# Systems Dynamics

Course ID: 267MI – Fall 2022

Thomas Parisini
Gianfranco Fenu

University of Trieste
Department of Engineering and Architecture

# 267MI –Fall 2022

## Lecture 1
## Generalities: Systems and Models

# Lecture 1: Table of Contents

## 1. Generalities: Systems and Models

# Systems Dynamics

# Systems

Inputs (”causes”)

$$u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix} \in \Re^m$$

Outputs (”effects”)

$$y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_m(t) \end{bmatrix} \in \Re^p$$

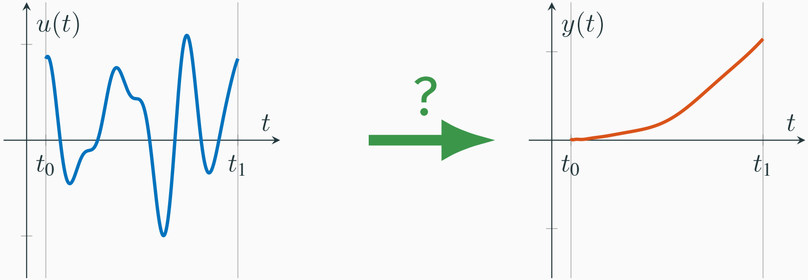

Definition of the ”system” entity to be analysed $\Longrightarrow$ Physical laws, a priori knowledge, heuristic considerations, statistical evidence, etc. $\Longrightarrow$ *Mathematical models*: algebraic and/or differential and/or difference equations

# Dynamic Systems Described by State Equations

# Dynamic Systems

Recalling from the *Fundamentals in Control* course

What is the meaning of ”Dynamic”?



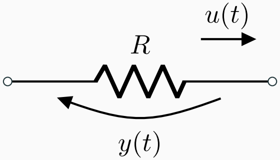

Can $y(t)$ be determined in a unique way?

If the answer is **“NO”** → The system is a **dynamic system**.

# Dynamic Systems Described by State Equations

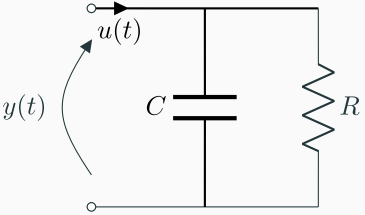
## Dynamic Systems

# Dynamic Systems: Examples




$$y(t) = R \cdot u(t)$$

The system is **NOT** dynamic

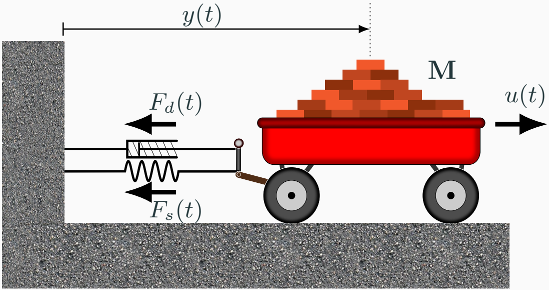



$$\left.\begin{array}{l} u(t),\ t \in [t_0, t_1] \\ y(t_0) \end{array}\right\}$$

$$\Longrightarrow y(t),\ t \in [t_0, t_1]$$

The system is **dynamic**

# Dynamic Systems: Examples




$$\left.\begin{array}{l} u(t),\ t \in [t_0, t_1] \\ y(t_0) \\ \dot{y}(t_0) \end{array}\right\} \implies y(t),\ t \in [t_0, t_1]$$

The system is **dynamic**

# State Variables: a Qualitative Definition

## State variables

Variables to be known at time $t = t_0$ in order to be able to determine the output $y(t)$, $t \geq t_0$ from the knowledge of the input $u(t)$, $t \geq t_0$:

$$x_i(t),\ i = 1, 2, \ldots, n \qquad \text{(state variables)}$$

$\cdots$ In more **rigorous** terms $\Longrightarrow$

# Dynamic Systems Described by State Equations

## Formal Definitions

# Dynamic Systems: Formal Definitions

A **dynamic system** is an abstract entity defined in axiomatic way:

$$\mathcal{S} = \{T, U, \Omega, X, Y, \Gamma, \varphi, \eta\}$$

- $T$ : set of **time instants** provided with an order relation
- $U$ : set of admissible **input** values
- $\Omega$ : set of admissible **control functions**
- $X$ : set of admissible **state** values
- $Y$ : set of admissible **output** values
- $\Gamma$ : set of admissible **output functions**

# Dynamic Systems: Formal Definitions (cont.)

**State transition function**:

$$\varphi : \, T \times T \times X \times \Omega \mapsto X \quad \Longrightarrow \quad x(t) = \varphi(t, t_0, x_0, u(\cdot))$$

1. **Consistency**: $\varphi(t_0, t_0, x_0, u(\cdot)) = x_0 \,, \; \forall \, (t_0, x_0, u(\cdot)) \in T \times X \times \Omega$
2. **Irreversibility**: $\varphi$ is defined $\forall \, t \geq t_0, \, t \in T$
3. **Composition**:

$$\varphi(t_2, t_0, x_0, u(\cdot)) = \varphi(t_2, t_1, \varphi(t_1, t_0, x_0, u(\cdot)), u(\cdot))$$
$$\forall \, (t_0, u(\cdot)) \in T \times \Omega \,, \; \forall \, t_0, t_1, t_2 \in T : \; t_0 < t_1 < t_2$$

4. **Causality**:

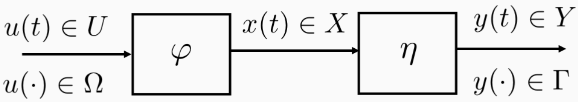
$$u'_{[t_0,t)}(\cdot) = u''_{[t_0,t)}(\cdot) \implies \varphi(t, t_0, x_0, u'(\cdot)) = \varphi(t, t_0, x_0, u''(\cdot)),$$
$$\forall (t, t_0, x_0) \in T \times T \times X$$

# Dynamic Systems: Formal Definitions (cont.)

**Output function**:

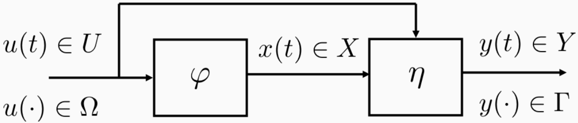
- Case 1: **strictly proper** system:

$$\eta : T \times X \mapsto Y \Longrightarrow y(t) = \eta(t, x(t)), \; \forall t \in T$$



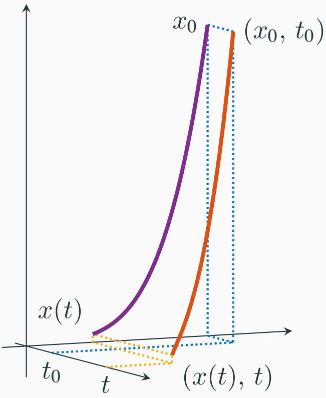

- Case 2: **non strictly proper** system:

$$\eta : T \times X \times U \mapsto Y \Longrightarrow y(t) = \eta(t, x(t), u(t)), \; \forall t \in T$$



# Dynamic Systems: Formal Definitions (cont.)

$(x, t) \in X \times T$ is defined as **event**




Given:

- $(x_0, t_0)$ initial event
- $u(\cdot)$ input function

One has:

$\varphi(\cdot, t_0, x_0, u(\cdot))$ state movement

$\varphi(t, t_0, x_0, u(\cdot)),\ t \geq t_0$ state trajectory

$\eta(\cdot, \varphi(\cdot, t_0, x_0, u(\cdot)))$ output movement

$\eta(t, \varphi(t, t_0, x_0, u(\cdot))),\ t \geq t_0$ output trajectory

# Dynamic Systems: Formal Definitions (cont.)

$\bar{x} \in X$ is an **equilibrium state** if $\forall\, t_0 \in T,\, \exists\, u(\cdot) \in \Omega$ such that

$$\varphi(t, t_0, \bar{x}, u(\cdot)) = \bar{x}\,,\ \forall\, t \geq t_0,\, t \in T$$

$\bar{y} \in Y$ is an **equilibrium output** if $\forall\, t_0 \in T,\, \exists \bar{x} \in X,\, \exists\, u(\cdot) \in \Omega$ such that

$$\eta(t, \varphi(t, t_0, \bar{x}, u(\cdot))) = \bar{y}\,,\ \forall\, t \geq t_0,\, t \in T$$

Notice that, in general:

- the specific input function $u(\cdot) \in \Omega$ depends on the choice of the initial time-instant $t_0 \in T$
- the fact that the state of a dynamic system is at equilibrium does not imply that the output is at equilibrium as well, unless $\eta(t, x(t))$ does not depend explicitly on time (in which case, the output function takes on the form $\eta(x(t))$)

# Dynamic Systems: Formal Definitions (cont.)

- A dynamic system is **invariant** if $T$ is an additive algebraic group and $\forall\, u(\cdot) \in \Omega$, $\forall\, \tau \in T$, letting $u^\tau(t) := u(t-\tau) \in \Omega$, it follows that

$$\begin{cases} \varphi(t, t_0, x_0, u(\cdot)) = \varphi(t+\tau, t_0+\tau, x_0, u^\tau(\cdot)), \ \forall\, t, \tau \in T \\ y(t) = \eta(t, x(t)) \end{cases}$$

- A dynamic system is **discrete-time** if $T$ is isomorphous with $\mathbb{Z}$
- A dynamic system is **continuous-time** if $T$ is isomorphous with $\mathbb{R}$
- A dynamic system is **finite-dimensional** (**lumped-parameter**) if $U, X, Y$ are finite-dimensional vector spaces
- A dynamic system is **infinite-dimensional** (**distributed-parameter**) if $U, X, Y$ are infinite-dimensional vector spaces

# Dynamic Systems Described by State Equations

## Interconnection of Dynamic Systems

# Interconnection of Dynamic Systems

We consider **interconnected systems**

$$\mathcal{S} = \{T, U, \Omega, X, Y, \Gamma, \varphi, \eta\}$$

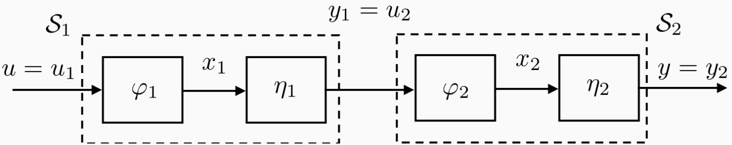
composed of $N$ subsystems

$$\mathcal{S}_i = \{T_i, U_i, \Omega_i, X_i, Y_i, \Gamma_i, \varphi_i, \eta_i\}\ ,\ \ i = 1, 2, \dots N$$

interacting with each other through their external variables such as inputs $u_i(\cdot) \in \Omega_i$ and outputs $y_i(\cdot) \in \Gamma_i$

**Assumption**. The interconnected system $\mathcal{S}$ satisfies the formal definition of dynamic system

## Cascade interconnection

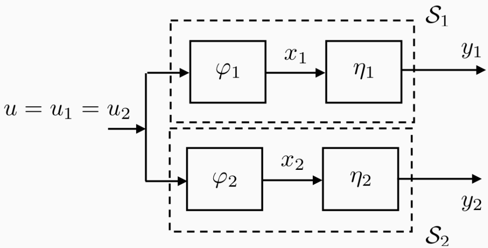



$$\mathcal{S} = \{T = T_1 = T_2, U = U_1, \Omega = \Omega_1, X = X_1 \times X_2, Y = Y_2, \Gamma = \Gamma_2\}$$

$$\begin{cases} \big(x_1(t), x_2(t)\big) = \Big(\varphi_1\big(t, t_0, x_1(t_0), u(\cdot)\big), \\ \qquad\qquad \varphi_2\Big(t, t_0, x_2(t_0), \eta_1\Big(t, \varphi_1\big(t, t_0, x_1(t_0), u(\cdot)\big)\Big)\Big)\Big) \\ y(t) = y_2(t) = \eta_2\big(t, x_2(t)\big) \end{cases}$$

# Interconnection of Dynamic Systems (cont.)

## Parallel interconnection



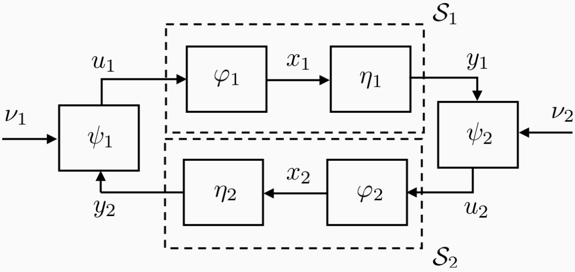

$$\mathcal{S} = \{T = T_1 = T_2, U = U_1 = U_2, \Omega = \Omega_1 = \Omega_2, X = X_1 \times X_2, Y = Y_1 \times Y_2, \\ \Gamma = \Gamma_1 \times \Gamma_2\}$$

$$\begin{cases} \big(x_1(t), x_2(t)\big) = \Big(\varphi_1\big(t, t_0, x_1(t_0), u(\cdot)\big), \varphi_2\big(t, t_0, x_2(t_0), u(\cdot)\big)\Big) \\ \\ \big(y_1(t), y_2(t)\big) = \Big(\eta_1\big(t, x_1(t)\big), \eta_2\big(t, x_2(t)\big)\Big) \end{cases}$$

# Interconnection of Dynamic Systems (cont.)

## Feedback interconnection

General scheme:



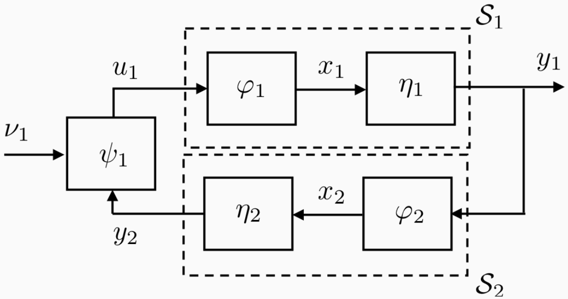

$$u_1(t) = \psi_1\Big(y_2(t), \nu_1(t), t\Big)$$

$$u_2(t) = \psi_2\Big(y_1(t), \nu_2(t), t\Big)$$

# Interconnection of Dynamic Systems (cont.)

## Feedback interconnection

Commonly used scheme:

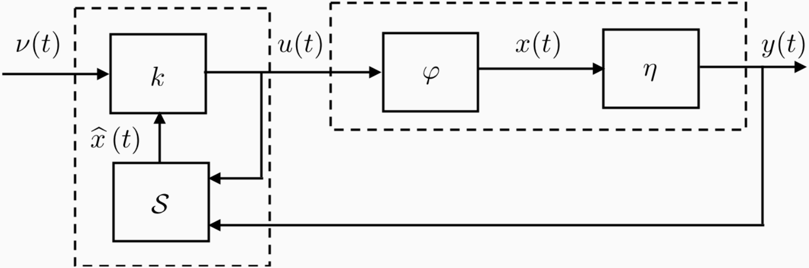



$$\mathcal{S} = \{T = T_1 = T_2, U = V_1, \Omega = \Omega_{\nu_1}, X = X_1 \times X_2, Y = Y_1, \Gamma = \Gamma_1\}$$

$$\begin{cases} \Big(x_1(t), x_2(t)\Big) = \Big(\varphi_1\big(t, t_0, x_1(t_0), \psi_1(\nu_1(\cdot), y_2(\cdot))\big), \varphi_2\big(t, t_0, x_2(t_0), y_1(\cdot)\big)\Big) \\ y(t) = y_1(t) = \eta_1\big(t, x_1(t)\big) \end{cases}$$

# Feedback Interconnection: a Notable Example

A notable example of feedback interconnection is the **state control law + state observer** scheme (will be dealt with in the *Control Theory* course)



