eq. di B. (7 variables!) → (2) Jeans eq. = properties

→ (1) solutions but not all! some examples

1

4.4 JEANS THEOREMS AND SPHERICAL SYSTEMS

integral motion of a given potential $\phi(\bar{x})$

cap 3. $I[\bar{x}(t_1), \bar{v}(t_1)] = I[(\bar{x}(t_2), \bar{v}(t_2))]$ along any orbit

$I[\bar{x}(t), \bar{v}(t)]$ is an integral of motion

if $\frac{d}{dt}[I[\bar{x}(t), \bar{v}(t)] = 0$ along any orbit

$$\frac{dI}{dt} = \bar{\nabla} I \cdot \frac{d\bar{x}}{dt} + \frac{\partial I}{\partial \bar{v}} \cdot \frac{d\bar{v}}{dt} \quad \text{or} \quad \bar{v}\,\bar{\nabla} I - \bar{\nabla}\phi \frac{\partial I}{\partial \bar{v}} = 0$$

cf. this with eq. of B. 4-13c $\cancel{\frac{\partial f}{\partial t}} + \bar{v}\,\bar{\nabla} f - \bar{\nabla}\phi \frac{\partial f}{\partial \bar{v}} = 0$

steady state solution f

→ **Jeans theorem**

f steady state solut. $f = f(I(\bar{x}, \bar{v}))$

function of I, NON NEGATIVE → is a f

strong Jeans theorem

potential is regular → system is $f(I_1, I_3, I_3)$

spherical potential $I \to E$, $\vec{L}$ (cap3)

energy, angular momentum

extension of the strong Jeans theorem, if a system is spherically symmetric in/all its

f spherical sys → $f = f(E, L)$

2/

stellar system provides potential ϕ
(self gravitating)

$$\nabla^2\phi = 4\pi G\rho \quad = 4\pi G \int f \, d^3\vec{v}$$

Poisson

4-101e

in spherical symmetry

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = 4\pi\rho = 4\pi G\int f\left(\tfrac{1}{2}v^2+\phi, |\vec{r}\times\vec{v}|\right) d^3\vec{v}$$

4-101b

Poisson eq.
fundamental eq.
governing spherical equilibrium
stellar systems

Relative potential Ψ e relative energy $\mathcal{E}$

$$\Psi \equiv -\phi + \phi_0 \qquad \mathcal{E} \equiv -E + \phi_0 = \overbrace{-\tfrac{1}{2}v^2 - \phi + \phi_0}\; = \Psi - \frac{1}{2}v^2$$

ϕ_0 is chosen $\rightarrow$ $f > 0$ for $\mathcal{E} > 0$
$f = 0$ for $\mathcal{E} \leq 0$

$\nabla^2\Psi = -4\pi G\rho$ with boundary conditions
$\Psi \to \phi_0$ as $|x| \to \infty$
that is $\phi \to 0$
as usual

SYSTEMS with isotropic dispersion tensors

$f = f(\mathcal{E})$

$$\overline{v_r^2} = \frac{1}{\rho}\int dv_r\, dv_\theta\, dv_\phi\, v_r^2 f\Big[\Psi - \tfrac{1}{2}\underbrace{(v_r^2 + v_\theta^2 + v_\phi^2)}_{v^2}\Big]$$

$$\overline{v_\theta^2} = \frac{1}{\rho}\int dv_r\, dv_\theta\, dv_\phi\, v_\theta^2 f\Big[\quad " \quad\Big]$$

are the same with a change of the variable of integration $\Longrightarrow$ $\overline{v_r^2} = \overline{v_\theta^2} = \overline{v_\theta^2}$

$\Longrightarrow$ isotropic tensor

$\Longrightarrow$ $f = f(\mathcal{E})$ $\Updownarrow$ isotropic $\overleftrightarrow{\sigma}_{\ell}^{2}$ tensor

[If not isotropic $f = f(\mathcal{E}, L)$]

4-101 b $\longrightarrow$ we chose Φ_0 in such way $f(\mathcal{E}) = 0$ for $\mathcal{E} < 0$ in 4.101 b

$$-\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right) = 4\pi G \underbrace{\int_0^{\sqrt{2\Psi}} f(\mathcal{E})\, 4\pi v^2 dv}_{\rho}$$

eq.

