1-\alpha}}\,dx$$

$$= -\frac{\operatorname{sen}\pi\alpha}{\pi}\left[\int_t^\infty \frac{df}{dx}\frac{dx}{(x-t)^{1-\alpha}} - \lim_{x\to\infty}\frac{f(x)}{(x-t)^{1-\alpha}}\right]$$

$\alpha = \frac{1}{2}$ $\quad x = R^2$ $\quad t = z^2$ $\quad f = \Sigma$ $\quad g = \nu$

$$\nu(z) = -\frac{1}{\pi}\left[\int_z^\infty \frac{d\Sigma}{2R\,dR}\cdot\frac{2R\,dR}{\sqrt{(R^2-z^2)}} - \cancel{\lim_{R^2\to\infty}\frac{\Sigma(R)}{\sqrt{R^2-z^2}}}\right]$$

almeno R^{-2} $\to 0$

$\to\infty$ come R

nei sistemi reali

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