# Dynamic Systems Described by State Equations

## Finite-dimensional Regular Systems

# Finite-dimensional Regular Systems

A dynamic systems is **regular** if:

- $U, \Omega, X, Y, \Gamma$ are normed vector spaces
- $\varphi(\cdot,\cdot,\cdot,\cdot)$ is a continuous function with respect its arguments
- $\dfrac{\mathrm{d}}{\mathrm{d}t}\varphi(t, t_0, x_0, u(\cdot))$ does exist and it is continuous for all values of the arguments where $u(\cdot)$ is continuous

The state movement $\varphi(t, t_0, x_0, u(\cdot))$ of a regular finite-dimensional dynamic system is the **unique solution** of a suitable vector differential equation

$$\begin{cases} \dot{x}(t) = f(x(t), u(t), t) \\ x(t_0) = x_0 \end{cases}$$

and

$$y(t) = g(x(t), u(t), t)$$

## Discrete-time dynamic systems obtain by sampling a continuous-time regular system

- $U, X, Y$ finite-dimensional normed vector spaces
- $\Omega = \{u(\cdot) : \text{ piecewise constant } u_i(\cdot)\,,\ i = 1, \ldots, m\}$
- Sampling time $\Delta T$:

$$u(k) = u(t)\,,\ t_0 + k\Delta T \le t < t_0 + (k+1)\Delta T,\ k = 0, 1, \ldots$$
$$y(k) = y(t_0 + k\Delta T),\ k = 0, 1, \ldots$$

Then:

$$\begin{cases} x(k+1) = f_d(x(k), u(k), k) \\ y(k) = g_d(x(k), u(k), k) \end{cases}$$

where (from composition property of $\varphi$):

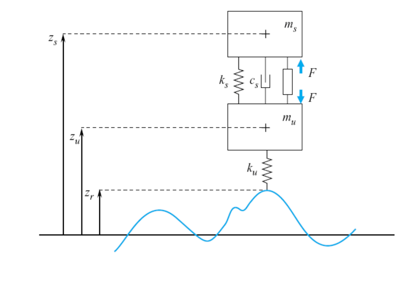
$$f_d(x(k), u(k), k) = \varphi(t_0 + (k+1)\Delta T, t_0 + k\Delta T, x(k), u(k))$$
$$g_d(x(k), u(k), k) = \eta(x(k), u(k), t_0 + k\Delta T)$$

# Dynamic Systems Described by State Equations

## An Example

# An Example: Continuous-Time Model of a Car Suspension


From a real vehicle ...

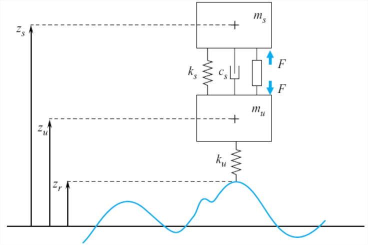



to a simplified *quarter-car model*

## *quarter-car* model hypotheses

- vehicle as assembly of four decoupled parts
- each part consists of
  - the *sprung mass*: a quarter of the vehicle mass, supported by a suspension actuator, placed between the vehicle and the tyre
  - the *unsprung mass*: the wheel/tyre sub-assembly
- the model allows only for vertical motion: the vehicle is moving forward with an almost constant speed

# Continuous-Time Model of a Car Suspension (cont.)




- state variables:
  - vertical positions of sprung and unsprung masses vs. the corresponding steady-state values
  - vertical speeds of masses
- inputs:
  - ground vertical position vs. the steady-state
  - active actuator force
- outputs:
  - sprung mass vertical acceleration
  - contact force between tyre and ground

$$
\begin{cases}
x_1(t) = z_s(t) - \bar{z}_s \\
x_2(t) = z_u(t) - \bar{z}_u \\
x_3(t) = \dot{x}_1(t) \\
x_4(t) = \dot{x}_2(t) \\
u_1(t) = z_r(t) - \bar{z}_r \\
u_2(t) = F(t) \\
y_1(t) = \ddot{x}_1 \\
y_2(t) = k_u\left(x_2(t) - u_1(t)\right)
\end{cases}
$$

# Continuous-Time Model of a Car Suspension (cont.)

$$
\begin{cases}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_s}{m_s} & \frac{k_s}{m_s} & -\frac{c_s}{m_s} & \frac{c_s}{m_s} \\ \frac{k_s}{m_u} & -\frac{k_s + k_u}{m_u} & \frac{c_s}{m_u} & -\frac{c_s}{m_u} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_s} \\ \frac{k_s}{m_u} & -\frac{1}{m_u} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -\frac{k_s}{m_s} & \frac{k_s}{m_s} & -\frac{c_s}{m_s} & \frac{c_s}{m_s} \\ 0 & k_u & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{m_s} \\ -k_u & 0 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
\end{cases}
$$

# Continuous-Time Car Suspension: an Example

Assuming

$$m_s = 400.0 \text{ kg} \qquad m_u = 50.0 \text{ kg} \qquad c_s = 2.0\,10^3 \text{ N s m}^{-1}$$
$$k_s = 2.0\,10^4 \text{ N m}^{-1} \qquad k_u = 2.5\,10^5 \text{ N m}^{-1}$$

the car suspension model becomes

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1.0 \\ -50.0 & 50.0 & -5.0 & 5.0 \\ 400.0 & -5400.0 & 40.0 & -40.0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 2.5\,10^{-3} \\ 5.0\,10^3 & -2.0\,10^{-2} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -50.0 & 50.0 & -5.0 & 5.0 \\ 0 & 2.5\,10^5 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 2.5\,10^{-3} \\ -2.5\,10^5 & 0 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \end{cases}$$

# Sampled-Time Car Suspension Models

Let's get a **sampled-time** description of the same dynamic system:

- How does the sampled-time description correlate with the continuous-time model?
- What happens if we increase or decrease the sampling rate? Does the sampled-time model change with the sampling time?
- Does the sampled-time model describe the behaviour of the continuous-time dynamic system for **any possible choice** of the sampling time value?

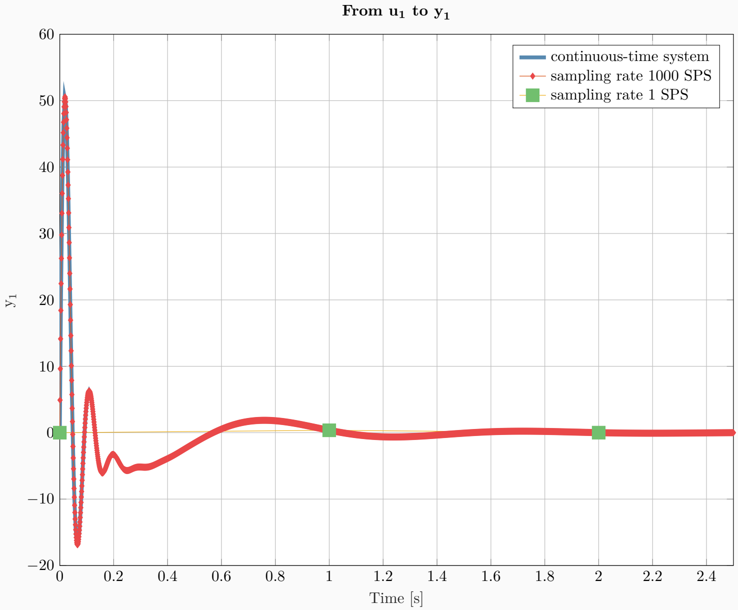
# Using *1000 samples per second* as sampling rate

$$
\begin{cases}
\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{bmatrix} = \begin{bmatrix} 9.98\,10^{-1} & 2.05 \cdot 10^{-5} & 9.98 \cdot 10^{-4} & 2.47 \cdot 10^{-6} \\ 1.97 \cdot 10^{-4} & 0.99 & 1.98 \cdot 10^{-5} & 9.80 \cdot 10^{-4} \\ -4.89 \cdot 10^{-2} & 3.65 \cdot 10^{-3} & 9.95 \cdot 10^{-1} & 4.91 \cdot 10^{-3} \\ 3.91 \cdot 10^{-1} & -5.29 & 3.93 \cdot 10^{-2} & 0.96 \end{bmatrix} \cdot \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix} \\
\qquad + \begin{bmatrix} 4.13 \cdot 10^{-6} & 1.23 \cdot 10^{-9} \\ 2.47 \cdot 10^{-3} & -9.85 \cdot 10^{-9} \\ 1.24 \cdot 10^{-2} & 2.44 \cdot 10^{-6} \\ 4.90 & -1.95 \cdot 10^{-5} \end{bmatrix} \cdot \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix} \\
\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} -50.0 & 50.0 & -5.0 & 5.0 \\ 0 & 2.5 \cdot 10^5 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k \end{bmatrix} \\
\qquad + \begin{bmatrix} 0 & 2.5 \cdot 10^{-3} \\ -2.5 \cdot 10^5 & 0 \end{bmatrix} \cdot \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}
\end{cases}
$$

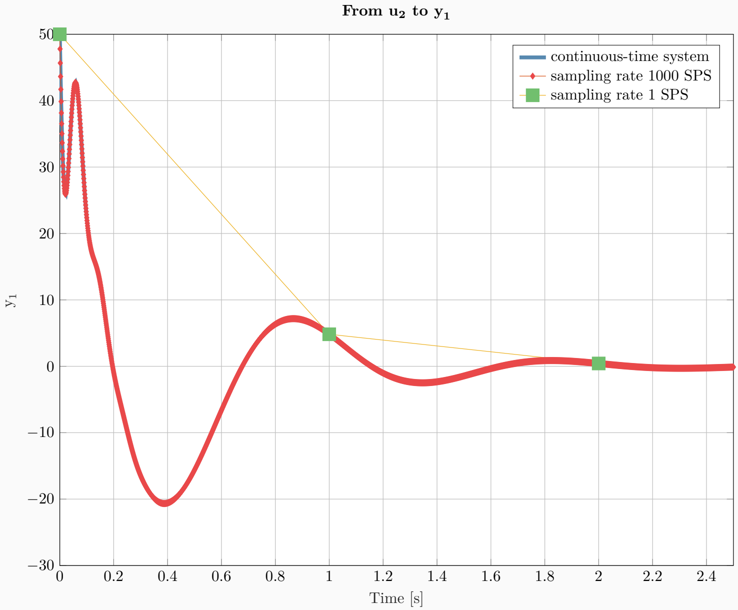
# Instead, using *1 sample per second* as sampling rate

$$
\begin{cases}
\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{bmatrix} = \begin{bmatrix} 1.17 \cdot 10^{-1} & -1.76 \cdot 10^{-2} & 4.65 \cdot 10^{-3} & 1.34 \cdot 10^{-4} \\ 7.75 \cdot 10^{-3} & -4.87 \cdot 10^{-3} & 1.07 \cdot 10^{-3} & 1.29 \cdot 10^{-5} \\ -1.79 \cdot 10^{-1} & -4.90 \cdot 10^{-1} & 9.94 \cdot 10^{-2} & 3.64 \cdot 10^{-4} \\ -4.84 \cdot 10^{-2} & -1.62 \cdot 10^{-2} & 2.91 \cdot 10^{-3} & -2.95 \cdot 10^{-5} \end{bmatrix} \cdot \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix} \\
\qquad + \begin{bmatrix} 9.00 \cdot 10^{-1} & 4.41 \cdot 10^{-5} \\ 9.97 \cdot 10^{-1} & -3.88 \cdot 10^{-7} \\ 6.70 \cdot 10^{-1} & 8.96 \cdot 10^{-6} \\ 6.46 \cdot 10^{-2} & 2.42 \cdot 10^{-6} \end{bmatrix} \cdot \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix} \\
\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} -50.0 & 50.0 & -5.0 & 5.0 \\ 0 & 2.5 \cdot 10^5 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k \end{bmatrix} \\
\qquad + \begin{bmatrix} 0 & 2.5 \cdot 10^{-3} \\ -2.5 \cdot 10^5 & 0 \end{bmatrix} \cdot \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}
\end{cases}
$$

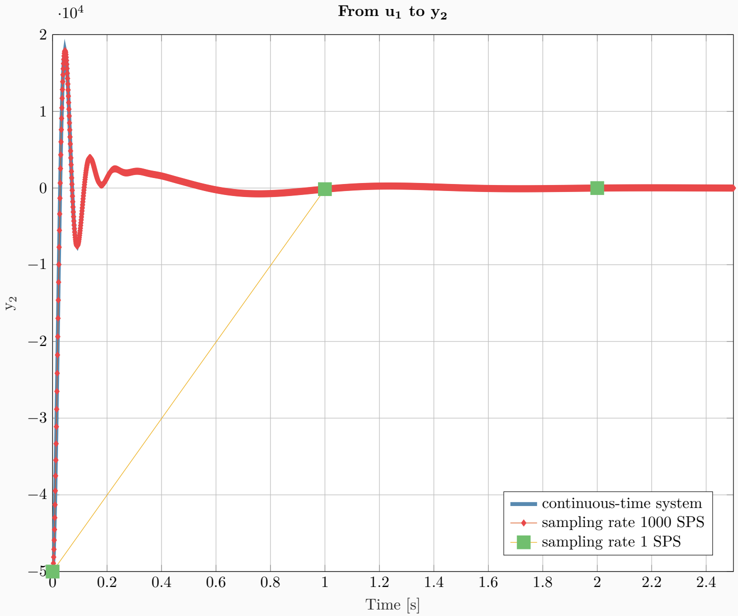
# Step Responses Comparison



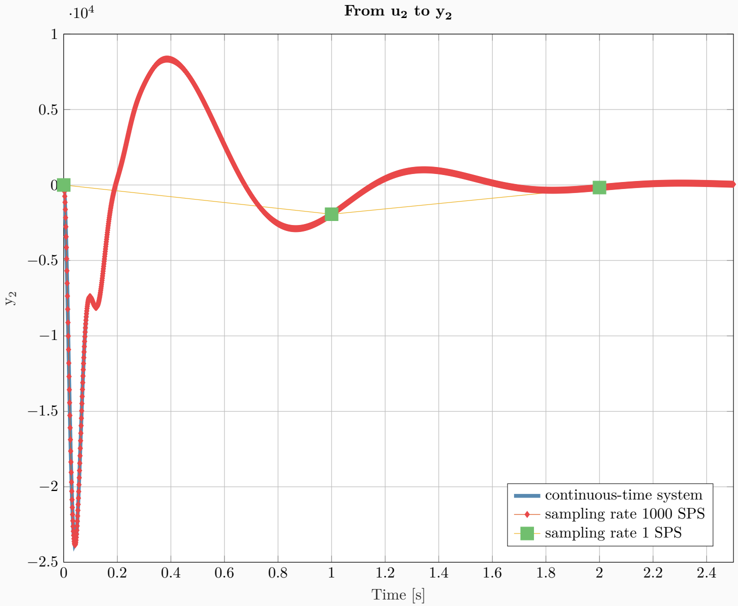
# Step Responses Comparison (cont.)



# Step Responses Comparison (cont.)



# Step Responses Comparison (cont.)



# Sampled-Time Car Suspension Description (cont.)

### Remarks

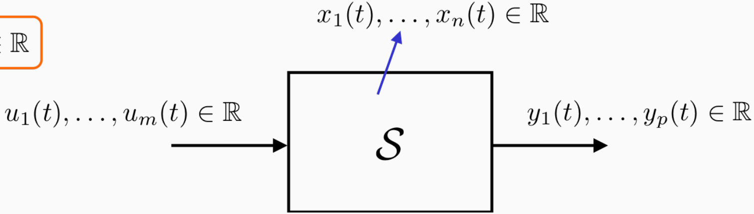
- by selecting **different sampling rate** we obtained **different representations** of the same continuous-time dynamic system
- **sampling** may **heavily distort the information**, giving a completely wrong discrete-time representation of the original continuous-time system: indeed the model obtained using *one sample per second* as the sampling rate is wrong!