$$\begin{cases} f(\mathcal{E}) = 0 & \mathcal{E} < 0 \\ f(\mathcal{E}) > 0 & \mathcal{E} \geq 0 \Rightarrow \Psi - \frac{1}{2}v^2 \geq 0 \Rightarrow v \leq \sqrt{2\Psi} \end{cases}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right) = -16\pi^2 G \int_0^{\sqrt{2\Psi}} f(\underbrace{\Psi - \tfrac{1}{2}v^2}_{\mathcal{E}})\, v^2 dv$$

change of integration variable $v \to \mathcal{E}$

$$v^2 = 2(\Psi - \mathcal{E}) \qquad v^2 dv = \frac{-2(\Psi - \mathcal{E})\,\cancel{2}\, d\mathcal{E}}{\cancel{2}\sqrt{2(\Psi - \mathcal{E})}}$$

$$2v\,dv = -2\,d\mathcal{E}$$

$v \to 0$, $\mathcal{E} \to \Psi$

$v \to \sqrt{2\Psi}$, $\mathcal{E} \to 0$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right) = (+)\,16\pi^2 G\int_0^{\Psi} f(\mathcal{E})\sqrt{2(\Psi-\mathcal{E})}\,d\mathcal{E} \qquad \boxed{4-104}$$

$(-) \times (-) \times (-)$

V integrale rovesciato

$$= 4\pi G\rho = 4\pi G\cdot\underbrace{\left(4\pi\int_0^{\Psi}\ldots\right)}_{\rho}$$

$f \to \Psi \to \rho$ (1A)

$\rho \to \Psi \to f$ (1B)

METHOD (1A) $\quad f \longrightarrow \rho$

METHOD 1A $f \to \rho$

a) Polytropes and Plummer's model

$$f(\mathcal{E}) = \begin{cases} F\,\mathcal{E}^{n-\frac{3}{2}} & (\mathcal{E}>0) \\ 0 & (\mathcal{E}\leq 0) \end{cases}$$ 4-105

model for stellar polytropes $\qquad \mathcal{E} = \Psi - \frac{1}{2}v^2$

do $\boxed{\text{eq.} *}$

$$\rho = 4\pi \int_0^{\sqrt{2\Psi}} f\left(\Psi - \frac{1}{2}v^2\right) v^2 dv = 4\pi F \int_0^{\sqrt{2\Psi}} \left(\Psi - \frac{1}{2}v^2\right)^{n-\frac{3}{2}} v^2 dv$$

substitution $\underline{v^2 = 2\Psi\cos^2\theta}$ $\qquad \Psi$ is indep. of θ

~~$\rho = 4\pi F$~~ $\quad v = 0 \to \cos\theta = 0 \to \theta = \frac{\pi}{2}$

$v = \sqrt{2\Psi} \to \cos\theta = 1 \to \theta = 0$

$2v\,dv = -4\Psi\cos\theta\,\text{sen}\theta\,d\theta = -4\Psi\,\text{sen}\theta\cos\theta\,d\theta$

$v^2 dv = -\underbrace{2\Psi\,\text{sen}\theta\cos\theta\,d\theta}_{v\,dv} \cdot \underbrace{\sqrt{2\Psi}\cos\theta}_{v} = -2\sqrt{2}\,\Psi^{3/2}\,\text{sen}\theta \cdot \cos^2\theta\,d\theta$

$$\rho = 4\pi F \int_0^{\pi/2} (-1) \underbrace{(\Psi - \Psi\cos^2\theta)}_{\Psi(1-\cos^2\theta)}{}^{n-3/2} (-2\sqrt{2})\Psi^{3/2}\,\text{sen}\theta\cos^2\theta\,d\theta =$$

$$= 4\pi F \int_0^{\pi/2} \Psi^n (\text{sen}\theta)^{2n-2} \cos^2\theta\,(+2\sqrt{2})\,d\theta =$$

$$= 8\pi F\left[\int_0^{\pi/2} \Psi^n (\text{sen}\,\theta)^{2n-2} - \int_0^{\pi/2} \Psi^n (\text{sen}\,\theta)^{2n} d\theta\right] =$$

$$= c_n \Psi^n$$

where $c_n = 8\sqrt{2}\,\pi F \cdot \left[\int_0^{\pi/2} (\text{sen}\,\theta)^{2n-2} d\theta + - \int (\text{sen}\,\theta)^{2n} d\theta\right]$ $\quad 2^{7/2}$

$$\boxed{\rho = c_n \Psi^n \qquad \Psi > 0}$$

$\boxed{4.107\,\text{e}}$

eq. of stellar polytrope

POLYTROPIC CASE

4.107e $\boxed{\rho = c_m \psi^m}$ $\quad \psi > 0$

$$C_m = \dots \frac{(2\pi)^{3/2}(m-\frac{3}{2})!\,F}{m!}$$

wolfram

$(m-\frac{3}{2})! \longrightarrow \Gamma(m-\frac{1}{2})$

$m! \longrightarrow \Gamma(m+1)$

$$\int_0^\infty x^{z-1} e^{-x}\,dx = \Gamma(z) = (z-1)!$$

c_m is finite if $\boxed{m > \frac{1}{2}}$

No polytropic stellar system is homogeneous since $\rho \propto \psi^0$ but $0 < \frac{1}{2}$

