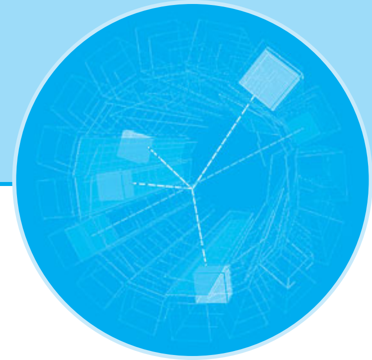

CHAPTER 5

The Wavelike Properties of Particles

In 1924, a French graduate student, Louis de Broglie,[1] proposed in his doctoral dissertation that the dual—i.e., wave-particle—behavior that was by then known to exist for radiation was also a characteristic of matter, in particular, electrons. This suggestion was highly speculative, since there was yet no experimental evidence whatsoever for any wave aspects of electrons or any other particles. What had led him to this seemingly strange idea? It was a "bolt out of the blue," like Einstein's "happy thought" that led to the principle of equivalence (see Chapter 2). De Broglie described it with these words:

> After the end of World War I, I gave a great deal of thought to the theory of quanta and to the wave-particle dualism. . . . It was then that I had a sudden inspiration. Einstein's wave-particle dualism was an absolutely general phenomenon extending to all physical nature.[2]

Since the visible universe consists entirely of matter and radiation, de Broglie's hypothesis is a fundamental statement about the grand symmetry of nature. (There is currently strong observational evidence that ordinary matter makes up only about 4 percent of the visible universe. About 22 percent is some unknown form of invisible "dark matter," and approximately 74 percent consists of some sort of equally mysterious "dark energy." See Chapter 13.)

5-1 The de Broglie Hypothesis

De Broglie stated his proposal mathematically with the following equations for the frequency and wavelength of the electron waves, which are referred to as the *de Broglie relations*:

$$f = \frac{E}{h} \qquad \textbf{5-1}$$

$$\lambda = \frac{h}{p} \qquad \textbf{5-2}$$

where E is the total energy, p is the momentum, and λ is called the *de Broglie wavelength* of the particle. For photons, these same equations result directly from

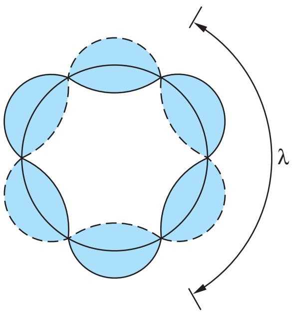

Figure 5-1 Standing waves around the circumference of a circle. In this case the circle is 3λ in circumference. If the vibrator were, for example, a steel ring that had been suitably tapped with a hammer, the shape of the ring would oscillate between the extreme positions represented by the solid and broken lines.

Einstein's quantization of radiation $E = hf$ and Equation 2-31 for a particle of zero rest energy $E = pc$ as follows:

$$E = pc = hf = \frac{hc}{\lambda}$$

By a more indirect approach using relativistic mechanics, de Broglie was able to demonstrate that Equations 5-1 and 5-2 also apply to particles with mass. He then pointed out that these equations lead to a physical interpretation of Bohr's quantization of the angular momentum of the electron in hydrogenlike atoms, namely, that the quantization is equivalent to a standing-wave condition. (See Figure 5-1.) We have

$$mvr = n\hbar = \frac{nh}{2\pi} \qquad \text{for} \qquad n = \text{integer}$$

$$2\pi r = \frac{nh}{mv} = \frac{nh}{p} = n\lambda = \text{circumference of orbit} \tag{5-3}$$

The idea of explaining discrete energy states in matter by standing waves thus seemed quite promising.

De Broglie's ideas were expanded and developed into a complete theory by Erwin Schrödinger late in 1925. In 1927, C. J. Davisson and L. H. Germer verified the de Broglie hypothesis directly by observing interference patterns, a characteristic of waves, with electron beams. We will discuss both Schrödinger's theory and the Davisson-Germer experiment in later sections, but first we have to ask ourselves why wavelike behavior of matter had not been observed before de Broglie's work. We can understand why if we first recall that the wave properties of light were not noticed either until apertures or slits with dimensions of the order of the wavelength of light could be obtained. This is because the wave nature of light is not evident in experiments where the primary dimensions of the apparatus are large compared with the wavelength of the light used. For example, if A represents the diameter of a lens or the width of a slit, then diffraction effects[3] (a manifestation of wave properties) are limited to angles θ around the forward direction ($\theta = 0°$) where $\sin\theta = \lambda/A$. In geometric (ray) optics $\lambda/A \to 0$, so $\theta \approx \sin\theta \to 0$, too. However, if a characteristic

Louis V. de Broglie, who first suggested that electrons might have wave properties. [*Courtesy of Culver Pictures.*]

dimension of the apparatus becomes of the order of (or smaller than) λ, the wavelength of light passing through the system, then $\lambda/A \rightarrow 1$. In that event $\theta \approx \lambda/A$ is readily observable, and the wavelike properties of light become apparent. Because Planck's constant is so small, the wavelength given by Equation 5-2 is extremely small for any macroscopic object. This point is among those illustrated in the following section.