# Dynamic Systems Described by State Equations

## Continuous-time State Equations

# Continuous-time State Equations

$\forall t \in \mathbb{R}$




State equations (dynamic)

$$\begin{cases} \dot{x}_1(t) = f_1(x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t), t) \\ \quad \vdots \\ \dot{x}_n(t) = f_n(x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t), t) \end{cases}$$

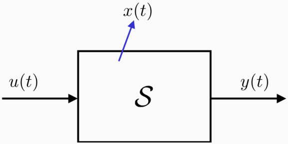
Output equations (algebraic)

$$\begin{cases} y_1(t) = g_1(x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t), t) \\ \quad \vdots \\ y_p(t) = g_p(x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t), t) \end{cases}$$

# Continuous-time State Equations (cont.)

$$u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix} \in \mathbb{R}^m, \quad y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_p(t) \end{bmatrix} \in \mathbb{R}^p$$

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \in \mathbb{R}^n$$




$$f(x,u,t) = \begin{bmatrix} f_1(x,u,t) \\ \vdots \\ f_n(x,u,t) \end{bmatrix} \in \mathbb{R}^n$$

$$g(x,u,t) = \begin{bmatrix} g_1(x,u,t) \\ \vdots \\ g_p(x,u,t) \end{bmatrix} \in \mathbb{R}^p$$

**Compact form**

$$\begin{cases} \dot{x}(t) = f(x(t), u(t), t) \\ y(t) = g(x(t), u(t), t) \end{cases}$$

# Linear Dynamic Systems

Consider the continuous-time dynamic system state-space representation:

$$\begin{cases} \dot{x}(t) = f(x(t), u(t), t) \\ y(t) = g(x(t), u(t), t) \end{cases}$$

This state-space equation describes a **linear system** if and only if the functions $f(\cdot)$ and $g(\cdot)$ are **linear with respect to their state and input vector arguments**:

$$\forall\, \alpha_1, \alpha_2 \in \mathbb{R},\, \forall\, x_1, x_2 \in \mathbb{R}^n,\, \forall\, u_1, u_2 \in \mathbb{R}^m \;:$$

$$\begin{aligned} f(\alpha_1 x_1 + \alpha_2 x_2, \alpha_1 u_1 + \alpha_2 u_2, t) &= \alpha_1 f(x_1, u_1, t) + \alpha_2 f(x_2, u_2, t) \\ g(\alpha_1 x_1 + \alpha_2 x_2, \alpha_1 u_1 + \alpha_2 u_2, t) &= \alpha_1 g(x_1, u_1, t) + \alpha_2 g(x_2, u_2, t) \end{aligned}$$

# Linear Dynamic Systems: Matrix Form

Consider the state-space representation:

$$\begin{cases} \dot{x}(t) = f(x(t), u(t), t) \\ y(t) = g(x(t), u(t), t) \end{cases}$$

and suppose that the linearity assumption holds. Then:

$$\begin{cases} f_1(x,u,t) = a_{11}(t)x_1 + \cdots + a_{1n}(t)x_n + b_{11}(t)u_1 + \cdots + b_{1m}(t)u_m \\ \vdots \\ f_n(x,u,t) = a_{n1}(t)x_1 + \cdots + a_{nn}(t)x_n + b_{n1}(t)u_1 + \cdots + b_{nm}(t)u_m \\ \\ y_1 = c_{11}(t)x_1 + \cdots + c_{1n}(t)x_n + d_{11}(t)u_1 + \cdots + d_{1m}(t)u_m \\ \vdots \\ y_p = c_{p1}(t)x_1 + \cdots + c_{pn}(t)x_n + d_{p1}(t)u_1 + \cdots + d_{pm}(t)u_m \end{cases}$$

where $a_{ij}(t), b_{ij}(t), c_{ij}(t), d_{ij}(t)$ are generic functions of the time instant $t$.

# Linear Dynamic Systems: Matrix Form (cont.)

Letting:

$$A(t) := \begin{bmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & \ddots & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{bmatrix} \; ; \quad B(t) := \begin{bmatrix} b_{11}(t) & \cdots & b_{1m}(t) \\ \vdots & \vdots & \vdots \\ b_{n1}(t) & \cdots & b_{nm}(t) \end{bmatrix}$$

$$C(t) := \begin{bmatrix} c_{11}(t) & \cdots & c_{1n}(t) \\ \vdots & \ddots & \vdots \\ c_{p1}(t) & \cdots & c_{pn}(t) \end{bmatrix} \; ; \quad D(t) := \begin{bmatrix} d_{11}(t) & \cdots & d_{1m}(t) \\ \vdots & \vdots & \vdots \\ d_{p1}(t) & \cdots & d_{pm}(t) \end{bmatrix}$$

$$x(t) := \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \; ; \quad u(t) := \begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix} \; ; \quad y(t) := \begin{bmatrix} y_1(t) \\ \vdots \\ y_p(t) \end{bmatrix}$$

One gets:

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ \\ y(t) = C(t)x(t) + D(t)u(t) \end{cases}$$

# Time-Invariant Linear Dynamic Systems

In the **time-invariant** scenario, the matrices $A(t), B(t), C(t), D(t)$ do not depend on the time-index $k$, that is are **constant** matrices $A, B, C, D$:

$$A := \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} ; \quad B := \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \vdots & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix}$$

$$C := \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{p1} & \cdots & c_{pn} \end{bmatrix} ; \quad D := \begin{bmatrix} d_{11} & \cdots & d_{1m} \\ \vdots & \vdots & \vdots \\ d_{p1} & \cdots & d_{pm} \end{bmatrix}$$

and thus:

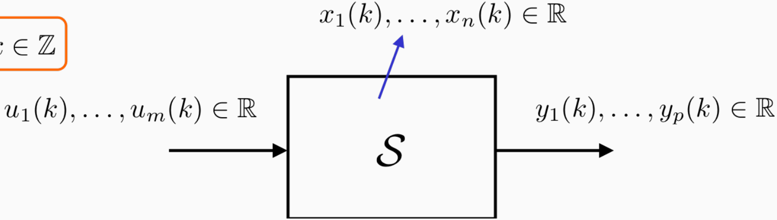
$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ \\ y(t) = Cx(t) + Du(t) \end{cases}$$

# Dynamic Systems Described by State Equations

## Discrete-time State Equations

# Discrete-time State Equations

$\forall\, k \in \mathbb{Z}$




State equations (dynamic)

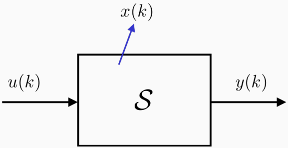
$$\begin{cases} x_1(k+1) = f_1(x_1(k), \ldots, x_n(k), u_1(k), \ldots, u_m(k), k) \\ \quad \vdots \\ x_n(k+1) = f_n(x_1(k), \ldots, x_n(k), u_1(k), \ldots, u_m(k), k) \end{cases}$$

Output equations (algebraic)

$$\begin{cases} y_1(k) = g_1(x_1(k), \ldots, x_n(k), u_1(k), \ldots, u_m(k), k) \\ \quad \vdots \\ y_p(k) = g_p(x_1(k), \ldots, x_n(k), u_1(k), \ldots, u_m(k), k) \end{cases}$$

# Discrete-time State Equations (cont.)

$$u(k) = \begin{bmatrix} u_1(k) \\ \vdots \\ u_m(k) \end{bmatrix} \in \mathbb{R}^m, \quad y(k) = \begin{bmatrix} y_1(k) \\ \vdots \\ y_p(k) \end{bmatrix} \in \mathbb{R}^p$$




$$x(k) = \begin{bmatrix} x_1(k) \\ \vdots \\ x_n(k) \end{bmatrix} \in \mathbb{R}^n$$

$$f(x, u, k) = \begin{bmatrix} f_1(x, u, k) \\ \vdots \\ f_n(x, u, k) \end{bmatrix} \in \mathbb{R}^n$$

$$g(x, u, k) = \begin{bmatrix} g_1(x, u, k) \\ \vdots \\ g_p(x, u, k) \end{bmatrix} \in \mathbb{R}^p$$

**Compact form**

$$\begin{cases} x(k+1) = f(x(k), u(k), k) \\ y(k) = g(x(k), u(k), k) \end{cases}$$

# Linear Dynamic Systems

Consider the discrete-time dynamic system state-space representation:

$$\begin{cases} x(k+1) = f(x(k), u(k), k) \\ y(k) = g(x(k), u(k), k) \end{cases}$$

This state-space equation describes a **linear system** if and only if the functions $f(\cdot)$ and $g(\cdot)$ are **linear with respect to their state and input vector arguments**:

$$\forall\, \alpha_1, \alpha_2 \in \mathbb{R},\ \forall\, x_1, x_2 \in \mathbb{R}^n,\ \forall\, u_1, u_2 \in \mathbb{R}^m \ :$$

$$f(\alpha_1 x_1 + \alpha_2 x_2, \alpha_1 u_1 + \alpha_2 u_2, k) = \alpha_1 f(x_1, u_1, k) + \alpha_2 f(x_2, u_2, k)$$
$$g(\alpha_1 x_1 + \alpha_2 x_2, \alpha_1 u_1 + \alpha_2 u_2, k) = \alpha_1 g(x_1, u_1, k) + \alpha_2 g(x_2, u_2, k)$$

# Linear Dynamic Systems: Matrix Form

Consider the state-space representation:

$$\begin{cases} x(k+1) = f(x(k), u(k), k) \\ y(k) = g(x(k), u(k), k) \end{cases}$$

and suppose that the linearity assumption holds. Then:

$$\begin{cases} f_1(x,u,k) = a_{11}(k)x_1 + \cdots + a_{1n}(k)x_n + b_{11}(k)u_1 + \cdots + b_{1m}(k)u_m \\ \vdots \\ f_n(x,u,k) = a_{n1}(k)x_1 + \cdots + a_{nn}(k)x_n + b_{n1}(k)u_1 + \cdots + b_{nm}(k)u_m \\ \\ y_1 = c_{11}(k)x_1 + \cdots + c_{1n}(k)x_n + d_{11}(k)u_1 + \cdots + d_{1m}(k)u_m \\ \vdots \\ y_p = c_{p1}(k)x_1 + \cdots + c_{pn}(k)x_n + d_{p1}(k)u_1 + \cdots + d_{pm}(k)u_m \end{cases}$$

where $a_{ij}(k), b_{ij}(k), c_{ij}(k), d_{ij}(k)$ are generic functions of the discrete-time index $k$.

# Linear Dynamic Systems: Matrix Form (cont.)

Letting:

$$A(k) := \begin{bmatrix} a_{11}(k) & \cdots & a_{1n}(k) \\ \vdots & \ddots & \vdots \\ a_{n1}(k) & \cdots & a_{nn}(k) \end{bmatrix} ; \quad B(k) := \begin{bmatrix} b_{11}(k) & \cdots & b_{1m}(k) \\ \vdots & \vdots & \vdots \\ b_{n1}(k) & \cdots & b_{nm}(k) \end{bmatrix}$$

$$C(k) := \begin{bmatrix} c_{11}(k) & \cdots & c_{1n}(k) \\ \vdots & \ddots & \vdots \\ c_{p1}(k) & \cdots & c_{pn}(k) \end{bmatrix} ; \quad D(k) := \begin{bmatrix} d_{11}(k) & \cdots & d_{1m}(k) \\ \vdots & \vdots & \vdots \\ d_{p1}(k) & \cdots & d_{pm}(k) \end{bmatrix}$$

$$x(k) := \begin{bmatrix} x_1(k) \\ \vdots \\ x_n(k) \end{bmatrix} ; \quad u(k) := \begin{bmatrix} u_1(k) \\ \vdots \\ u_m(k) \end{bmatrix} ; \quad y(k) := \begin{bmatrix} y_1(k) \\ \vdots \\ y_p(k) \end{bmatrix}$$

One gets:

$$\begin{cases} x(k+1) = A(k)x(k) + B(k)u(k) \\ y(k) = C(k)x(k) + D(k)u(k) \end{cases}$$

# Time-Invariant Linear Dynamic Systems

In the **time-invariant** scenario, the matrices $A(k), B(k), C(k), D(k)$ do not depend on the time-index $k$, that is are **constant** matrices $A, B, C, D$:

$$A := \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \; ; \quad B := \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \vdots & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix}$$

$$C := \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{p1} & \cdots & c_{pn} \end{bmatrix} \; ; \quad D := \begin{bmatrix} d_{11} & \cdots & d_{1m} \\ \vdots & \vdots & \vdots \\ d_{p1} & \cdots & d_{pm} \end{bmatrix}$$

and thus:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ \\ y(k) = Cx(k) + Du(k) \end{cases}$$

# Dynamic Systems Described by State Equations

## More Definitions and Properties

# More Definitions and Properties

- **Time-invariant Dynamic Systems**

$$\begin{cases} \dot{x}(t) = f(x(t), u(t), \cancel{t}) \\ y(t) = g(x(t), u(t), \cancel{t}) \end{cases} \implies \begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = g(x(t), u(t)) \end{cases}$$

$$\begin{cases} x(k+1) = f(x(k), u(k), \cancel{k}) \\ y(k) = g(x(k), u(k), \cancel{k}) \end{cases} \implies \begin{cases} x(k+1) = f(x(k), u(k)) \\ y(k) = g(x(k), u(k)) \end{cases}$$

- **Strictly Proper Dynamic Systems**

$$\begin{cases} \dot{x}(t) = f(x(t), u(t), t) \\ y(t) = g(x(t), \cancel{u(t)}, t) \end{cases} \implies \begin{cases} \dot{x}(t) = f(x(t), u(t), t) \\ y(t) = g(x(t), t) \end{cases}$$

$$\begin{cases} x(k+1) = f(x(k), u(k), k) \\ y(k) = g(x(k), \cancel{u(k)}, k) \end{cases} \implies \begin{cases} x(k+1) = f(x(k), u(k), k) \\ y(k) = g(x(k), k) \end{cases}$$

# More Definitions and Properties (cont.)

- **Forced and Free Dynamic Systems**

$$\begin{cases} \dot{x}(t) = f(x(t),\ \xcancel{u(t)}\ ,t) \\ y(t) = g(x(t),\ \xcancel{u(t)}\ ,t) \end{cases} \implies \begin{cases} \dot{x}(t) = f(x(t),t) \\ y(t) = g(x(t),t) \end{cases}$$

$$\begin{cases} x(k+1) = f(x(k),\ \xcancel{u(k)}\ ,k) \\ y(k) = g(x(k),\ \xcancel{u(k)}\ ,k) \end{cases} \implies \begin{cases} x(k+1) = f(x(k),k) \\ y(k) = g(x(k),k) \end{cases}$$

It is worth noting that in case the input function $u(t),\ \forall\, t$ or input sequence $u(k),\ \forall\, k$ are **known beforehand**, the dynamic system can be re-written as a free one:

$$\begin{cases} \dot{x}(t) = f(x(t),u(t),t) = \widetilde{f}(x(t),t) \\ y(t) = g(x(t),u(t),t) = \widetilde{g}(x(t),t) \end{cases}$$

$$\begin{cases} x(k+1) = f(x(k),u(k),k) = \widetilde{f}(x(k),k) \\ y(k) = g(x(k),u(k),k) = \widetilde{g}(x(k),k) \end{cases}$$