4.107e + From Poisson eq. 4-107b

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) + 4\pi G c_m \psi^m = 0$$

change variables $s \equiv \frac{r}{b}$ $\quad \varphi = \frac{\psi}{\psi_0}$

$$\frac{1}{s^2 b^2}\frac{d\left(s^2 b^2 \frac{\psi_0}{b}\frac{d\varphi}{ds}\right)}{b\,ds} + 4\pi G c_m (\psi_0 \varphi)^m = 0$$

$$\underbrace{\frac{\psi_0}{4\pi G \psi_0^{m} b^{2} c_m}}_{\equiv 1} \cdot \frac{1}{s^2}\frac{d}{ds}\left(s^2\frac{d\varphi}{ds}\right) + \varphi^m = 0$$

$$\equiv 1 \longrightarrow b = (4\pi G \psi_0^{m-1} c_m)^{-1/2}$$

Poisson $\longrightarrow$

$$\frac{1}{s^2}\frac{d}{ds}\left(s^2\frac{d\varphi}{ds}\right) = \begin{cases} -\varphi^m & \varphi > 0 \\ 0 & \varphi \le 0 \end{cases} \qquad \boxed{4.108c}$$

LANE-EMDEN EQ.

at $r = 0$ $\quad \varphi = 1$ by definition

$\frac{d\varphi}{ds} = 0$ no gravitat. force at the

Binney - Tremaine

FOR Eq. 4-107 b (p.223)

$$\rho = 4\pi F \int_0^{\sqrt{2\Psi}} \left(\Psi - \frac{1}{2} v^2 \right)^{n-3/2} v^2 \, dv$$

$v^2 = 2\Psi \cos^2\theta \qquad v=0 \longrightarrow \cos\theta = \emptyset \quad \theta = \frac{\pi}{2}$

$\qquad\qquad\qquad\qquad v = \sqrt{2\Psi} \longrightarrow \cos\theta = 1 \quad \theta = \emptyset$

$$2v\,dv = -2\Psi \underbrace{2\cos\theta \operatorname{sen}\theta \, d\theta}_{v\,dv} = -4\Psi \operatorname{sen}\theta \cos\theta \, d\theta$$

$\sqrt{2}$

$$v^2 dv = -2\Psi \operatorname{sen}\theta \cos\theta \, d\theta \cdot \sqrt{2\Psi} \cos\theta =$$

$$= -2\sqrt{2}\,\Psi^{3/2} \operatorname{sen}\theta \cos^2\theta \, d\theta$$

$\operatorname{sen}^{2n-3}$

$$\rho = 4\pi F \int_0^{\pi/2} (-1) \left[\Psi (1 - \cos^2\theta) \right]^{n-3/2} (-2\sqrt{2}) \Psi^{3/2} \cdot$$

$$\cdot \operatorname{sen}\theta \cos^2\theta \, d\theta =$$

$$= 4\pi F \int_0^{\pi/2} \Psi^n (\operatorname{sen}\theta)^{2n-2} \cos^2\theta \, (+2\sqrt{2}) \, d\theta =$$

$$= 8\sqrt{2}\pi F \left[\int_0^{\pi/2} \Psi^n (\operatorname{sen}\theta)^{2n-2} (1 - \operatorname{sen}^2\theta) \, d\theta \right] =$$

$$= 8\sqrt{2}\pi F \left[\int_0^{\pi/2} \Psi^n (\operatorname{sen}\theta)^{2n-2} d\theta - \int_0^{\pi/2} \Psi^n (\operatorname{sen}\theta)^{2n} d\theta \right] =$$

$$= C_n \Psi^n$$

$$C_n = \underbrace{8\sqrt{2}\pi}_{2^{7/2}} F \left[\int_0^{\pi/2} (\operatorname{sen}\theta)^{2n-2} d\theta - \int_0 (\operatorname{sen}\theta)^{2n} d\theta \right]$$

4-107 b (p.223)

6bis