5-2 Measurements of Particle Wavelengths

Although we now have diffraction systems of nuclear dimensions, the smallest-scale systems to which de Broglie's contemporaries had access were the spacings between the planes of atoms in crystalline solids, about 0.1 nm. This means that even for an extremely small macroscopic particle, such as a grain of dust ($m \approx 0.1$ mg) moving through air with the average kinetic energy of the atmospheric gas molecules, the smallest diffraction systems available would have resulted in diffraction angles τ only of the order of 10^{-10} radian, far below the limit of experimental detectability. The small magnitude of Planck's constant ensures that λ will be smaller than any readily accessible aperture, placing diffraction beyond the limits of experimental observation. For objects whose momenta are larger than that of the dust particle, the possibility of observing *particle*, or *matter waves*, is even less, as the following example illustrates.

EXAMPLE 5-1 De Broglie Wavelength of a Ping-Pong Ball What is the de Broglie wavelength of a Ping-Pong ball of mass 2.0 g after it is slammed across the table with speed 5 m/s?

SOLUTION

$$\lambda = \frac{h}{mv} = \frac{6.63 \times 10^{-34}\ \text{J}\cdot\text{s}}{(2.0 \times 10^{-3}\ \text{kg})(5\ \text{m/s})}$$
$$= 6.6 \times 10^{-32}\ \text{m} = 6.6 \times 10^{-23}\ \text{nm}$$

This is 17 orders of magnitude smaller than typical nuclear dimensions, far below the dimensions of any possible aperture.

The case is different for low-energy electrons, as de Broglie himself realized. At his *soutenance de thèse* (defense of the thesis), de Broglie was asked by Perrin[4] how his hypothesis could be verified, to which he replied that perhaps passing particles, such as electrons, through very small slits would reveal the waves. Consider an electron that has been accelerated through V_0 volts. Its kinetic energy (nonrelativistic) is then

$$E = \frac{p^2}{2m} = eV_0$$

Solving for p and substituting into Equation 5-2,

$$\lambda = \frac{h}{p} = \frac{hc}{pc} = \frac{hc}{(2mc^2eV_0)^{1/2}}$$

Using $hc = 1.24 \times 10^3$ eV $\cdot$ nm and $mc^2 = 0.511 \times 10^6$ eV, we obtain

$$\lambda = \frac{1.226}{V_0^{1/2}}\text{nm} \qquad \text{for} \qquad eV_0 \sim mc^2 \tag{5-4}$$

The following example computes an electron de Broglie wavelength, giving a measure of just how small the slit must be.

EXAMPLE 5-2 De Broglie Wavelength of a Slow Electron Compute the de Broglie wavelength of an electron whose kinetic energy is 10 eV.

SOLUTION

1. The de Broglie wavelength is given by Equation 5-2:

$$\lambda = \frac{h}{p}$$

2. *Method 1:* Since a 10-eV electron is nonrelativistic, we can use the classical relation connecting the momentum and the kinetic energy:

$$E_k = \frac{p^2}{2m}$$

or

$$\begin{aligned} p &= \sqrt{2mE_k} \\ &= \sqrt{(2)(9.11 \times 10^{-31}\ \text{kg})(10\ \text{eV})(1.60 \times 10^{-19}\ \text{J/eV})} \\ &= 1.71 \times 10^{-24}\ \text{kg} \cdot \text{m/s} \end{aligned}$$

3. Substituting this result into Equation 5-2:

$$\begin{aligned} \lambda &= \frac{6.63 \times 10^{-34}\ \text{J} \cdot \text{s}}{1.71 \times 10^{-24}\ \text{kg} \cdot \text{m/s}} \\ &= 3.88 \times 10^{-10}\ \text{m} = 0.39\ \text{nm} \end{aligned}$$

4. *Method 2:* The electron's wavelength can also be computed from Equation 5-4 with $V_0 = 10$ V:

$$\begin{aligned} \lambda &= \frac{1.226}{V^{1/2}} = \frac{1.226}{\sqrt{10}} \\ &= 0.39\ \text{nm} \end{aligned}$$

Remarks: *Though this wavelength is small, it is just the order of magnitude of the size of an atom and of the spacing of atoms in a crystal.*

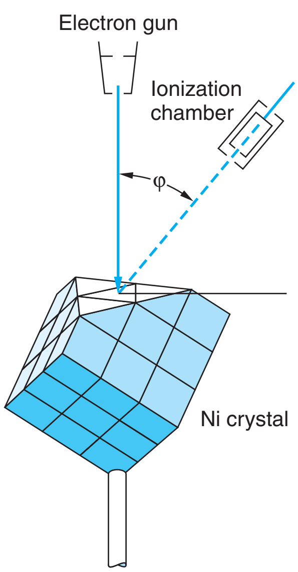

Figure 5-2 The Davisson-Germer experiment. Low-energy electrons scattered at angle φ from a nickel crystal are detected in an ionization chamber. The kinetic energy of the electrons could be varied by changing the accelerating voltage on the electron gun.