# More Definitions and Properties (cont.)

- **Free Movement**

$$\begin{aligned}&\dot{x}(t) = f(x(t), u(t), t)\\ &y(t) = g(x(t), u(t), t)\\ &\text{with:}\\ &\quad x(t_0) = x_0\,;\;\; u(t) = 0\,,\;\; \forall\, t\end{aligned} \qquad \Longrightarrow \qquad \begin{gathered}\{\, (x_l(t), t),\; t \in [t_0, t_1]\,\}\\ \text{free movement}\end{gathered}$$

$$\begin{aligned}&x(k+1) = f(x(k), u(k), k)\\ &y(k) = g(x(k), u(k), k)\\ &\text{with:}\\ &\quad x(k_0) = x_0\,;\;\; u(k) = 0\,,\;\; \forall\, k\end{aligned} \qquad \Longrightarrow \qquad \begin{gathered}\{\, (x_l(k), k),\; k \in [k_0, k_1]\,\}\\ \text{free movement}\end{gathered}$$

# More Definitions and Properties (cont.)

- **Forced Movement**

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), t) \\ y(t) &= g(x(t), u(t), t) \\ \text{with:} & \\ x(t_0) &= 0 \end{aligned} \quad \Longrightarrow \quad \begin{gathered} \{\, (x_f(t), t),\ t \in [t_0, t_1] \,\} \\ \text{forced movement} \end{gathered}$$

$$\begin{aligned} x(k+1) &= f(x(k), u(k), k) \\ y(k) &= g(x(k), u(k), k) \\ \text{with:} & \\ x(k_0) &= 0 \end{aligned} \quad \Longrightarrow \quad \begin{gathered} \{\, (x_f(k), k),\ k \in [k_0, k_1] \,\} \\ \text{forced movement} \end{gathered}$$

# Dynamic Systems Described by State Equations

## Discrete-time Systems

# Discrete-time Systems

Consider:

$$\begin{array}{l} x(k+1) = f(x(k), u(k), k) \\ y(k) = g(x(k), u(k), k) \end{array} \quad , \quad k > k_0,\ x(k_0) = x_0$$

Clearly, by iterating the state equations:

$$\begin{aligned}
x(k_0) \quad &= x_0 \\
x(k_0+1) \quad &= f(x(k_0), u(k_0), k_0) \\
x(k_0+2) \quad &= f(x(k_0+1), u(k_0+1), k_0+1) \\
&= f(f(x(k_0), u(k_0), k_0), u(k_0+1), k_0+1) \\
x(k_0+3) \quad &= f(x(k_0+2), u(k_0+2), k_0+2) \\
&= f(f(f(x(k_0), u(k_0), k_0), u(k_0+1), k_0+1), u(k_0+2), k_0+2)
\end{aligned}$$

and so on. Hence, the **state transition function** has the form

$$x(k) = \varphi(k, k_0, x_0, \{u(k_0), \ldots, u(k-1)\})$$

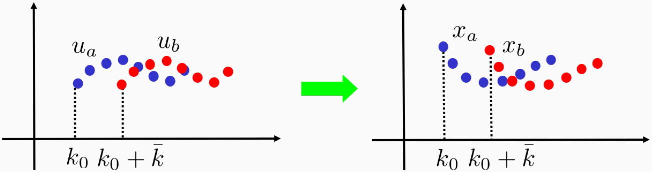
thus enhancing the **causality property**.

# Time-invariant Discrete-time Systems

$$\begin{aligned} x(k+1) &= f(x(k), u(k)) \\ y(k) &= g(x(k), u(k)) \end{aligned} \quad , \; x(k_0) = x_0, \; u_a(k) = u(k), \; k \in \{k_0, \dots, k_1\}$$

yields the state sequence $x_a(k), \; k \in \{k_0, \dots, k_1\}$. Let's shift the initial time by $\bar{k}$ and the input sequence as well:

$$\begin{aligned} & x(k_0 + \bar{k}) = x_0 \\ & u_b(k) = u_a(k - \bar{k}), \\ & \quad k \in \{k_0 + \bar{k}, \dots, k_1 + \bar{k}\} \end{aligned} \quad \Longrightarrow \quad \begin{aligned} & x_b(k) = x_a(k - \bar{k}), \\ & \quad k \in \{k_0 + \bar{k}, \dots, k_1 + \bar{k}\} \end{aligned}$$




**Conventionally**, we set $k_0 = 0$.

# Equilibrium Analysis: Equilibrium States and Outputs

- A state $\bar{x} \in \mathbb{R}^n$ is an **equilibrium state** if $\forall\, k_0$, $\exists\, \{\bar{u}(k) \in \mathbb{R}^m, k \geq k_0\}$ such that

$$\begin{aligned} x(k_0) &= \bar{x} \\ u(k) &= \bar{u}(k),\ \forall\, k \geq k_0 \end{aligned} \implies x(k) = \bar{x},\ \forall\, k > k_0$$

- An output $\bar{y} \in \mathbb{R}^p$ is an **equilibrium output** if $\forall\, k_0$, $\exists\, \{\bar{u}(k) \in \mathbb{R}^m, k \geq k_0\}$ such that

$$\begin{aligned} x(k_0) &= \bar{x} \\ u(k) &= \bar{u}(k),\ \forall\, k \geq k_0 \end{aligned} \implies y(k) = \bar{y},\ \forall\, k > k_0$$

In general:

- The input sequence $\{\bar{u}(k) \in \mathbb{R}^m, k \geq k_0\}$ depends on the initial time $k_0$
- The fact that the state is of equilibrium does **not** imply that the corresponding output coincides with an equilibrium output

# Equilibrium Analysis in the Time-invariant Case

In the time-invariant case, **all equilibrium states** can be determined by imposing **constant** input sequences.

A state $\bar{x} \in \mathbb{R}^n$ is an equilibrium state if $\exists\, \bar{u} \in \mathbb{R}^m$ such that

$$\begin{matrix} x(k_0) = \bar{x} \\ u(k) = \bar{u},\, \forall\, k \geq k_0 \end{matrix} \implies x(k) = \bar{x},\, \forall\, k > k_0$$

All equilibrium states $\bar{x} \in \mathbb{R}^n$ can thus be obtained by finding all solutions of the algebraic equation

$$\bar{x} = f(\bar{x}, \bar{u})\,, \quad \forall\, \bar{u} \in \mathbb{R}^m$$

The following sets are also introduced:

$$\begin{aligned} \bar{X}_{\bar{u}} &= \{\bar{x} \in \mathbb{R}^n :\ \bar{x} = f(\bar{x}, \bar{u})\} \\ \bar{X} &= \{\bar{x} \in \mathbb{R}^n :\ \exists\, \bar{u} \in \mathbb{R}^m \text{ such that } \bar{x} = f(\bar{x}, \bar{u})\} \end{aligned}$$

# Dynamic Systems Described by State Equations

## State Space Description: Criteria and Examples

**But … How to determine a state space description?**

Recall:

**State variables**

Variables to be known at time $t = t_0$ in order to be able to determine the output $y(t)$, $t \geq t_0$ from the knowledge of the input $u(t)$, $t \geq t_0$:

$$x_i(t),\ i = 1, 2, \dots, n \qquad \text{(state variables)}$$

# State Space Descriptions (cont.)

**A ”physical” criterion**

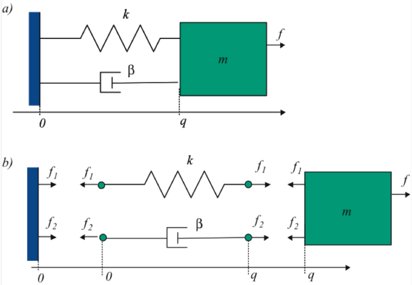
State variables can be defined as entities associated with storage of mass, energy, etc. . . .

For example:

- **Passive electrical systems**: voltages on capacitors, currents on inductors
- **Translational mechanical systems**: linear displacements and velocities of each independent mass
- **Rotational mechanical systems**: angular displacements and velocities of each independent inertial rotating mass
- **Hydraulic systems**: pressure or level of fluids in tanks
- **Thermal systems**: temperatures
- . . .

# State Space Descriptions: Example 1 (continuous-time)

## A mechanical system

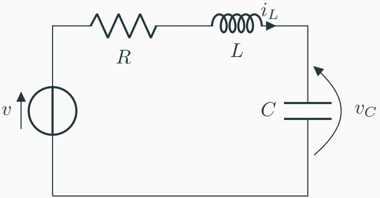


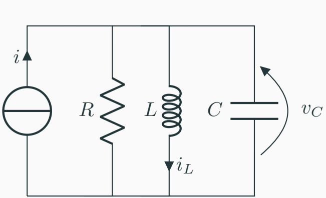

$$m\ddot{q} + \beta\dot{q} + kq = f$$

$$\begin{aligned} x_1 &:= q \\ x_2 &:= \dot{q} \end{aligned} \quad \Longrightarrow \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \quad \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \ddot{q} = -\dfrac{k}{m}x_1 - \dfrac{\beta}{m}x_2 + \dfrac{1}{m}f \end{cases}$$

# State Space Descriptions: Example 2 (continuous-time)

## Electrical systems




$$L\frac{di_L}{dt} = v - Ri_L - v_C$$

$$C\frac{dv_C}{dt} = i - \frac{1}{R}v_C - i_L$$

$$C\frac{dv_C}{dt} = i_L$$

$$L\frac{di_L}{dt} = v_C$$

$$x_1 := i_L\,;\;\; x_2 := v_C$$

$$\begin{cases} \dot{x}_1 = -\frac{R}{L}x_1 - \frac{1}{L}x_2 + \frac{1}{L}v \\ \dot{x}_2 = \frac{1}{C}x_1 \end{cases}$$

$$\begin{cases} \dot{x}_1 = \frac{1}{L}x_2 \\ \dot{x}_2 = -\frac{1}{C}x_1 - \frac{1}{RC}x_2 + \frac{1}{C}iv \end{cases}$$

# State Space Descriptions: Example 3 (discrete-time)

**Student dynamics: 3-years undergraduate course**

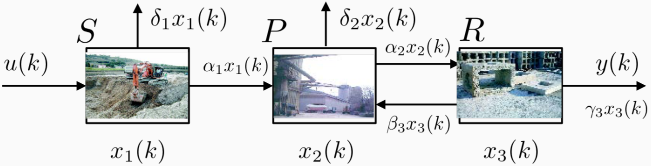
- percentages of students promoted, repeaters, and dropouts are roughly constant
- direct enrolment in 2nd and 3rd academic year is not allowed
- students cannot enrol for more than 3 years

$$\begin{cases} x_1(k+1) = \beta_1 x_1(k) + u(k) \\ x_2(k+1) = \alpha_1 x_1(k) + \beta_2 x_2(k) \\ x_3(k+1) = \alpha_2 x_2(k) + \beta_3 x_3(k) \\ y(k) = \alpha_3 x_3(k) \end{cases}$$

- $x_i(k)$: number of students enrolled in year $i$ at year $k$, $i = 1, 2, 3$
- $u(k)$: number of freshmen at year $k$
- $y(k)$: number of graduates at year $k$
- $\alpha_i$: promotion rate during year $i$, $\alpha_i \in [0, 1]$
- $\beta_i$: failure rate during year $i$, $\beta_i \in [0, 1]$
- $\gamma_i$: dropout rate during year $i$, $\gamma_i = 1 - \alpha_i - \beta_i \geq 0$

# State Space Descriptions: Example 4 (discrete-time)

## Supply chain




- $S$ purchases the quantity $u(k)$ of raw material at each month $k$
- A fraction $\delta_1$ of raw material is discarded, a fraction $\alpha_1$ is shipped to producer $P$
- A fraction $\alpha_2$ of product is sold by $P$ to retailer $R$, a fraction $\delta_2$ is discarded
- Retailer $R$ returns a fraction $\beta_3$ of defective products every month, and sells a fraction $\gamma_3$ to customers

# State Space Descriptions: Example 4 (discrete-time) (cont.)

$$\begin{cases} x_1(k+1) = (1-\alpha_1-\delta_1)x_1(k) + u(k) \\ x_2(k+1) = \alpha_1 x_1(k) + (1-\alpha_2-\delta_2)x_2(k) \\ \qquad\qquad\qquad +\beta_3 x_3(k) \\ x_3(k+1) = \alpha_2 x_2(k) + (1-\beta_3-\gamma_3)x_3(k) \\ y(k) = \gamma_3 x_3(k) \end{cases}$$

- $k$: month counter
- $x_1(k)$: raw material in $S$
- $x_2(k)$: products in $P$
- $x_3(k)$: products in $R$
- $y(k)$: products sold to customers

# State Space Descriptions (cont.)

## A ”mathematical” criterion

- **Continuous-time case**. An input-out differential equation model of the system is available:

$$\frac{\mathrm{d}^n y}{\mathrm{d}t^n} = \varphi\left(\frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}}, \ldots, \frac{\mathrm{d}y}{\mathrm{d}t}, y, u, t\right)$$

- **Discrete-time case**. An input-out difference equation model of the system is available:

$$y(k+n) = \varphi\left(y(k+n-1), y(k+n-2), \ldots, y(k), u(k), k\right)$$

Suitable state variables – without necessarily a physical meaning – are **defined** to represent ”mathematically” the differential equation or the difference equation models of the dynamic system

# State Space Descriptions (cont.)

**Continuous-time case**:

$$\frac{\mathrm{d}^n y}{\mathrm{d}t^n} = \varphi\left(\frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}}, \dots, \frac{\mathrm{d}y}{\mathrm{d}t}, y, u, t\right)$$

Letting:

$$\begin{cases} x_1(t) := y(t) \\ x_2(t) := \dfrac{\mathrm{d}y}{\mathrm{d}t} \\ \vdots \\ x_n(t) := \dfrac{\mathrm{d}^n y}{\mathrm{d}t^n} \end{cases} \implies \quad x := \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$$

one gets:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_n = \varphi(x, u, t) \\ y = x_1 \end{cases}$$

# State Space Descriptions (cont.)

**Discrete-time case**:

$$y(k+n) = \varphi\left(y(k+n-1), y(k+n-2), \ldots, y(k), u(k), k\right)$$

Letting:

$$\begin{cases} x_1(k) := y(k) \\ x_2(k) := y(k+1) \\ \vdots \\ x_n(k) := y(k+n-1) \end{cases} \quad \Longrightarrow \quad x := \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$$

one gets:

$$\begin{cases} x_1(k+1) = x_2(k) \\ x_2(k+1) = x_3(k) \\ \vdots \\ x_n(k) = \varphi(x, u, k) \\ y(k) = x_1(k) \end{cases}$$

# State Space Descriptions (cont.)

**Example (discrete-time)**:

$$w(k) - 3w(k-1) + 2w(k-2) - w(k-3) = 6u(k)$$

Letting:

$$\begin{cases} x_1(k) := w(k-3) \\ x_2(k) := w(k-2) \\ x_3(k) := w(k-1) \end{cases} \implies x := \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

one gets:

$$\begin{cases} x_1(k+1) = x_2(k) \\ x_2(k+1) = x_3(k) \\ x_3(k+1) = 3x_3(k) - 2x_2(k) + x_1(k) + 6u(k) \\ y(k) = x_3(k) \end{cases}$$

# State Space Descriptions (cont.)

## The state space description is not unique

- The fact that physical and non-physical approaches can be followed to describe the **same dynamic system** in state-space form clearly reveals the **non-uniqueness** of this representation
- Later on some more details will be given concerning **equivalent** state space descriptions

# Sampling and Reconstructing

# From Continuous-Time to Discrete-Time

**Remarks**

- Till now we carried out a general treatment of dynamic systems considering both the continuous-time and the discrete-time cases
- Since the course is intended to cover **data-based** system dynamics, analysis and estimation, from now on only the discrete-time case will be dealt with
- However, before doing this, the issue of **conversion of a continuous-time into a discrete-time by sampling** has to be dealt with in some detail

# Sampling and Reconstructing

## Sampling and Reconstructing in Time Domain

# From Continuous-Time to Discrete-Time: Signal Taxonomy

## Continuous-time vs. discrete-time signals

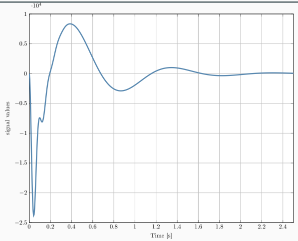
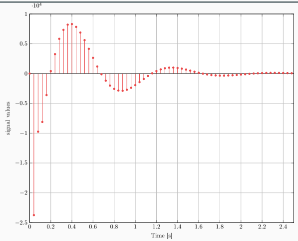
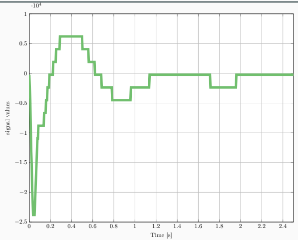
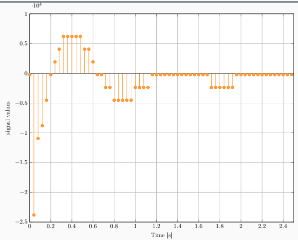
- **continuous-time signal**: a *function of time* (independent variable) $x = x(t)$, such that the independent variable **time is continuous**
  - the domain of the function $x = x(t)$ has the cardinality of the real numbers set $\mathbb{R}$.
- **discrete-time signal**: a signal $y = y(k)$, **specified** only **for discrete values of time** (the independent variable)
  - the domain of the function $y = y(k)$ has the cardinality of the integer numbers set $\mathbb{Z}$.
  - a discrete-time signal is usually called *sequence*

# Signal Taxonomy (cont.)

## Analog vs. digital signals

- **analog signal**: the **amplitude** of the signal may vary in a **continuous** range
  - an analog signal can be both continuous-time and discrete-time signal.
- **digital signal**: a signal whose **amplitude is quantized**, i.e. the amplitude of a digital signal can take only a finite number of values.
  - a digital signal can be both continuous-time and discrete-time signal.

