PROVA SCRITTA DI SISTEMI DINAMICI
A.A. 2020/2021

10 settembre 2021

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|---|---|
| **Nome e Cognome:** | |
| **gruppo:** | Gruppo A |
| **esercizio:** | Esercizio 1 |

**Note:** Scrivere le risposte su un singolo foglio bianco usando penna nera. Non scrivere con inchiostro blu o a matita. Non consegnare fogli aggiuntivi. La chiarezza e precisione nelle risposte sarà oggetto di valutazione.

Dichiaro che le risposte a questo esercizio sono frutto del mio e solo del mio lavoro e che non mi sono consultato con altri.

Soluzione esercizio 2

**Domanda 2**

Siano $x(\cdot)$ e $z(\cdot)$ due **processi stazionari di rumore bianco gaussiano, indipendenti tra loro**, con

$$\mathrm{E}[x(t)] = 5.0\,,\ \sigma_x^2 = 0.4 \quad \mathrm{E}[z(t)] = 0.0\,,\ \sigma_z^2 = 6.25 \qquad \forall t$$

Si supponga di poter osservare la variabile

$$y(t) = 2.3\,x(t) + 4.2\,z(t)$$

e di aver osservato i valori seguenti:

$$y(1) = 4.815 \quad y(2) = 1.623$$

1. Facendo uso della **formula di Bayes**, determinare lo **stimatore ottimo** di $x(1)$ in base alla sola osservazione $y(1)$.

   Quanto vale la varianza della stima?

2. Sempre facendo uso della **formula di Bayes**, determinare lo **stimatore ottimo** di $x(2)$ in base ad entrambe le osservazioni $y(1)\,,\ y(2)\,$.

   Quanto vale la varianza della stima stavolta? E quanto vale l'innovazione di $y(2)$ rispetto all'osservazione $y(1)$ ?

## punto 1: stima a partire dalla 1ª osservazione

dalla Teoria

### Bayes Estimation: Generalisations

- If $E(d) = d_m\,,\;\; E(\vartheta) = \vartheta_m$, then:

$$\begin{cases} \hat{\vartheta} = \vartheta_m + \dfrac{\lambda_{\vartheta d}}{\lambda_{dd}}\,(d - d_m) \\ \operatorname{var}\left(\vartheta - \hat{\vartheta}\right) = \lambda_{\vartheta\vartheta} - \dfrac{\lambda_{\vartheta d}^2}{\lambda_{dd}} \end{cases}$$

- If $d$ and $\vartheta$ are vectors with $E(d) = d_m\,,\;\; E(\vartheta) = \vartheta_m$ and

$$\operatorname{var}\left(\begin{bmatrix} d \\ \vartheta \end{bmatrix}\right) = \begin{bmatrix} \Lambda_{dd} & \Lambda_{d\vartheta} \\ \Lambda_{\vartheta d} & \Lambda_{\vartheta\vartheta} \end{bmatrix} \qquad \Lambda_{d\vartheta} = \Lambda_{\vartheta d}^{\top}$$

Then:

$$\begin{cases} \hat{\vartheta} = \vartheta_m + \Lambda_{\vartheta d}\,{\Lambda_{dd}}^{-1}\,(d - d_m) \\ \operatorname{var}\left(\vartheta - \hat{\vartheta}\right) = \Lambda_{\vartheta\vartheta} - \Lambda_{\vartheta d}\,{\Lambda_{dd}}^{-1}\Lambda_{d\vartheta} \end{cases}$$



Bisogna valutare le varianze e covarianze presenti nelle formule:

$\Lambda_{\vartheta\vartheta} = var(x) = 0,9$ $\quad$ $\Lambda_{dd} = var\, y$ ← da calcolare

$\Lambda_{\vartheta d} = cov(x,y)$ ← da calcolare

$var(y) = (2.3)^2 var(x) + (4.2)^2 var(z) = 112,3660$

$$d_{\mu} = E[y] = 2.3\,E[x] + 4.2\,E[z] = 11,500$$

$$\Lambda_{\theta d} = \mathrm{cov}(x,y) = E\left[(x-\bar{x})(y-\bar{y})\right] =$$