The Davisson-Germer Experiment

In a brief note in the August 14, 1925, issue of the journal *Naturwissenschaften*, Walter Elsasser, at the time a student of J. Franck's (of the Franck-Hertz experiment), proposed that the wave effects of low-velocity electrons might be detected by scattering them from single crystals. The first such measurements of the wavelengths of electrons were made in 1927 by C. J. Davisson[5] and L. H. Germer, who were studying electron reflection from a nickel target at Bell Telephone Laboratories, unaware of either Elsasser's suggestion or de Broglie's work. After heating their target to remove an oxide coating that had accumulated during an accidental break in their vacuum system, they found that the scattered electron intensity as a function of the scattering angle showed maxima and minima. The surface atoms of their nickel target had, in the process of cooling, formed relatively large single crystals, and they were observing electron diffraction. Recognizing the importance of their accidental discovery, they then prepared a target consisting of a single crystal of nickel and extensively investigated the scattering of electrons from it. Figure 5-2 illustrates their experimental arrangement. Their data for 54-eV electrons, shown in Figure 5-3, indicate a strong maximum of scattering at $\varphi = 50°$. Consider the scattering from a set of Bragg

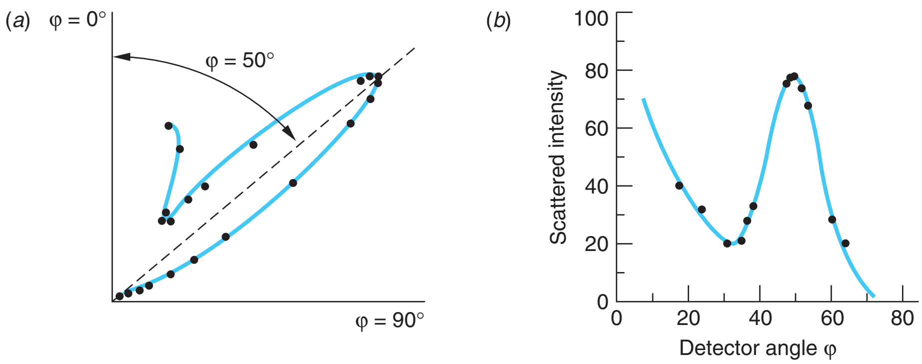

Figure 5-3 Scattered intensity vs. detector angle for 54-eV electrons. (*a*) Polar plot of the data. The intensity at each angle is indicated by the distance of the point from the origin. Scattering angle φ is plotted clockwise starting at the vertical axes. (*b*) The same data plotted on a Cartesian graph. The intensity scales are arbitrary but the same on both graphs. In each plot there is maximum intensity at $\varphi = 50°$, as predicted for Bragg scattering of waves having wavelength $\lambda = h/p$. [*From Nobel Prize Lectures: Physics (Amsterdam and New York: Elsevier, © Nobel Foundation, 1964).*]

planes, as shown in Figure 5-4. The Bragg condition for constructive interference is $n\lambda = 2d \sin\theta = 2d \cos\alpha$. The spacing of the Bragg planes d is related to the spacing of the atoms D by $d = D \sin\alpha$; thus

$$n\lambda = 2D \sin\alpha \cos\alpha = D \sin 2\alpha$$

or

$$n\lambda = D \sin\varphi \qquad \textbf{5-5}$$

where $\varphi = 2\alpha$ is the scattering angle.

The spacing D for Ni is known from x-ray diffraction to be 0.215 nm. The wavelength calculated from Equation 5-5 for the peak observed at $\varphi = 50°$ by Davisson and Germer is, for $n = 1$,

$$\lambda = 0.215 \sin 50° = 0.165 \text{ nm}$$

The value calculated from the de Broglie relation for 54-eV electrons is

$$\lambda = \frac{1.226}{(54)^{1/2}} = 0.167 \text{ nm}$$

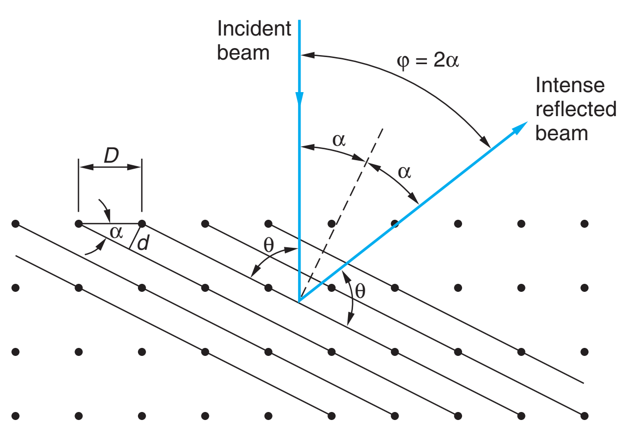

Figure 5-4 Scattering of electrons by a crystal. Electron waves are strongly scattered if the Bragg condition $n\lambda = 2d \sin\theta$ is met. This is equivalent to the condition $n\lambda = D \sin\varphi$.