# Signal Taxonomy: Graphical Summary

| | continuous-time | discrete-time |
|---|---|---|
| **analog** |  |  |
| **digital** |  |  |

# Sampling & Digital Coding: Main Issues

$$e(t) \rightsquigarrow e_k = e(k\,T_s) \rightsquigarrow 0\ 1\ 10\ 10 \ldots$$

The conversion of an analog, continuous-time signal $e = e(t)$ to a digital, discrete-time sequence is subject to two main issues:

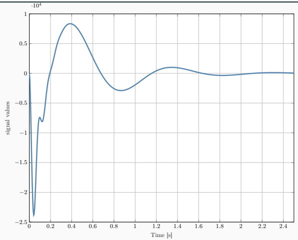
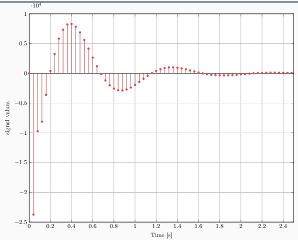
- **loss of information**, due to the conversion from continuous-time to discrete-time (more details later)
- **quantisation noise and distortion**, due to the analog to digital conversion process

**Sampling issues taken into account**

- sampling and the loss of information, a glimpse on the theoretical motivations of, and how to cope with this issue are discussed topics
- quantisation and coding issues are not taken into account

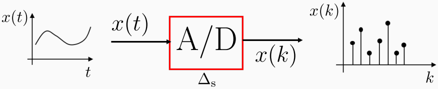
From now on, consider the **sampling procedure** simply as a **conversion** from an analog, **continuous-time signal** to an analog, **discrete-time signal**.

Moreover, hereafter each time-based signal will be labelled just as continuous-time or discrete-time signal.

| | **continuous-time** | **discrete-time** |
|---|---|---|
| **analog** |   |   |

# The Ideal Sampler

How to convert a continuous-time signal to a discrete-time one?




## Periodic sampling using an *ideal sampler*

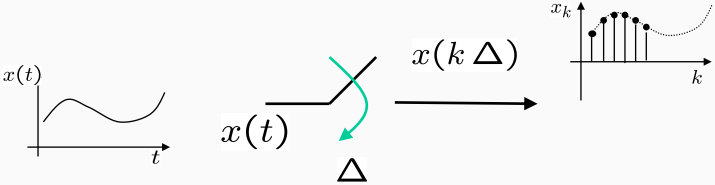
- the aim of the A/D converter is to transform a continuous-time signal $x(t)$ into a discrete-time sequence $x(k)$
- given a time interval $\Delta$, called **sampling period**, applying a *periodic sampling* means to extract and collect, creating a *sequence*, values of the signal corresponding to time instants, integer multiples of the sampling period

$$\{x(k)\}_{k\in\mathbb{Z}} \Longrightarrow \{x(t)\ :\ t = k\,\Delta\,,\ k \in \mathbb{Z}\}$$

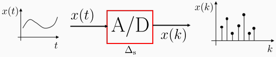
# The Ideal Sampler (cont.)

An ideal sampler acts as an **ideal** electrical **switch**

- the switch commutes between the two states “open” and “closed”, driven by a periodic pulse signal (called the *clock signal*), with the time period equal to the sampling period $\Delta$;
- when a clock pulse occurs, the switch closes instantaneously, the actual sample of the input signal can be “copied” into the sampler output and then the switch commutes (instantaneously) to the “open” state, waiting for the next clock pulse.



# The Ideal Sampler (cont.)




**Sampling rate**
Given the sampling period $\Delta$, let's define the *rate of conversion* from continuous to discrete time using

- **sampling angular frequency**

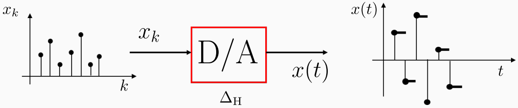
$$\Omega_s = \frac{2\pi}{\Delta} \quad [\text{rad/s}]$$

- **sampling frequency**

$$f_s = \frac{1}{\Delta} \quad [\text{Hz}]$$

# The Reconstructor

Consider now the backward operation: how to characterize the conversion of a discrete-time signal to a continuous-time one?




## Reconstruction using a data-holder

- the purpose of the D/A subsystem is to reconstruct the sampled signal into a form that resembles the original signal, before sampling.
- the simplest D/A subsystem [indeed the most common one] is the so-called **zero-order-hold (ZOH)**.

# The Reconstructor (cont.)

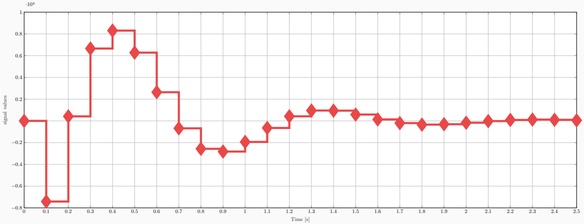
## Reconstruction using a D/A converter (cont.)

- the ZOH clamps the output signal to a value corresponding to that of the input sequence at the current clock pulse, until the next clock pulse arrives.

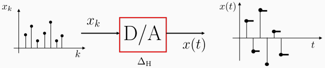
$$x(t) = x(k)\,,\ k\,\Delta_{\mathsf{H}} \leq t < (k+1)\ \Delta_{\mathsf{H}} \quad k \in \mathbb{Z}$$

- the time period $\Delta_{\mathsf{H}}$ is called **holding period**.

Note that the output signal of a ZOH is a stair-wise signal



# The Reconstructor (cont.)




**Holding rate**
Given the holding period $\Delta_H$, let's define the *rate of conversion* for a D/A device using

- **holding angular frequency**

$$\Omega_H = \frac{2\pi}{\Delta_H} \quad [\text{rad/s}]$$

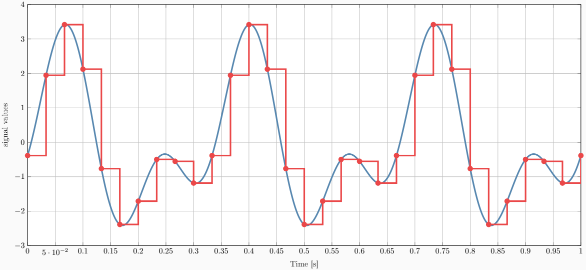
- **holding frequency**

$$f_H = \frac{1}{\Delta_H} \quad [\text{Hz}]$$

Usually the sampling and holding frequencies have the same value.

# Sampling and Reconstructing

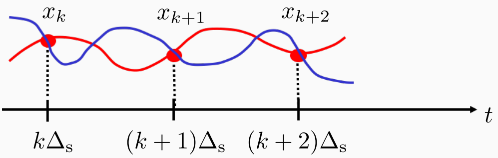
- What happens if a continuous-time signal is firstly sampled and then reconstructed? How is the output signal of the ZOH w.r.t the original continuous-time signal? The same or?
- Indeed, the output of the ZOH is a stair-wise signal, so **the reconstructed signal is different from the original one**: sampling and reconstruction are just approximately the opposite function of each other.



# Sampling and Loss of Information

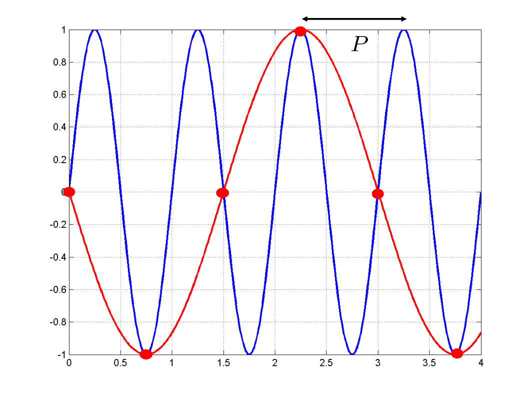
$$\left. \begin{array}{c} x_k\,,\ k \in \mathbb{Z} \\ + \\ \text{a priori knowledge} \\ \text{of the signal features} \end{array} \right\} \longrightarrow x(t) = \textbf{?}$$

- In general, **reconstructing** the continuous-time signal starting from the samples is an **ill-posed problem**: the reconstruction may be ambiguous.



# Sampling a Sinusoidal Signal

Consider the signal $x(t) = \sin\left(\bar{\Omega}t\right) \quad P = \dfrac{2\pi}{\bar{\Omega}}$



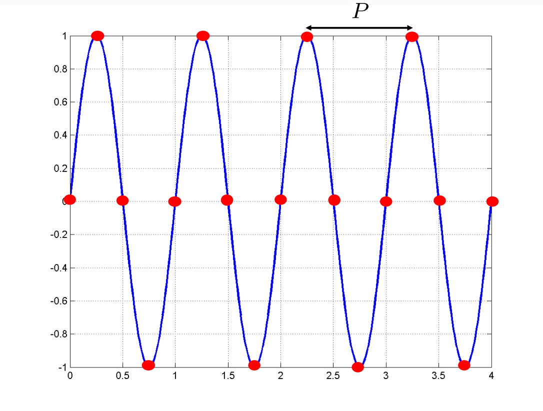

Select as sampling period the value

$$\Delta = \frac{3}{4}P = \frac{3\pi}{2\bar{\Omega}}$$

Indeed, it's easy to determine sinusoidal signals, with period $\hat{P} > P$, that may generate the same values, obtained by sampling $x(t)$.

Note: the frequency of the ambiguous signal is lower than the frequency of the original signal. This effect is called **frequency aliasing** (or *frequency fold-over*).

# Sampling a Sinusoidal Signal (cont.)




Reducing the sampling period (i.e. increasing the sampling frequency) the ambiguity disappears: no more frequency fold-over effect.

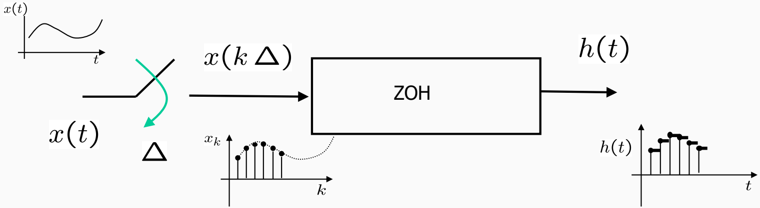
$$\Delta = \frac{P}{4} = \frac{\pi}{2\,\bar{\Omega}}$$

By choosing properly the sampling period, the frequency aliasing effect has been avoided. Note: the effective sampling frequency is much higher than the signal time frequency.

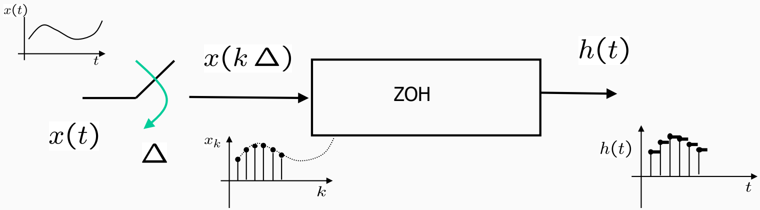
# Ideal Sampler & ZOH: Mathematical Model

- So far, it has been illustrated by examples that, when sampling a simple sinusoidal signal, choosing properly the sampling period grants to avoid the aliasing effect.
- How to generalize? What is the effect of the sampling procedure? How does the choice of the sampling period influence the frequency aliasing effect?

The influence of the sampling period on the aliasing effect will be explained by modelling the direct connection of an ideal sampler to a ZOH ($\Delta$ is the sampling period)



# Ideal Sampler & ZOH: Mathematical Model (cont.)




The output of the ZOH is a continuous-time signal, expressed as

$$h(k\Delta + \tau) = x(k\Delta)\,, \quad 0 \leq \tau < \Delta\,,\ k = 0,\, 1,\, 2,\, 3\, \cdots$$ a stair-wise signal

$$h(t) = x(0)\,[1(t) - 1(t-\Delta)] + x(\Delta)\,[1(t-\Delta) - 1(t-2\Delta)] + \cdots$$

$$= \sum_{k=0}^{+\infty} x(k\Delta)\,[1(t-k\Delta) - 1\,(t-(k+1)\Delta)]$$

# Ideal Sampler & ZOH: Mathematical Model (cont.)

Applying the Laplace transform

$$\mathcal{L}\left\{1(t - k\Delta)\right\} = \frac{e^{-k\Delta s}}{s}$$

$$\mathcal{L}\left\{h(t)\right\} = H(s) = \sum_{k=0}^{+\infty} x(k\Delta)\, \frac{e^{-k\Delta s} - e^{-(k+1)\Delta s}}{s}$$

$$= \frac{1 - e^{-\Delta s}}{s} \cdot \sum_{k=0}^{+\infty} x(k\Delta)\, e^{-k\Delta s}$$

function only of $\Delta$

function of input signal $x(t)$ and sampling period $\Delta$

# Ideal Sampler & ZOH: Mathematical Model (cont.)

a transfer function model for the ZOH

the Laplace transform of ideal sampler's output as continuous-time signal

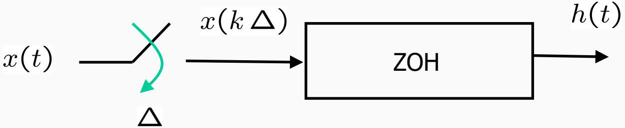
$$H(s) = \frac{1 - e^{-\Delta s}}{s} \cdot \sum_{k=0}^{+\infty} x(k\Delta)\, e^{-k\Delta s} = G_{\mathsf{ZOH}}(s)\, X^*(s)$$