$\bar{x} = E(x)$

$\bar{y} = E(y)$

$$= E\left\{(x-\bar{x})\left[2.3(x-\bar{x}) + 4.2(z-\bar{z})\right]\right\}$$

$$= 2.3\,\underbrace{E\left[(x-\bar{x})^2\right]}_{\mathrm{var}(x)} + 4.2\,\underbrace{E\left[(x-\bar{x})(z-\bar{z})\right]}_{\mathrm{cov}(x,z)=0}$$

sono indipendenti

$$= 2.3\,\mathrm{var}(x) = 0,9200$$

A questo punto

$$\hat{x} = \bar{x} + \Lambda_{\theta d} \cdot \Lambda_{dd}^{-1} \cdot \left[y(1) - \bar{y}\right] = 4,9453$$

$$\mathrm{var}(x - \hat{x}) = \Lambda_{\theta\theta} - \Lambda_{\theta d}^{2} \cdot \Lambda_{dd}^{-1} = 0,3925$$

Punto 2: stima usando 2 osservazioni (

dalla teoria

## Recursive form of Bayes estimation

- For now, denote by $\vartheta$ the unknown to be estimated and by $d$ the observed data.
- Suppose (just for simplicity and without loss of generality) that
  - $\vartheta$ scalar
  - $d(1),\ d(2)$ two scalar data
  - $\mathrm{E}(\vartheta) = 0,\ \mathrm{E}[d(1)] = 0,\ \mathrm{E}[d(2)] = 0$
- Then

$$\begin{bmatrix} \vartheta \\ d(1) \\ d(2) \end{bmatrix} \sim G\left( \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \lambda_{\vartheta\vartheta} & \lambda_{\vartheta 1} & \lambda_{\vartheta 2} \\ \lambda_{1\vartheta} & \lambda_{11} & \lambda_{12} \\ \lambda_{2\vartheta} & \lambda_{21} & \lambda_{22} \end{bmatrix} \right)$$

where $\lambda_{\vartheta\vartheta} = \mathrm{E}(\vartheta^2),\ \lambda_{\vartheta 1} = \mathrm{E}[\vartheta d(1)],\ \ldots$



## Recursive form of Bayes estimation (cont.)

- The estimate of $\vartheta$ based on the single data point $d(1)$ is given by

$$\mathrm{E}[\vartheta \mid d(1)] = \frac{\lambda_{\vartheta 1}}{\lambda_{11}} d(1)$$

- Instead, the estimate of $\vartheta$ based on two data points $d(1),\ d(2)$ is

$$\mathrm{E}[\vartheta \mid d(1), d(2)] = [\lambda_{\vartheta 1} \ \ \lambda_{\vartheta 2}] \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix}^{-1} \begin{bmatrix} d(1) \\ d(2) \end{bmatrix}$$

where $\lambda_{12} = \lambda_{21}$ But

$$\begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix}^{-1} = \frac{1}{\lambda_{11}\lambda_{22} - \lambda_{12}^2} \begin{bmatrix} \lambda_{22} & -\lambda_{12} \\ -\lambda_{12} & \lambda_{11} \end{bmatrix}$$

and hence

$$\mathrm{E}[\vartheta \mid d(1), d(2)] = \frac{1}{\lambda_{11}\lambda_{22} - \lambda_{12}^2} [(\lambda_{\vartheta 1}\lambda_{22} - \lambda_{\vartheta 2}\lambda_{12})\, d(1) + \\ + (-\lambda_{\vartheta 1}\lambda_{12} + \lambda_{\vartheta 2}\lambda_{11})\, d(2)]$$