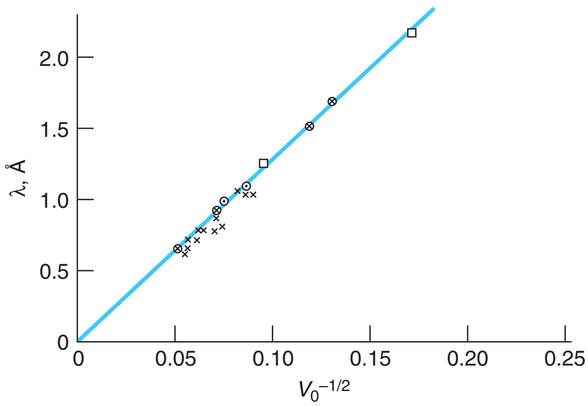

Figure 5-5 Test of the de Broglie formula $\lambda = h/p$. The wavelength is computed from a plot of the diffraction data plotted against $V_0^{-1/2}$, where V_0 is the accelerating voltage. The straight line is $1.226V_0^{-1/2}$ nm as predicted from $\lambda = h(2mE)^{-1/2}$. These are the data referred to in the quotation from Davisson's Nobel lecture. (× From observations with diffraction apparatus; ⊗ same, particularly reliable; □ same, grazing beams. ⊙ From observations with reflection apparatus.) [*From Nobel Prize Lectures: Physics (Amsterdam and New York: Elsevier, © Nobel Foundation, 1964).*]

The agreement with the experimental observation is excellent! With this spectacular result Davisson and Germer then conducted a systematic study to test the de Broglie relation using electrons up to about 400 eV and various experimental arrangements. Figure 5-5 shows a plot of measured wavelengths versus $V_0^{-1/2}$. The wavelengths measured by diffraction are slightly lower than the theoretical predictions because the refraction of the electron waves at the crystal surface has been neglected. We have seen from the photoelectric effect that it takes work of the order of several eV to remove an electron from a metal. Electrons entering a metal thus gain kinetic energy; therefore, their de Broglie wavelength is slightly less inside the crystal.[6]

A subtle point must be made here. Notice that the wavelength in Equation 5-5 depends only on D, the interatomic spacing of the crystal, whereas our derivation of that equation included the interplane spacing as well. The fact that the structure of the crystal really is essential shows up when the energy is varied, as was done in collecting the data for Figure 5-5. Equation 5-5 suggests that a change in λ, resulting from a change in the energy, would mean only that the diffraction maximum would occur at some other value of φ such that the equation remains satisfied. However, as can be seen from examination of Figure 5-4, the value of φ is determined by α, the angle of the planes determined by the crystal structure. Thus, if there are no crystal planes making an angle $\alpha = \varphi/2$ with the surface, then setting the detector at $\varphi = \sin^{-1}(\lambda/D)$ will not result in constructive interference and strong reflection for that value of λ, even though Equation 5-5 is satisfied. This is neatly illustrated by Figure 5-6, which shows a series of polar graphs (like Figure 5-3*a*) for electrons of energies from 36 eV through 68 eV. The building to a strong reflection at $\varphi = 50°$ is evident for $V_0 = 54$ V, as we have already seen. But Equation 5-5 by itself would also lead us to expect, for example, a strong reflection at $\varphi = 64°$ when $V_0 = 40$ V, which obviously does not occur.

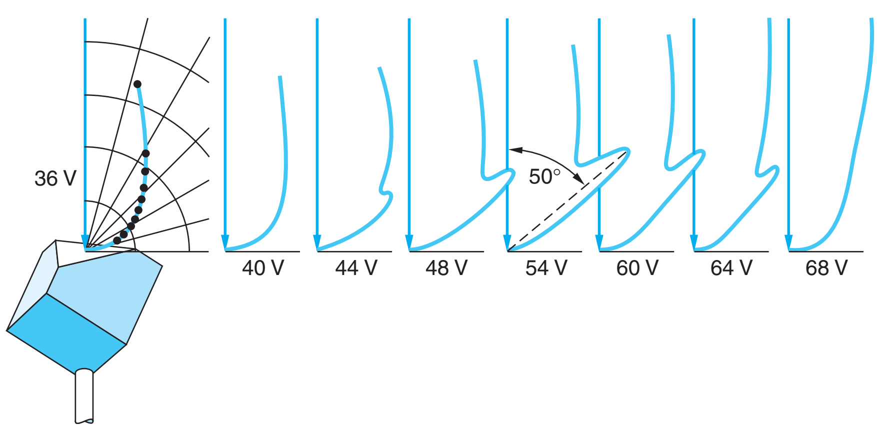

Figure 5-6 A series of polar graphs of Davisson and Germer's data at electron accelerating potentials from 36 V to 68 V. Note the development of the peak at $\varphi = 50°$ to a maximum when $V_0 = 54$ V.