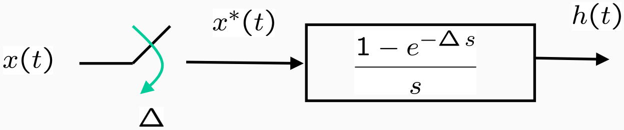
where

$$G_{\mathsf{ZOH}}(s) = \frac{1 - e^{-\Delta s}}{s} \qquad X^*(s) \triangleq \sum_{k=0}^{+\infty} x(k\Delta)\, e^{-k\Delta s}$$

# Ideal Sampler & ZOH: Mathematical Model (cont.)

So far, we demonstrated the equivalence between the following two structures




where $x^*(t) = \mathcal{L}^{-1}\{X^*(s)\}$ $\qquad X^*(s) \triangleq \sum_{k=0}^{+\infty} x(k\Delta)\, e^{-k\Delta s}$

# Ideal Sampler as Impulse Modulator

Note: $x^*(t)$ is a continuous-time representation of the ideal sampler output (indeed a sequence of samples)

$$x^*(t) = \mathcal{L}^{-1}\{X^*(s)\} = \mathcal{L}^{-1}\left\{\sum_{k=0}^{+\infty} x(k\Delta)\, e^{-k\Delta s}\right\}$$

Now, recalling the main properties of the Dirac *delta function*

$$\mathcal{L}^{-1}\left\{e^{-k\Delta s}\right\} = \delta\left(t - k\Delta\right) \qquad \delta(t) = \begin{cases} 0 & \forall t \neq 0 \\ \text{undef.} & t = 0 \end{cases}$$

$$\int_{-\infty}^{+\infty} \delta(t)\, dt = 1 \qquad \int_{-\infty}^{+\infty} f(t)\, \delta(t-\tau)\, dt = f(\tau)$$

$x^*(t)$ can be expressed as

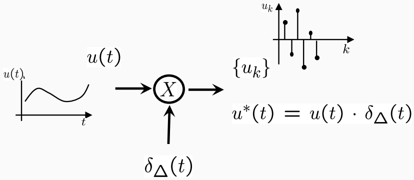
$$x^*(t) = \mathcal{L}^{-1}\left\{\sum_{k=0}^{+\infty} x(k\Delta)\, e^{-k\Delta s}\right\} = \sum_{k=0}^{+\infty} x(k\Delta)\, \delta(t - k\Delta)$$

# Ideal Sampler as Impulse Modulator (cont.)

$$
\begin{aligned}
x^*(t) &= \sum_{k=0}^{+\infty} x(k\Delta)\,\delta(t-k\Delta) \\
&= \sum_{k=0}^{+\infty} x(t)\,\delta(t-k\Delta) \\
&= x(t) \cdot \sum_{k=0}^{+\infty} \delta(t-k\Delta) \\
&= x(t) \cdot \delta_\Delta(t)
\end{aligned}
$$

where

$$
\delta_\Delta(t) = \sum_{k=0}^{+\infty} \delta(t-k\Delta)
$$




- $x^*(t)$ can be expressed as the result of the modulation of the original signal $x(t)$ with a train of Dirac impulses
- owing to this result, the ideal sampler is also referred as an **impulse modulator**

# Sampling and Reconstructing

## Sampling and Reconstructing using Laplace- and Z- Transform

# Laplace- & Z- Transform of Ideal Sampler Output

- Since the output of the impulse modulator may be described as a continuous-time signal $x^*(t)$ but also as a discrete-time sequence $x(k\Delta)$, how to correlate such representations?
- Consider the Laplace-transform of $x^*(t)$ and the Z-transform of the sequence $x(k\Delta)$

$$\mathcal{L}\left\{x^*(t)\right\} = X^*(s) = \sum_{k=0}^{+\infty} x(k\Delta) e^{-k\Delta s}$$

$$\mathcal{Z}\left\{x(k\Delta)\right\} = X(z) = \sum_{k=0}^{+\infty} x(k\Delta)\, z^{-k}$$

It's easy to find that using the substitutions

$$z = e^{s\,\Delta} \quad \Longleftrightarrow \quad s = \frac{1}{\Delta}\, \log z$$

the Laplace transform may be rewritten as Z-transform and vice-versa.

# Properties of $X^*(s)$: $X^*(s)$ vs $X(s)$

## Definition: starred transform

The function $X^*(s) = \mathcal{L}\{x^*(t)\}$ is usually called the starred transform.

## Property 1: the starred transform $X^*(s)$ vs. $X(s)$

The starred transform may be expressed as a scaled summation of infinite copies of the Laplace transform of the original analog signal $X(s) = \mathcal{L}\{x(t)\}$, shifted each other by $j\Omega_s$ (where $\Omega_s = \frac{2\pi}{\Delta}$ and $\Delta$ is the sampling period)

$$X^*(s) = \frac{1}{\Delta} \sum_{k=-\infty}^{k=+\infty} X\left(s - jk\Omega_s\right), \quad \Omega_s = \frac{2\pi}{\Delta}, \quad X(s) = \mathcal{L}\{x(t)\}$$

# 1$^{st}$ Property of Starred Transform - Sketch of Proof

**Proof - a sketch.**

Recall the ideal sampler output expression

$$x^*(t) = \sum_{k=0}^{+\infty} x(k\Delta)\,\delta(t - k\Delta)$$

Remember: the original, analog signal $x(t)$ is a *causal signal*. Owing to this property, the summation may be modified

$$x(t) \equiv 0 \ \forall t < 0 \quad \Longrightarrow \ x^*(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\,\delta(t - k\Delta)$$

According to this modification, let's redefine also the *impulse train*

$$\delta_\Delta(t) = \sum_{k=-\infty}^{+\infty} \delta(t - k\Delta)$$

# $1^{st}$ Property of Starred Transform - Sketch of Proof (cont.)

Now represent the ***impulse train*** as **Fourier series**

$$\delta_\Delta(t) = \sum_{k=-\infty}^{k=+\infty} C_\Delta(k) e^{jk\Omega_s t} \qquad \Omega_s = \frac{2\pi}{\Delta}$$

$$C_\Delta(k) = \frac{1}{\Delta} \int_{-\frac{\Delta}{2}}^{+\frac{\Delta}{2}} \delta_\Delta(t)\, e^{-jk\Omega_s t}\, dt$$

$$= \frac{1}{\Delta} \int_{-\frac{\Delta}{2}}^{+\frac{\Delta}{2}} \delta(t)\, e^{-jk\Omega_s t}\, dt = \frac{1}{\Delta}$$

Thus

$$\delta_\Delta(t) = \frac{1}{\Delta} \sum_{k=-\infty}^{k=+\infty} e^{jk\Omega_s t}$$

# 1st Property of Starred Transform - Sketch of Proof (cont.)

By substitution of the impulse train expression into the ideal sampler output $x^*(t)$, we obtain

$$x^*(t) = x(t) \cdot \delta_\Delta(t) = \frac{1}{\Delta} \sum_{k=-\infty}^{k=+\infty} x(t)\, e^{jk\Omega_s t}$$

Applying the Laplace transform

$$X^*(s) = \mathcal{L}\left\{x^*(t)\right\} = \frac{1}{\Delta} \sum_{k=-\infty}^{k=+\infty} \int_{-\infty}^{+\infty} \left[x(t)\, e^{jk\Omega_s}\right] e^{-st}\, dt$$

Let's apply the bilateral Laplace transform to $x^*(t)$: remember, we rewrote $x^*(t)$ as it is non-causal signal

# 1st Property of Starred Transform - Sketch of Proof (cont.)

Thus

$$X^*(s) = \frac{1}{\Delta} \sum_{k=-\infty}^{k=+\infty} \int_{-\infty}^{+\infty} \left[x(t)\, e^{jk\Omega_s}\right] e^{-st}\, dt$$

Recall the Laplace transform property

$$\mathcal{L}\left\{e^{kt}\, f(t)\right\} = F\,(s-k) \quad \forall k \in \mathbb{C}\,, \qquad F(s) = \mathcal{L}\left\{f(t)\right\}$$

Finally

$$X^*(s) = \frac{1}{\Delta} \sum_{k=-\infty}^{k=+\infty} X\,(s - jk\Omega_s)\,, \quad k \in \mathbb{Z}\,, \quad \Omega_s = \frac{2\pi}{\Delta}$$

□

# Properties of $X^*(s)$: Periodicity of the Starred Transform

**Property 2: the starred transform is periodic in $s$, with period $j\Omega_s$**

$$X^*(s) = X^*\left(s + jn\Omega_s\right)\ , \quad n \in \mathbb{N}\,,\ \Omega_s = \frac{2\pi}{\Delta}$$

**Proof.**

$$X^*\left(s + jn\Omega_s\right) = \sum_{k=0}^{+\infty} x(k\Delta)\, e^{-k\Delta(s+jn\Omega_s)}$$

Since $\Omega_s \cdot \Delta = 2\pi$, applying the Euler's relationship $e^{j\theta} = \cos\theta + j\sin\theta$

$$e^{-jnk\Delta\Omega_s} = e^{-jnk2\pi} = 1 \quad \forall n\,,\ k \in \mathbb{N}$$

thus

$$X^*\left(s + jn\Omega_s\right) = \sum_{k=0}^{+\infty} x(k\Delta)e^{-k\Delta s} = X^*\left(s\right)$$

□

# Properties of $X^*(s)$: Poles of the Starred Transform

**Property 3: poles of the starred transform vs poles of $X(s)$**

If $X(s)$ has a pole at $s = \hat{s}$,
then $X^*(s)$ must have poles at $s = \hat{s} + jk\Omega_s \; , \quad k \in \mathbb{Z}$

**Proof.**

Rewrite the result of “Property 1”

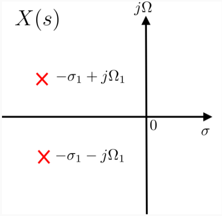
$$
\begin{aligned}
X^*(s) =& \frac{1}{\Delta} \sum_{k=-\infty}^{k=+\infty} X\left(s - jk\Omega_s\right) \\
=& \frac{1}{\Delta} \Big[ X(s) + X\left(s - j\Omega_s\right) + X\left(s - 2j\Omega_s\right) + \cdots \\
& + X\left(s + j\Omega_s\right) + X\left(s + 2j\Omega_s\right) + \cdots \Big]
\end{aligned}
$$

If $X(s)$ has a pole at $s = \hat{s}$, then each term of the latter expression will contribute with a pole at $s = \hat{s} - jk\Omega_s \; , \quad k \in \mathbb{Z}$.

□

# Properties of $X^*(s)$: Poles of the Starred Transform (cont.)

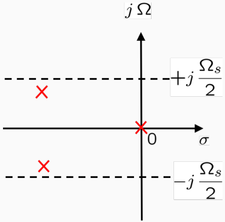
## Poles map of the starred transform




- if $X(s)$ has a pole in $s = -\sigma_1 + j\Omega_1$, then the sampling operation will generate poles for $X^*(s)$ in $s = -\sigma_1 + j\Omega_1 \pm jk\Omega_s \ , \quad k \in \mathbb{Z}$
- on the contrary, if $X(s)$ has a pole in $s = -\sigma_1 + j(\Omega_1 + \Omega_s)$, then $X^*(s)$ will have a pole in $s = -\sigma_1 + j\Omega_1$
- pole locations in $X(s)$ at $s = -\sigma_1 + j(\Omega_1 \pm k\Omega_s) \ , \quad k \in \mathbb{Z}$ will result in identical pole locations in $X^*(s)$

# Properties of $\mathrm{X}^*(\mathrm{s})$: Poles of the Starred Transform (cont.)

## Primary and complementaries strips in the $s$-plane

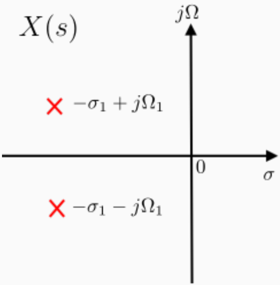


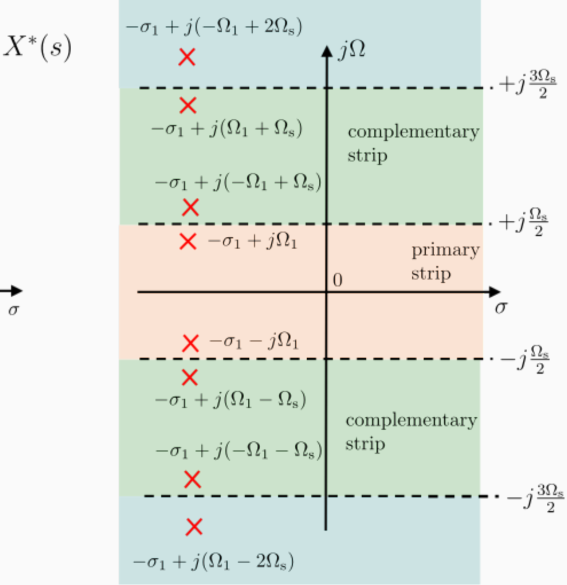

- consider the $s$-plane of the starred transform and divide it into strips
  - the **primary strip** is defined as the strip for which

$$\left\{ s:\ s \in \mathbb{C}\,,\ s = \sigma + j\Omega\,,\ -\frac{\Omega_s}{2} \leq \Omega \leq +\frac{\Omega_s}{2} \right\}$$

- if the pole-zero locations for the starred transform are known in the primary strip, then the pole-zero locations for $X^*(s)$ in the entire $s$-plane are known.

# Properties of $X^*(s)$: Poles Map of Starred Transform



# Properties of $X^*(s)$: Poles of The Starred Transform (cont.)

**What about zeros of starred transform?**

Indeed, the zeros of $X(s)$ **do not uniquely determine** the location of zeros of the starred transform $X^*(s)$. However, the zero locations of $X^*(s)$ are periodic, with period $j\Omega_s$ (Property 2).

# Sampling and Reconstructing

## Sampling, Reconstructing and Aliasing in the Frequency Domain

# Laplace & Fourier Transform of a Causal, Continuous-time Signal

Consider a causal, continuous-time signal $x(t)$. The **unilateral Laplace transform** of such a signal is defined as

$$\mathcal{L}\{x(t)\} = X(s) = \int_0^{+\infty} x(\tau)e^{-s\tau}\, d\tau$$

whereas the **Fourier transform** is

$$\mathcal{F}\{x(t)\} = X(\Omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\Omega\tau}\, d\tau$$

Exploiting the signal causality, the Fourier transform may be rewritten as

$$\mathcal{F}\{x(t)\} = X(\Omega) = \int_0^{+\infty} x(t)e^{-j\Omega\tau}\, d\tau = \mathcal{L}\{x(t)\}|_{s=j\Omega}$$

provided that both transforms exist.

# Sampling and Aliasing in the Frequency Domain

### Starred transform result in the frequency domain

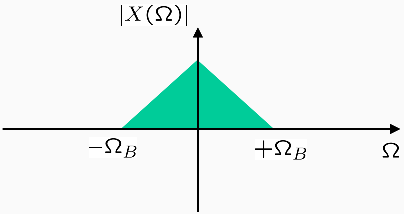
Analysing the starred signal $x^*(t)$ by applying **the Fourier transform** (instead of the Laplace one), provides the same result:

the Fourier transform of the starred signal may be expressed as a scaled summation of infinite copies of the Fourier transform of the original analog signal

$$X^*(\Omega) = \frac{1}{\Delta} \sum_{k=-\infty}^{k=+\infty} X\left(\Omega - k\Omega_\mathsf{s}\right) , \quad \Omega_\mathsf{s} = \frac{2\pi}{\Delta} , \quad X(\Omega) = \mathcal{F}\left\{x(t)\right\}$$

# Band-limited Signals

Suppose that the signal $x(t)$ is a so-called **band-limited signal**, i.e. the amplitude spectrum $|X(\Omega)|$ of the signal is non zero only if $|\Omega| \leq \Omega_B$ (where $X(\Omega) = \mathcal{F}\{x(t)\}$).