## Recursive form of Bayes estimation (cont.)

recursion

$$
\begin{aligned}
\mathrm{E}[\vartheta \mid d(1), d(2)] =& \frac{1}{\lambda^2}\left(\lambda_{\vartheta 2} - \lambda_{\vartheta 1}\frac{\lambda_{12}}{\lambda_{11}}\right) d(2) \\
&+ \frac{1}{\lambda^2}\left(\lambda_{\vartheta 1}\frac{\lambda_{22}}{\lambda_{11}} - \lambda_{\vartheta 2}\frac{\lambda_{12}}{\lambda_{11}} - \lambda_{\vartheta 1}\frac{\lambda^2}{\lambda_{11}}\right) d(1) + \boxed{\frac{\lambda_{\vartheta 1}}{\lambda_{11}}\, d(1)}
\end{aligned}
$$

- substituting $\lambda^2 = \lambda_{22} - \dfrac{\lambda_{12}^2}{\lambda_{11}}$ we have

$$
\boxed{
\begin{aligned}
\mathrm{E}[\vartheta \mid d(1), d(2)] =& \frac{\lambda_{\vartheta 1}}{\lambda_{11}}\, d(1) \\
&+ \frac{1}{\lambda^2}\left(\lambda_{\vartheta 2} - \lambda_{\vartheta 1}\frac{\lambda_{12}}{\lambda_{11}}\right)\left[d(2) - \frac{\lambda_{12}}{\lambda_{11}} d(1)\right]
\end{aligned}
}
$$

- **Definition**. Given two random variables $d(1)$ and $d(2)$ we call innovation of $d(2)$ with respect to $d(1)$ the quantity:

$$
\boxed{e = d(2) - \mathrm{E}[d(2) \mid d(1)] = d(2) - \frac{\lambda_{12}}{\lambda_{11}} d(1)}
$$



## Generalization to the vector case

- Now, if $\vartheta$, $d(1)$, $d(2)$ are zero-mean vectors we have:

$$
\begin{bmatrix} \vartheta \\ d(1) \\ d(2) \end{bmatrix} \sim G\left( \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \Lambda_{\vartheta\vartheta} & \Lambda_{\vartheta 1} & \Lambda_{\vartheta 2} \\ \Lambda_{1\vartheta} & \Lambda_{11} & \Lambda_{12} \\ \Lambda_{2\vartheta} & \Lambda_{21} & \lambda_{22} \end{bmatrix} \right)
$$

  where $\Lambda_{\vartheta 1} = \Lambda_{1\vartheta}^\top$, $\Lambda_{\vartheta 2} = \Lambda_{2\vartheta}^\top$, $\Lambda_{21} = \Lambda_{12}^\top$

- We obtain:

$$
e = d(2) - \mathrm{E}[d(2) \mid d(1)] = d(2) - \Lambda_{21}\Lambda_{11}^{-1}\, d(1)
$$

  and hence:

$$
\begin{aligned}
\mathrm{E}[\vartheta \mid d(1), d(2)] =& \mathrm{E}[\vartheta \mid d(1)] + \mathrm{E}[\vartheta \mid e] \\
=& \Lambda_{\vartheta 1}\Lambda_{11}^{-1}\, d(1) + \Lambda_{\vartheta e}\Lambda_{ee}^{-1}\, e
\end{aligned}
$$

## Generalization to the non-zero mean case

- Now, if $\vartheta$, $d(1)$, $d(2)$ are non-zero mean vectors we have:

$$\begin{bmatrix} \vartheta \\ d(1) \\ d(2) \end{bmatrix} \sim G\left( \begin{bmatrix} \vartheta_m \\ d(1)_m \\ d(2)_m \end{bmatrix}, \begin{bmatrix} \Lambda_{\vartheta\vartheta} & \Lambda_{\vartheta 1} & \Lambda_{\vartheta 2} \\ \Lambda_{1\vartheta} & \Lambda_{11} & \Lambda_{12} \\ \Lambda_{2\vartheta} & \Lambda_{21} & \lambda_{22} \end{bmatrix} \right)$$

- We obtain:

$$\begin{aligned} E[\vartheta \mid d(1), d(2)] &= E[\vartheta \mid d(1)] + E[\vartheta \mid e] - \vartheta_m \\ &= \vartheta_m + \Lambda_{\vartheta\,1}\Lambda_{11}^{-1}\,[d(1) - d(1)_m] + \Lambda_{\vartheta e}\Lambda ee^{-1}\,e \end{aligned}$$

where, in analogy with the zero-mean scalar case we have:

- $\mathrm{E}(e) = 0$
- $\Lambda_{1e} = \mathrm{E}\left\{ [d(1) - d(1)_m]^\top \, e \right\} = 0$
- $\Lambda_{\vartheta e} = \Lambda_{\vartheta 2} - \Lambda_{\vartheta 1}\,\Lambda_{11}^{-1}\,\Lambda_{12}$



Va determinata la matrice di covarianza delle osservazioni:

$$\begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} = \begin{bmatrix} var[y(1)] & cov[y(1), y(2)] \\ cov[y(2), y(1)] & var[y(2)] \end{bmatrix}$$

$$var[y(1)] = var[y]$$

$$var[y(2)] = var[y]$$

il processo di $y(\cdot)$ è stazionario!

$$E\left\{\left[y(1)-\underbrace{E(y(1))}_{\bar{y}}\right]\left[y(2)-\underbrace{E[y(2)]}_{\bar{y}}\right]\right\}=\mathrm{cov}[y(1),y(2)]$$

$$E\left\{\left[y(1)-\bar{y}\right]\cdot\left[y(2)-\bar{y}\right]\right\} \qquad y=2.3x+4.2z$$

$$=E\left\{y(1)y(2)-\bar{y}\left[y(1)+y(2)\right]+\bar{y}^2\right\}=$$

$$\to E\left\{\left[2.3x(1)+4.2z(1)\right]\left[2.3x(2)+4.2z(2)\right]\right\}+$$

$$\underbrace{\underbrace{-\bar{y}\left[\underbrace{E(y(1))}_{\bar{y}}+\underbrace{E(y(2))}_{\bar{y}}\right]}_{-2\bar{y}^2}+\bar{y}^2}_{\to -\bar{y}^2}$$

$$=(2.3)^2E\{x(1)\cdot x(2)\}+(4.2)^2E\{z(1)\cdot z(2)\}+$$

$$+(2.3\cdot 4.2)\,\underbrace{\cancel{E[x(1)\cdot z(2)]}}_{0}+$$

$$+(2.3\cdot 4.2)\,\underbrace{\cancel{E[x(2)\cdot z(1)]}}_{0}-\bar{y}^2$$

$x(\cdot)$ e $z(\cdot)$ sono indipendenti

$$=$$

$\bar{z}=0$

$$\bar{y}^2=(2.3\bar{x}+4.2\bar{z})^2$$

$$=(2.3)^2(\bar{x})^2$$

rumore bianco

$$\operatorname{cov}[y(1), y(2)] =$$

$$= (2.3)^2 \, E\{x(1)\cdot x(2)\} + (4,2)^2 \, \underbrace{E\{z(1)\cdot z(2)\}}_{\operatorname{cov}[z(1),\, z(2)]} \overset{\emptyset}{} +$$

$$- (2.3)^2 (\bar{x})^2$$

$$= (2,3)^2 \left\{ E[x(1)\cdot x(2)] - (\bar{x})^2 \right\} = \emptyset$$

$$\operatorname{cov}[x(1), x(2)] = \emptyset$$

rumore bianco

Per definizione

$$\Lambda_{dd} = \begin{bmatrix} \operatorname{var}(y(1)) & \operatorname{cov}[y(1), y(2)] \\ \operatorname{cov}[y(2), y(1)] & \operatorname{var}[y(2)] \end{bmatrix}$$

$$= \begin{bmatrix} \operatorname{var} y & \emptyset \\ \emptyset & \operatorname{var} y \end{bmatrix}$$

Quindi

$$\begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} = \begin{bmatrix} \mathrm{var}\, y & 0 \\ 0 & \mathrm{var}\, y \end{bmatrix}$$