Clinton J. Davisson (left) and Lester H. Germer at Bell Laboratories, where electron diffraction was first observed. [*Bell Telephone Laboratories, Inc.*]

In order to show the dependence of the diffraction on the inner atomic layers, Davisson and Germer kept the detector angle φ fixed and varied the accelerating voltage rather than search for the correct angle for a given λ. Writing Equation 5-5 as

$$\lambda = \frac{D \sin \varphi}{n} = \frac{D \sin(2\alpha)}{n} \tag{5-6}$$

and noting that $\lambda \propto V_0^{-1/2}$, a graph of intensity versus $V_0^{1/2} (\propto 1/\lambda)$ for a given angle φ should yield (1) a series of equally spaced peaks corresponding to successive values of the integer n, if $\alpha = \varphi/2$ is an existing angle for atomic planes, or (2) no diffraction peaks if $\varphi/2$ is not such an angle. Their measurements verified the dependence upon the interplane spacing, the agreement with the prediction being about ± 1 percent. Figure 5-7 illustrates the results for $\varphi = 50°$. Thus, Davisson and Germer showed conclusively that particles with mass moving at speeds $v \ll c$ do indeed have wavelike properties, as de Broglie had proposed.

The diffraction pattern formed by high-energy electron waves scattered from nuclei provides a means by which nuclear radii and the internal distribution of the nuclear charge (the protons) are measured. See Chapter 11.

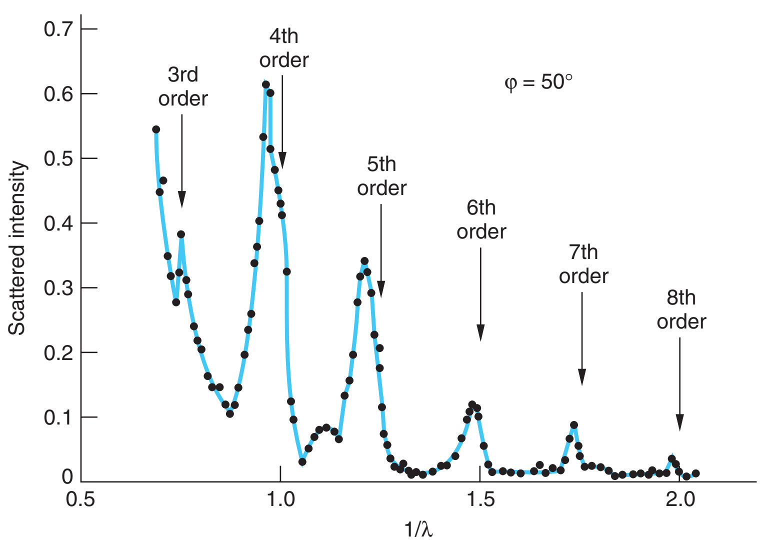

Figure 5-7 Variation of the scattered electron intensity with wavelength for constant φ. The incident beam in this case was 10° from the normal, the resulting refraction causing the measured peaks to be slightly shifted from the positions computed from Equation 5-5, as explained in note 6. [*After C. J. Davisson and L. H. Germer, Proceedings of the National Academy of Sciences, **14**, 619 (1928).*]

Here is Davisson's account of the connection between de Broglie's predictions and their experimental verification:

> Perhaps no idea in physics has received so rapid or so intensive development as this one. De Broglie himself was in the van of this development, but the chief contributions were made by the older and more experienced Schrödinger. It would be pleasant to tell you that no sooner had Elsasser's suggestion appeared than the experiments were begun in New York which resulted in a demonstration of electron diffraction—pleasanter still to say that the work was begun the day after copies of de Broglie's thesis reached America. The true story contains less of perspicacity and more of chance. . . It was discovered, purely by accident, that the intensity of elastic scattering [of electrons] varies with the orientations of the scattering crystals. Out of this grew, quite naturally, an investigation of elastic scattering by a single crystal of predetermined orientation. . . Thus the New York experiment was not, at its inception, a test of wave theory. Only in the summer of 1926, after I had discussed the investigation in England with Richardson, Born, Franck and others, did it take on this character.[7]

A demonstration of the wave nature of relativistic electrons was provided in the same year by G. P. Thomson, who observed the transmission of electrons with energies in the range of 10 to 40 keV through thin metallic foils (G. P. Thomson, the son of J. J. Thomson, shared the Nobel Prize in 1937 with Davisson). The experimental arrangement (Figure 5-8*a*) was similar to that used to obtain Laue patterns with x rays (see Figure 3-11). Because the metal foil consists of many tiny crystals randomly oriented, the diffraction pattern consists of concentric rings. If a crystal is oriented at an angle θ with the incident beam, where θ satisfies the Bragg condition, this crystal will strongly scatter at an equal angle θ; thus there will be a scattered beam making an angle 2θ with the incident beam. Figure 5-8*b* and *c* show the similarities in patterns produced by x rays and electron waves.