What happens when we acquire such a signal by sampling? In particular, what if $\Omega_s > 2\Omega_B$ , $\Omega_s = 2\Omega_B$ or $\Omega_s < 2\Omega_B$ ?

# Band-limited Signals (cont.)

## Band-limited signal

In rigorous terms, a signal is called a *band-limited signal* if

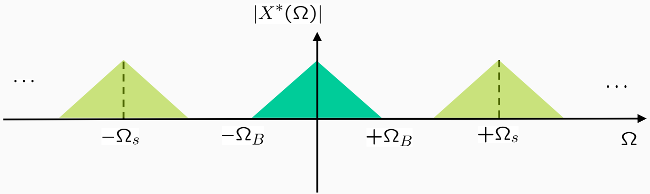
$$x(t) = \sum_{k=1}^{k=N} \alpha_k \sin\left(\Omega_k t + \varphi_k\right) \ , \quad \Omega_k \leq \Omega_\mathsf{B} \quad \forall k$$

or

$$x(t) = \int_0^{\Omega_\mathsf{B}} \alpha(\Omega) \sin\left[\Omega t + \varphi(\Omega)\right] \, d\Omega \ , \quad \Omega \in [0 \ , \quad \Omega_\mathsf{B}]$$

# Sampling and Aliasing in the Frequency Domain (cont.)

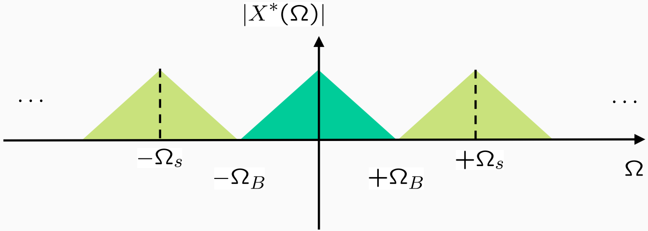
$$\Omega_s > 2\Omega_B$$




- no overlapping of spectra, so **no aliasing**
- to **reconstruct** the original signal (to isolate the original spectrum) a **realizable** (causal) **low-pass filter** is needed

# Sampling and Aliasing in the Frequency Domain (cont.)

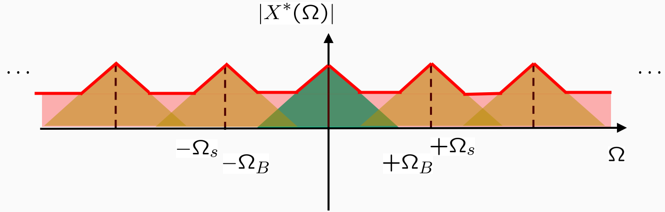
$$\Omega_\mathsf{s} = 2\Omega_\mathsf{B}$$




- still no overlapping of spectra, so **no aliasing**
- to reconstruct the original signal (to isolate the original spectrum) an **ideal (non-causal) low-pass filter** is needed

# Sampling and Aliasing in the Frequency Domain (cont.)

$$\Omega_\mathrm{s} < 2\Omega_\mathrm{B}$$




- overlapping of spectra, so **aliasing**
- **no way to reconstruct** the original signal (to isolate the original spectrum)

# Sampling and Reconstructing

## The Sampling Theorem

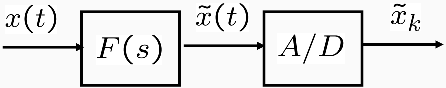
# Sampling Theorem

### Nyquist-Shannon theorem

A continuous-time signal which contains no frequency components greater than $\Omega_{\mathsf{B}}$ rad/s, is uniquely determined by the signal samples

$$\left\{ x_k = x(k\Delta)\ , \quad k \in \mathbb{Z}\ , \quad \Delta :\ \Omega_{\mathsf{s}} = \frac{2\pi}{\Delta} > 2\Omega_{\mathsf{B}} \right\}$$

# Anti-aliasing Filter




- How to guarantee the band limitedness of a signal?
- From a practical point of view, how to restrict the bandwidth of the signal to the band of interest, with the aim to satisfy the sampling theorem?
- **anti-aliasing filter**: a realizable low-pass filter

$$F(s) = \frac{1}{1 + \frac{s}{\bar{\Omega}}} \,, \quad B = [0\,,\bar{\Omega}] \quad \Omega_s = \frac{2\pi}{\Delta} > 2\bar{\Omega}$$

# Sampling and Reconstructing

## Aliasing in the Laplace Transform Domain

# Aliasing in the $s$-plane

Recall the relationship between the starred transform and the Laplace transform of the original continuous-time signal

$$X^*(s) = \frac{1}{\Delta} \sum_{k=-\infty}^{k=+\infty} X\left(s - jk\Omega_{\mathsf{s}}\right) , \quad \Omega_{\mathsf{s}} = \frac{2\pi}{\Delta}, \quad X(s) = \mathcal{L}\left\{x(t)\right\}$$

and the relationship between the starred transform and the Z-transform of the sampled sequence

$$z = e^{s\,\Delta} \quad \Longleftrightarrow \quad s = \frac{1}{\Delta}\,\log z$$

The aliasing effect may be analysed also in the $s$-plane of the starred transform, exploiting such relations.

# Aliasing in the $s$-plane (cont.)

Consider two values into the $s$-plane of the starred transform, such that

$$s_p = s_q + jk\Omega_\mathsf{s} \ , \quad k \in \mathbb{Z}$$

- The sampling relationship $z = e^{s\,\Delta}$ gives

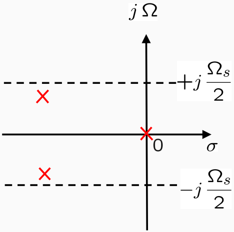
$$z_p \equiv z_q \quad \forall k \in \mathbb{Z}$$

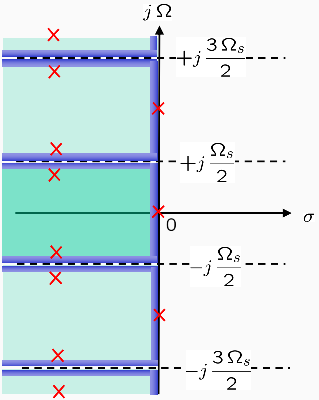
- Different values in the $s$-plane correspond to the same value in the $z$-plane!

# Aliasing in the $s$-plane (cont.)

- There is no bijective correspondence between $s$- and $z$-plane. Indeed, the $s$-plane may be divided into horizontal, $\Omega_s$ wide strips and the $s$-plane points, belonging to each of these strips, correspond one-to-one to a unique point into the $z$-plane.
- The effect of sampling may be explained as transforming the $s$-plane of the original signal's Laplace transform into a series of shifted strips (the $s$-plane of the starred signal), each of them with the same zero and pole locations and finally folding these strips on each other, in order to map the resulting folded $s$-plane into the $z$-plane of the sampled signal's Z-transform.

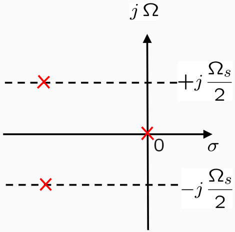
# Aliasing in the $s$-plane: $\Omega_s > 2\Omega_B$

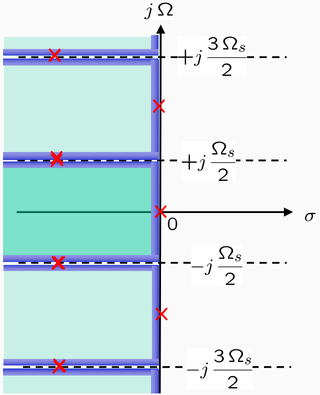



- the primary strip contains the whole set of pole location of the Laplace transform of the original continuous-time signal
- **no aliasing**

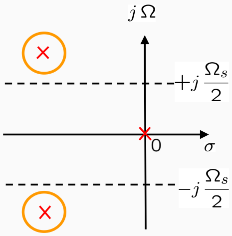
# Aliasing in the $s$-plane: $\Omega_s = 2\Omega_B$

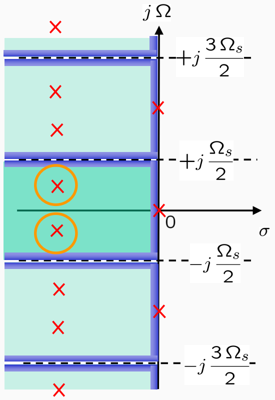



- some pole locations may lay on the border between primary and complementary strips
- still **no aliasing**

# Aliasing in the $s$-plane: $\Omega_s < 2\Omega_B$




- overlapping of pole location configurations
- **aliasing**

Note: the alias appear as poles with time constant values lower that the original ones!

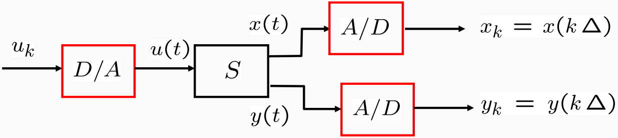
# Sampling and LTI Systems: from Continuous-Time to Discrete-Time Systems

# Sampling and LTI Systems: from Continuous-Time to Discrete-Time Systems

## The Step-Invariant Transform

# C2d with Sampler & Hold

Consider the scheme




- How to obtain a discrete-time description of a linear, time-invariant, continuous-time dynamic system?
- Both state variables and outputs are sampled by means of an **ideal sampler**
- The inputs to the LTI systems are converted from discrete- to continuous-time using a **ZOH**

# C2d with Sampler & Hold (cont.)

- Consider a LTI dynamic system, described by means of state equations

$$\begin{cases} \dot{x}(t) = & A_c x(t) + B_c u(t) \\ y(t) = & C_c x(t) + D_c u(t) \end{cases}$$

- The following expression holds

$$x(t) = e^{A_c(t-t_0)} x(t_0) + \int_{t_0}^{t} e^{A_c(t-\tau)} B_c u(\tau)\, d\tau$$

( from "*Fundamentals in Control*") where

$$e^{A_c t} = \mathcal{L}^{-1}\left\{(sI - A_c)^{-1}\right\} = I + A_c t + \frac{A_c^2 t^2}{2} + \frac{A_c^3 t^3}{3!} + \cdots$$

# C2d with Sampler & Hold (cont.)

- Remember the stairwise behaviour of the output of a ZOH device

$$u(t) = u_k = u(k\Delta)\,,\; k\,\Delta \leq t < (k+1)\;\Delta \quad k \in \mathbb{Z}$$

- Evaluate the state movement expression in a time interval between two successive sampling instants $k\Delta$ and $(k+1)\Delta$

$$x\left[(k+1)\Delta]\right) = e^{A_c\Delta} x\left(k\Delta\right) + \left\{ \int_{k\,\Delta}^{(k+1)\,\Delta} e^{A_c(t-\tau)} B_c)\, d\tau \right\} u\left(k\Delta\right)$$

the input $u(t)$ is a constant signal during the considered time interval

# C2d with Sampler & Hold (cont.)

- Substitute $r = (k+1)\Delta - \tau$ into the integral term and rewrite the last expression,

$$x\left[(k+1)\Delta]\right) = e^{A_c\Delta}\, x\,(k\Delta) + \left\{\int_0^{\Delta} e^{A_c r} B_c\, dr\right\} u\,(k\Delta)$$

- By comparison with the expression of the discrete-time state equations for the dynamic system considered

$$\begin{cases} x\left[(k+1)\Delta\right] = A_d\, x\,(k\Delta) + B_d\, u\,(k\Delta) \\ \\ y\,(k\Delta) = C_d\, x\,(k\Delta) + D_d\, u\,(k\Delta) \end{cases}$$

finally we obtain the **continuous to discrete-time conversion rule, applying ZOH** (the so-called *step–invariant transform*)

# C2d with Sampler & Hold (cont.)

### Step–invariant transform

Starting from a continuous-time LTI dynamic system

$$\begin{cases} \dot{x}(t) = A_c x(t) + B_c u(t) \\ y(t) = C_c x(t) + D_c u(t) \end{cases}$$

the corresponding discrete-time description, using a ZOH for inputs and ideal samplers for state and output signals is given by

$$A_d = e^{A_c \Delta} \qquad B_d = \int_0^{\Delta} e^{A_c r} B_c \, dr$$

$$C_d = C_c \qquad D_d = D_c$$

# C2d with Sampler & Hold: an Example

Consider

$$\begin{cases} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & -a \end{bmatrix} x + \begin{bmatrix} 0 \\ K \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{cases}$$

and let's determine the discrete-time description, by sampling with ZOH and ideal samplers.

$$(sI - A_c)^{-1} = \frac{1}{s(s+a)} \begin{bmatrix} s+a & 1 \\ 0 & s \end{bmatrix} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s(s+a)} \\ 0 & \frac{1}{s+a} \end{bmatrix}$$

# C2d with Sampler & Hold: an Example (cont.)

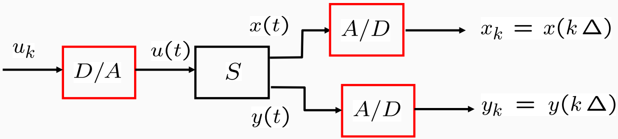
Applying the step-invariant transform

$$e^{A_c\,t} = \mathcal{L}^{-1}\left\{(sI - A_c)^{-1}\right\} = \begin{bmatrix} 1(t) & \frac{1}{a}\cdot 1(t) - \frac{1}{a}e^{-at}\cdot 1(t) \\ 0 & e^{-at}\cdot 1(t) \end{bmatrix}$$

$$e^{A_c\,\Delta} = \begin{bmatrix} 1 & \frac{1}{a}\left(1 - e^{-a\,\Delta}\right) \\ 0 & e^{-a\,\Delta} \end{bmatrix}$$

$$B_d = \int_0^{\Delta} e^{A_c\,r}\,B_c\,dr = \begin{bmatrix} \int_0^{\Delta} \frac{K}{a}\left(1 - e^{-a\,r}\right)\,dr \\ \int_0^{\Delta} K e^{-a\,r}\,dr \end{bmatrix}$$

# Why Do They Call It the Step-Invariant Transform?




## Step response of the sampled-time LTI system

The considered conversion technique from continuous-time to discrete-time LTI systems is usually called **step invariant transform**, due to a peculiar feature of the conversion rule itself:

- **the conversion rule preserves the step response** of the dynamic system, i.e. the values of the step response of the discrete-time description of the LTI system are exactly the samples of the step response of the effective continuous-time LTI system. [Hint: what is the output of a ZOH if the input is a discrete-time step sequence?]

# Why Do They Call It the Step-Invariant Transform? (cont.)

## Continuous- and discrete-time step responses: a comparison

- Let's consider the continuous-time LTI system

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \cdot x(t) + \begin{bmatrix} 0 \\ 10 \end{bmatrix} \cdot u(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot x(t) \end{cases}$$

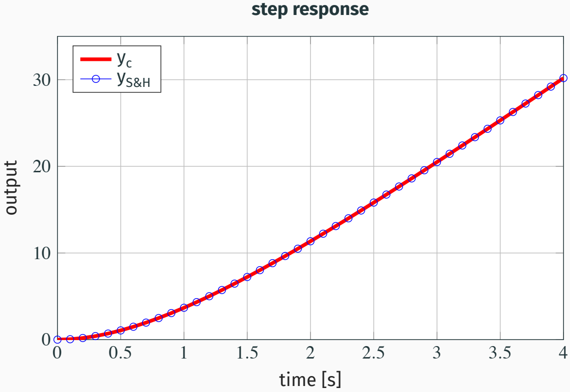
- Chosen as sampling period the value $\Delta = 0.1$ s, the discrete-time description of the considered LTI system is

$$\begin{cases} x(k+1) = \begin{bmatrix} 1 & 0.09516 \\ 0 & 0.9048 \end{bmatrix} \cdot x(k) + \begin{bmatrix} 0.04837 \\ 0.9516 \end{bmatrix} \cdot u(k) \\ y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot x(k) \end{cases}$$

# Why Do They Call It the Step-Invariant Transform? (cont.)

## Continuous- and discrete-time step responses: a comparison (cont.)

- Let's compare visually the step responses of the continuous-time LTI system and of the discrete-time description



# Sampling and LTI Systems: from Continuous-Time to Discrete-Time Systems

## Practical Issues

# C2d with Sampler & Hold: Practical Issues

$$A_d = e^{A_c \Delta} \qquad B_d = \int_0^{\Delta} e^{A_c r} B_c \, dr$$

$$C_d = C_c \qquad D_d = D_c$$

- How does one **determine in practice** the matrices described into the step-invariant transform?
- Are exact solutions or approximate expressions available?