Rimangono da calcolare $d_{\theta_1}$ e $d_{\theta_2}$

$$d_{\theta_1} = \mathrm{cov}\left[x, y(1)\right] = E\left[(x-\bar{x})(y(1)-\bar{y})\right] = 2.3\ \mathrm{var}\, x$$

passaggi nei 2 fogli seguenti

$$d_{\theta_2} = \mathrm{cov}\left[x, y(2)\right] = E\left[(x-\bar{x})(y(2)-\bar{y})\right] = 2.3\ \mathrm{var}\, x$$

A questo punto:

$$e = \left[d(2) - d(2)_m\right] - \frac{d_{12}}{d_{11}}\left[d(1) - d(1)_m\right]$$

ma $d_{12} = 0$

$$e \triangleq y(2) - \bar{y}$$

$y(\cdot)$ è combinazione di rumori bianchi indip.! È WN!

$$\gamma_{01} = \operatorname{cov}[x, y(1)] = E[(x(1) - \bar{x})(y(1) - \bar{y})]$$

NB quando osservo $y(1)$, la v.a. $x$ assume val $x(1)$

$$= E[x(1)\,y(1) - x(1)\,\bar{y} - \bar{x}\,y(1) + \bar{x}\,\bar{y}] =$$

$$= E[x(1)\,y(1)] - \bar{y}\,\underbrace{E(x(1))}_{\bar{x}} - \bar{x}\,\overbrace{E[y(1)]}^{\bar{y}} + \bar{x}\,\bar{y} =$$

$$= E[x(1)\cdot y(1)] - \bar{x}\,\bar{y} - \cancel{\bar{x}\,\bar{y}} + \cancel{\bar{x}\,\bar{y}}$$

$$= E[x(1)\cdot(2.3\,x(1) + 4.2\,z(1))] - \bar{x}\,\bar{y} =$$

indipendenti

$$= 2.3\,E[x(1)\cdot x(1)] + \underbrace{\cancel{4.2\,E[x(1)\cdot z(1)]}}_{\emptyset} - \bar{x}\,\bar{y} =$$

$$= 2.3\,E[(x(1))^2] - 2.3\,\bar{x}^2$$

$$\bar{y} = 2.3\,\bar{x} + \underbrace{\cancel{4.2\,\bar{z}}}_{\emptyset} = 2.3\,\bar{x}$$

$$\sigma_{d_1} = 2.3\sqrt{E\left[(x(1))^2\right] - \left[E\left[x(1)\right]\right]^2}$$

$$E(x^2) - \left[E(x)\right]^2 = \mathrm{var}\, x$$

$$= 2.3\ \mathrm{var}(x)$$

Analoghi passaggi per $\sigma_{d_2}$

Per quanto riguarda la stima

$$E\left[x \mid y(1), y(2)\right] = E(x) + \frac{d_{\vartheta 1}}{d_{11}}\left[y(1) - \bar{y}\right] + \frac{d_{\vartheta e}}{d_{ee}}\, e$$

$d_{12} = 0$

$$e = y(2) - \bar{y}$$

$$d_{\vartheta e} = d_{\vartheta 2} - d_{\vartheta 1}\frac{d_{12}}{d_{11}} = d_{\vartheta 2}$$

$$d_{ee} = d_{22} - \frac{d_{12}^2}{d_{11}} = d_{22}$$

In definitiva:

$$\hat{x}(2) = 5{,}0 + \frac{0{,}92}{112{,}3660}\cdot(-6{,}6850) + \frac{0{,}92}{112{,}3660}\cdot(-9{,}8770) \simeq 4{,}8644$$

$$\operatorname{var}\left[x(2)-\hat{x}(2)\right] = \operatorname{var} x + -\left[d_{\theta_1} \quad d_{\theta_2}\right] \begin{bmatrix} d_{11} & d_{12} \\ d_{12} & d_{22} \end{bmatrix}^{-1} \begin{bmatrix} d_{\theta 1} \\ d_{\theta_2} \end{bmatrix} = 0{,}3899$$

NB $\operatorname{var}\left[x(1)-\hat{x}(1)\right] > \operatorname{var}\left[x(2)-\hat{x}(2)\right]$