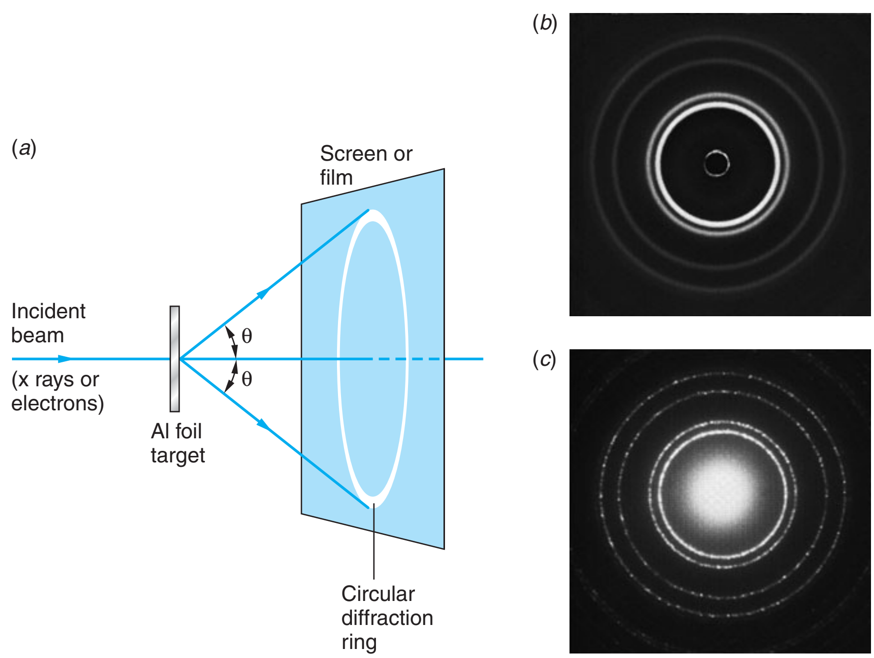

Figure 5-8 (*a*) Schematic arrangement used for producing a diffraction pattern from a polycrystalline aluminum target. (*b*) Diffraction pattern produced by x rays of wavelength 0.071 nm and an aluminum foil target. (*c*) Diffraction pattern produced by 600-eV electrons (de Broglie wavelength of about 0.05 nm) and an aluminum foil target. The pattern has been enlarged by 1.6 times to facilitate comparison with (*b*). [*Courtesy of Film Studio, Education Development Center.*]

Diffraction of Other Particles

The wave properties of neutral atoms and molecules were first demonstrated by O. Stern and I. Estermann in 1930 with beams of helium atoms and hydrogen molecules diffracted from a lithium fluoride crystal. Since the particles are neutral, there is no possibility of accelerating them with electrostatic potentials. The energy of the molecules was that of their average thermal motion, about 0.03 eV, which implies a de Broglie wavelength of about 0.10 nm for these molecules, according to Equation 5-2. Because of their low energy, the scattering occurs just from the array of atoms on the surface of the crystal, in contrast to Davisson and Germer's experiment. Figure 5-9 illustrates the geometry of the surface scattering, the experimental arrangement, and the results. Figure 5-9*c* indicates clearly the diffraction of He atom waves.

The diffraction patterns formed by helium atom waves are used to study impurities and defects on the surfaces of crystals. Being a noble gas, helium does not react chemically with molecules on the surface nor "stick" to the surface.

Since then, diffraction of other atoms, of protons, and of neutrons has been observed (see Figures 5-10, 5-11, and 5-12 on page 194). In all cases the measured wavelengths agree with de Broglie's prediction. There is thus no doubt that all matter has wavelike, as well as particlelike, properties, in symmetry with electromagnetic radiation.

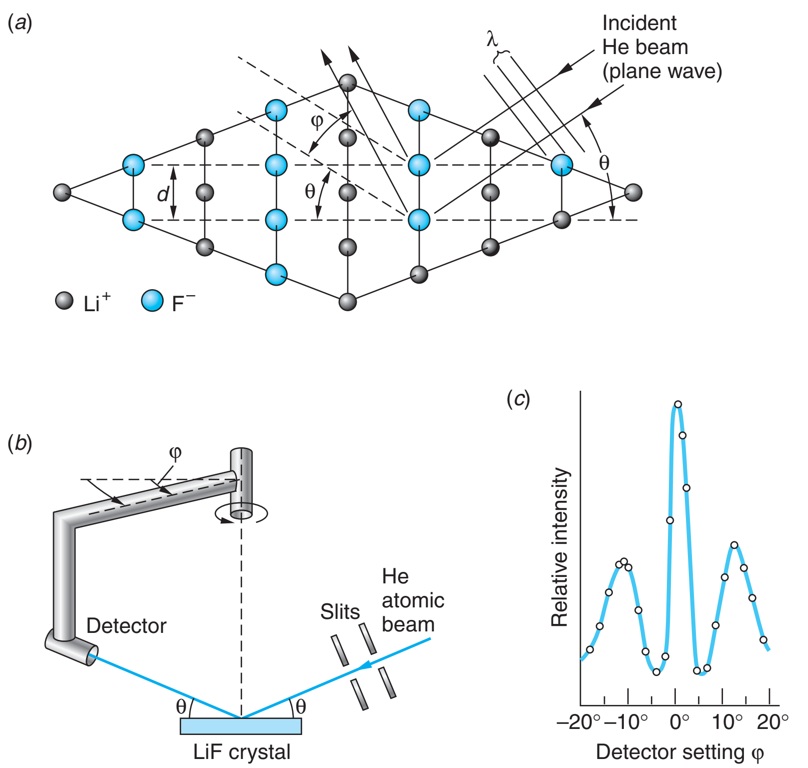

Figure 5-9 (*a*) He atoms impinge upon the surface of the LiF crystal at angle θ (θ = 18.5° in Estermann and Stern's experiment). The reflected beam also makes the same angle θ with the surface but is also scattered at azimuthal angles φ relative to an axis perpendicular to the surface. (*b*) The detector views the surface at angle θ but can scan through the angle φ. (*c*) At angle φ where the path difference ($d \sin\varphi$) between adjacent "rays" is $n\lambda$, constructive interference, i.e., a diffraction peak, occurs. The $n = 1$ peaks occur on either side of the $n = 0$ maximum.

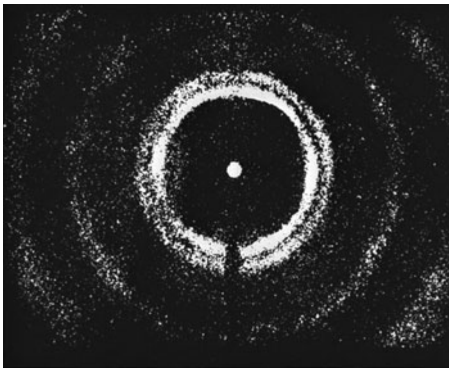

Figure 5-10 Diffraction pattern produced by 0.0568-eV neutrons (de Broglie wavelength of 0.120 nm) and a target of polycrystalline copper. Note the similarity in the patterns produced by x rays, electrons, and neutrons. [*Courtesy of C. G. Shull.*]

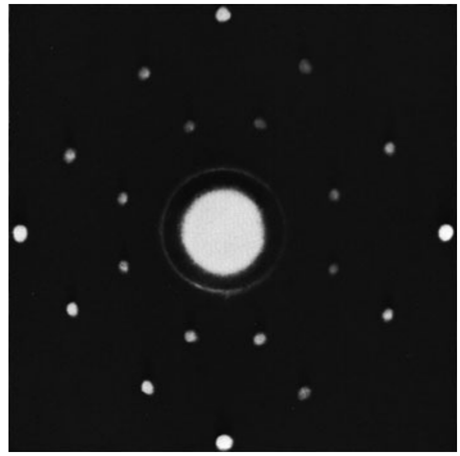

Figure 5-11 Neutron Laue pattern of NaCl. Compare this with the x-ray Laue pattern in Figure 3-14. [*Courtesy of E. O. Wollan and C. G. Shull.*]

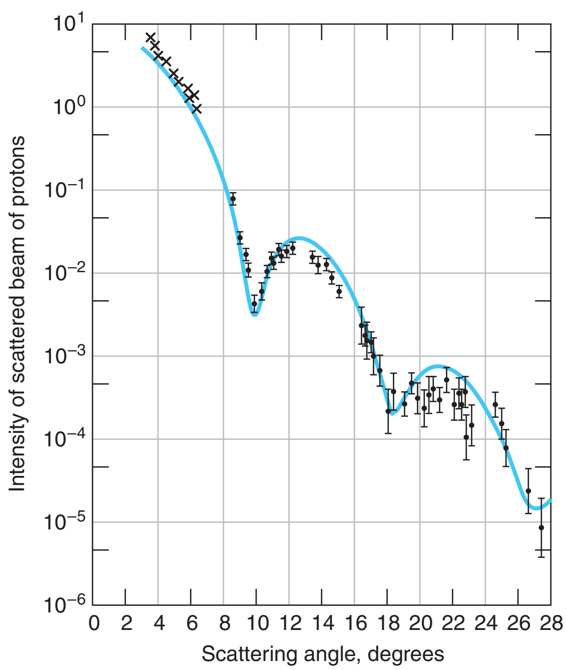

Figure 5-12 Nuclei provide scatterers whose dimensions are of the order of 10^{-15} m. Here the diffraction of 1-GeV protons from oxygen nuclei result in a pattern similar to that of a single slit.

An Easy Way to Determine de Broglie Wavelengths

It is frequently helpful to know the de Broglie wavelength for particles with a specific kinetic energy. For low energies where relativistic effects can be ignored, the equation leading to Equation 5-4 can be rewritten in terms of the kinetic energy as follows:

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE_k}} \tag{5-7}$$

To find the equivalent expression that covers both relativistic and nonrelativistic speeds, we begin with the relativistic equation relating the total energy to the momentum:

$$E^2 = (pc)^2 + (mc^2)^2 \qquad \textbf{2-31}$$

Writing E_0 for the rest energy mc^2 of the particle for convenience, this becomes

$$E^2 = (pc)^2 + E_0^2 \qquad \textbf{5-8}$$

Since the total energy $E = E_0 + E_k$, Equation 5-8 becomes

$$(E_0 + E_k)^2 = (pc)^2 + E_0^2$$

that, when solved for p, yields

$$p = \frac{(2E_0E_k + E_k^2)^{1/2}}{c}$$

from which Equation 5-2 gives

$$\lambda = \frac{hc}{(2E_0E_k + E_k^2)^{1/2}} \qquad \textbf{5-9}$$

This can be written in a particularly useful way applicable to any particle of any energy by dividing the numerator and denominator by the rest energy $E_0 = mc^2$ as follows:

$$\lambda = \frac{hc/mc^2}{(2E_0E_k + E_k^2)^{1/2}/E_0} = \frac{h/mc}{[2(E_k/E_0) + (E_k/E_0)^2]^{1/2}}$$

Recognizing h/mc as the Compton wavelength λ_c of the particle of mass m (see Section 3-4), we have that, for any particle,

$$\lambda/\lambda_c = \frac{1}{[2(E_k/E_0) + (E_k/E_0)^2]^{1/2}} \qquad \textbf{5-10}$$

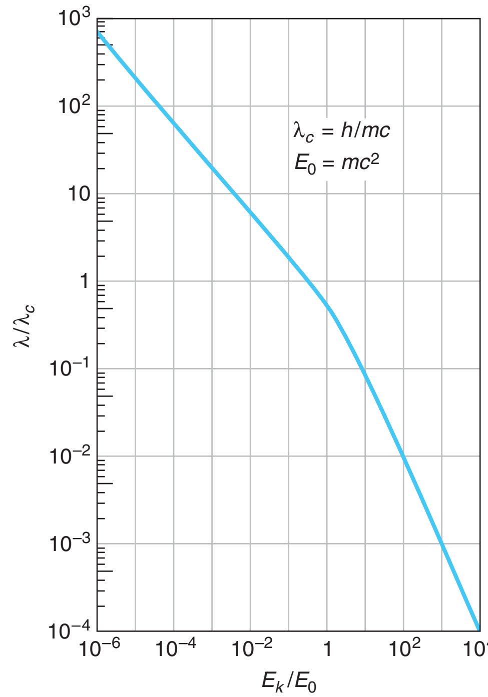

Figure 5-13 The de Broglie wavelength λ expressed in units of the Compton wavelength λ_c for a particle of mass m versus the kinetic energy of the particle E_k expressed in units of its rest energy $E_0 = mc^2$. For protons and neutrons $E_0 = 0.938$ GeV and $\lambda_c = 1.32$ fm. For electrons $E_0 = 0.511$ MeV and $\lambda_c = 0.00234$ nm.

A log-log graph of λ/λ_c versus E_k/E_0 is shown in Figure 5-13. It has two sections of nearly constant slope, one for $E_k \ll mc^2$ and the other for $E_k \gg mc^2$, connected by a curved portion lying roughly between $0.1 < E_k/E_0 < 10$. The following example illustrates the use of Figure 5-13.

EXAMPLE 5-3 The de Broglie Wavelength of a Cosmic Ray Proton Detectors on board a satellite measure the kinetic energy of a cosmic ray proton to be 150 GeV. What is the proton's de Broglie wavelength, as read from Figure 5-13?

SOLUTION

The rest energy of the proton is $mc^2 = 0.938$ GeV and the proton's mass is 1.67×10^{-27} kg. Thus, the ratio E_k/E_0 is

$$\frac{E_k}{E_0} = \frac{150\ \text{GeV}}{0.938\ \text{GeV}} = 160$$