# C2d with Sampler & Hold: Practical Issues (cont.)

## Exact formulas for the step-invariant transform

$$A_d = e^{A_c \Delta} \quad \Longleftarrow \quad e^{A_c t} = \mathcal{L}^{-1}\left\{(sI - A_c)^{-1}\right\}$$

$$B_d = \int_0^{\Delta} e^{A_c r} B_c \, dr = A_c^{-1} \cdot \left[e^{A_c \Delta} - I\right] \cdot B_c$$ if $A_c$ is nonsingular

## Approximate expressions

$$A_d = e^{A_c \Delta} \approx I + A_c \Delta + \frac{A_c^2 \Delta^2}{2!} + \frac{A_c^3 \Delta^3}{3!} + \cdots$$

$$B_d = \int_0^{\Delta} e^{A_c r} B_c \, dr \approx \left[\Delta + \frac{A_c \Delta^2}{2!} + \frac{A_c^2 \Delta^3}{3!} + \cdots\right] \cdot B_c$$

# Equivalent State-Space Representations

# Equivalent State-Space Representations

Consider the discrete-time dynamic system state-space representation:

$$\begin{cases} x(k+1) = f(x(k), u(k), k) \\ y(k) = g(x(k), u(k), k) \end{cases}$$

Let $\hat{x} := Tx$, where $T \in \mathbb{R}^{n \times n}$ is a generic non-singular $n \times n$ matrix ($\det(T) \neq 0$).

Then, the equivalent state-space description is given by:

$$\begin{cases} \hat{x}(k+1) = Tx(k+1) = Tf(T^{-1}\hat{x}(k), u(k), k) = \hat{f}(\hat{x}(k), u(k), k) \\ y(k) = g(T^{-1}\hat{x}(k), u(k), k) = \hat{g}(\hat{x}(k), u(k), k) \end{cases}$$

by suitably defining functions $\hat{f}$ and $\hat{g}$.

# Linear Dynamic Systems

# Linear Dynamic Systems

Consider the discrete-time dynamic system state-space representation:

$$\begin{cases} x(k+1) = f(x(k), u(k), k) \\ y(k) = g(x(k), u(k), k) \end{cases}$$

This state-space equation describes a **linear system** if and only if the functions $f(\cdot)$ and $g(\cdot)$ are **linear with respect to their state and input vector arguments**:

$$\forall\, \alpha_1, \alpha_2 \in \mathbb{R},\ \forall\, x_1, x_2 \in \mathbb{R}^n,\ \forall\, u_1, u_2 \in \mathbb{R}^m \ :$$

$$\begin{aligned} f(\alpha_1 x_1 + \alpha_2 x_2, \alpha_1 u_1 + \alpha_2 u_2, k) &= \alpha_1 f(x_1, u_1, k) + \alpha_2 f(x_2, u_2, k) \\ g(\alpha_1 x_1 + \alpha_2 x_2, \alpha_1 u_1 + \alpha_2 u_2, k) &= \alpha_1 g(x_1, u_1, k) + \alpha_2 g(x_2, u_2, k) \end{aligned}$$

# Linear Dynamic Systems: Matrix Form

Consider the state-space representation:

$$\begin{cases} x(k+1) = f(x(k), u(k), k) \\ y(k) = g(x(k), u(k), k) \end{cases}$$

and suppose that the linearity assumption holds. Then:

$$\begin{cases} f_1(x,u,k) = a_{11}(k)x_1 + \cdots + a_{1n}(k)x_n + b_{11}(k)u_1 + \cdots + b_{1m}(k)u_m \\ \vdots \\ f_n(x,u,k) = a_{n1}(k)x_1 + \cdots + a_{nn}(k)x_n + b_{n1}(k)u_1 + \cdots + b_{nm}(k)u_m \\ \\ y_1 = c_{11}(k)x_1 + \cdots + c_{1n}(k)x_n + d_{11}(k)u_1 + \cdots + d_{1m}(k)u_m \\ \vdots \\ y_p = c_{p1}(k)x_1 + \cdots + c_{pn}(k)x_n + d_{p1}(k)u_1 + \cdots + d_{pm}(k)u_m \end{cases}$$

where $a_{ij}(k), b_{ij}(k), c_{ij}(k), d_{ij}(k)$ are generic functions of the discrete-time index $k$.

# Linear Dynamic Systems: Matrix Form (cont.)

Letting:

$$A(k) := \begin{bmatrix} a_{11}(k) & \cdots & a_{1n}(k) \\ \vdots & \ddots & \vdots \\ a_{n1}(k) & \cdots & a_{nn}(k) \end{bmatrix} \; ; \quad B(k) := \begin{bmatrix} b_{11}(k) & \cdots & b_{1m}(k) \\ \vdots & \vdots & \vdots \\ b_{n1}(k) & \cdots & b_{nm}(k) \end{bmatrix}$$

$$C(k) := \begin{bmatrix} c_{11}(k) & \cdots & c_{1n}(k) \\ \vdots & \ddots & \vdots \\ c_{p1}(k) & \cdots & c_{pn}(k) \end{bmatrix} \; ; \quad D(k) := \begin{bmatrix} d_{11}(k) & \cdots & d_{1m}(k) \\ \vdots & \vdots & \vdots \\ d_{p1}(k) & \cdots & d_{pm}(k) \end{bmatrix}$$

$$x(k) := \begin{bmatrix} x_1(k) \\ \vdots \\ x_n(k) \end{bmatrix} \; ; \quad u(k) := \begin{bmatrix} u_1(k) \\ \vdots \\ u_m(k) \end{bmatrix} \; ; \quad y(k) := \begin{bmatrix} y_1(k) \\ \vdots \\ y_p(k) \end{bmatrix}$$

One gets:

$$\begin{cases} x(k+1) = A(k)x(k) + B(k)u(k) \\ \\ y(k) = C(k)x(k) + D(k)u(k) \end{cases}$$

# Linear Dynamic Systems

## Time-Invariant Linear Dynamic Systems

# Time-Invariant Linear Dynamic Systems

In the **time-invariant** scenario, the matrices $A(k), B(k), C(k), D(k)$ do not depend on the time-index $k$, that is are **constant** matrices $A, B, C, D$:

$$A := \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \; ; \quad B := \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \vdots & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix}$$

$$C := \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{p1} & \cdots & c_{pn} \end{bmatrix} \; ; \quad D := \begin{bmatrix} d_{11} & \cdots & d_{1m} \\ \vdots & \vdots & \vdots \\ d_{p1} & \cdots & d_{pm} \end{bmatrix}$$

and thus:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ \\ y(k) = Cx(k) + Du(k) \end{cases}$$

# Time-Invariant Linear Dynamic Systems: Equilibrium States

Consider a linear time-invariant dynamic system:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ \\ y(k) = Cx(k) + Du(k) \end{cases}$$

and consider a constant input sequence $u(k) = \bar{u},\ k \geq 0$. Hence, one has to solve the following equation for $x$:

$$x = Ax + B\bar{u} \implies (I - A)x = B\bar{u}$$

The following two cases have to be considered:

- $\det(I - A) \neq 0$
- $\det(I - A) = 0$

# Time-Invariant Linear Dynamic Systems: Equilibrium States

- $\det(I - A) \neq 0$. In this case, one gets:

$$\bar{x} = (I - A)^{-1} B\bar{u} \quad \Longrightarrow \quad \bar{x} \quad \text{is} \quad \textbf{unique} \ \forall\, \bar{u} \in \mathbb{R}^m$$

  Accordingly, the equilibrium output is given by:

$$\bar{y} = C\bar{x} + D\bar{u} = \left[ C(I - A)^{-1} B + D \right] \bar{u}$$

  Matrix $\left[ C(I - A)^{-1} B + D \right]$ is defined as **static gain**.

- $\det(I - A) = 0$. In this case, two different situations may occur:
  - $\exists\, \infty$ equilibrium states $\bar{x}$, $\exists\, \infty$ equilibrium outputs $\bar{y}$
  - $\nexists$ equilibrium states $\bar{x}$, $\nexists$ equilibrium outputs $\bar{y}$

# Equivalent State-Space Representations: LTI

Consider the discrete-time linear time-invariant (LTI) dynamic system state-space representation:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases}$$

Let $\hat{x} := T^{-1}x$, where $T \in \mathbb{R}^{n\times n}$ is a generic non-singular $n \times n$ matrix ( $\det(T) \neq 0$ ). Then, the equivalent state-space description is given by:

$$\begin{cases} \hat{x}(k+1) = T^{-1}x(k+1) = T^{-1}AT\hat{x}(k) + T^{-1}Bu(k) = \hat{A}\hat{x}(k) + \hat{B}u(k) \\ y(k) = CT\hat{x}(k) + Du(k) = \hat{C}\hat{x}(k) + Du(k) \end{cases}$$

Hence:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases} \iff \begin{cases} \hat{x}(k+1) = \hat{A}\hat{x}(k) + \hat{B}u(k) \\ y(k) = \hat{C}\hat{x}(k) + Du(k) \end{cases}$$

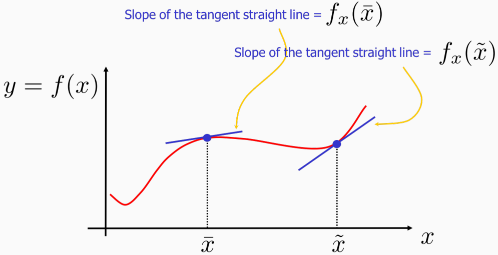
# Linear Dynamic Systems

## Linear Systems Obtained by Linearization

# Linear Systems Obtained by Linearization

## Basic Concept

- **Linear** systems are provided with numerous **analytical tools** that are not available for nonlinear systems
- **Approximating nonlinear systems by linear ones** in a "neighbourhood" of a nominal state movement may result very useful in practice




$$y(x) \simeq y(\bar{x}) + f_x(\bar{x})(x - \bar{x})$$

# Linear Systems Obtained by Linearization (cont.)

- Consider the nonlinear system:

$$\begin{cases} x(k+1) = f(x(k), u(k), k) \\ y(k) = g(x(k), u(k), k) \end{cases}$$

- Moreover, consider a **nominal state movement** $\bar{x}(k),\ k \geq k_0$ obtained by the initial state $x(k_0) = \bar{x}_0$ and the input sequence $u(k) = \bar{u}(k),\ k \geq k_0$.
- Let us **perturb** the initial state and the nominal input sequence, thus getting a **perturbed state movement**:

$$x(k_0) = \bar{x}_0 + \delta x_0\,;\ \ u(k) = \bar{u}(k) + \delta u(k) \implies x(k) = \bar{x}(k) + \delta x(k)$$

- Hence:

$$\begin{aligned} x(k+1) = \bar{x}(k+1) + \delta x(k+1) &= f(\bar{x}(k) + \delta x(k), \bar{u}(k) + \delta u(k), k) \\ &\simeq f(\bar{x}(k), \bar{u}(k), k) + f_x(\bar{x}(k), \bar{u}(k))\delta x(k) + f_u(\bar{x}(k), \bar{u}(k))\delta u(k) \end{aligned}$$

# Linear Systems Obtained by Linearization (cont.)

- Since the nominal state sequence $\bar{x}(k)$ is the solution of the difference equation $\bar{x}(k+1) = f(\bar{x}(k), \bar{u}(k), k)$, it follows that

$$\begin{aligned} \delta x(k+1) \simeq \quad & f_x(\bar{x}(k), \bar{u}(k))\delta x(k) + f_u(\bar{x}(k), \bar{u}(k))\delta u(k) \\ & = A(k)\delta x(k) + B(k)\delta u(k) \end{aligned}$$

where $A(k) \in \mathbb{R}^{n\times n},\ B(k) \in \mathbb{R}^{n\times m},\ k \geq k_0$ are defined as:

$$A(k) = f_x(\bar{x}(k), \bar{u}(k), k) = \left[ \begin{array}{ccc} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial f_n}{\partial x_1} & \cdots & \dfrac{\partial f_n}{\partial x_n} \end{array} \right]_{x(k)=\bar{x}(k), u(k)=\bar{u}(k)}$$

$$B(k) = f_u(\bar{x}(k), \bar{u}(k), k) = \left[ \begin{array}{ccc} \dfrac{\partial f_1}{\partial u_1} & \cdots & \dfrac{\partial f_1}{\partial u_m} \\ \vdots & & \vdots \\ \dfrac{\partial f_n}{\partial u_1} & \cdots & \dfrac{\partial f_n}{\partial u_m} \end{array} \right]_{x(k)=\bar{x}(k), u(k)=\bar{u}(k)}$$

# Linear Systems Obtained by Linearization (cont.)

- Concerning the **perturbed output** one has:

$$\bar{y}(k) = g(\bar{x}(k), \bar{u}(k), k)\,; \quad y(k) = \bar{y}(k) + \delta y(k)$$

Hence

$$\begin{aligned} y(k) &= g(x(k), u(k), k) = g(\bar{x}(k) + \delta x(k), \bar{u}(k) + \delta u(k), k) \\ &\simeq g(\bar{x}(k), \bar{u}(k), k) + g_x(\bar{x}(k), \bar{u}(k))\delta x(k) + g_u(\bar{x}(k), \bar{u}(k))\delta u(k) \end{aligned}$$

and then

$$\begin{aligned} \delta y(k) \simeq \quad & g_x(\bar{x}(k), \bar{u}(k))\delta x(k) + g_u(\bar{x}(k), \bar{u}(k))\delta u(k) \\ & = C(k)\delta x(k) + D(k)\delta u(k) \end{aligned}$$

where $C(k) \in \mathbb{R}^{p \times n}$, $D(k) \in \mathbb{R}^{p \times m}$, $k \geq k_0$ are defined as:

# Linear Systems Obtained by Linearization (cont.)

$$C(k) = g_x(\bar{x}(k), \bar{u}(k), k) = \begin{bmatrix} \dfrac{\partial g_1}{\partial x_1} & \cdots & \dfrac{\partial g_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial g_p}{\partial x_1} & \cdots & \dfrac{\partial g_p}{\partial x_n} \end{bmatrix}_{x(k)=\bar{x}(k), u(k)=\bar{u}(k)}$$

$$D(k) = g_u(\bar{x}(k), \bar{u}(k), k) = \begin{bmatrix} \dfrac{\partial g_1}{\partial u_1} & \cdots & \dfrac{\partial g_1}{\partial u_m} \\ \vdots & & \vdots \\ \dfrac{\partial g_p}{\partial u_1} & \cdots & \dfrac{\partial g_p}{\partial u_m} \end{bmatrix}_{x(k)=\bar{x}(k), u(k)=\bar{u}(k)}$$

**Summing up**: the linear system obtained by linearization around a given nominal state movement $\bar{x}(k),\ k \geq k_0$ obtained by the initial state $x(k_0) = \bar{x}_0$ and the input sequence $u(k) = \bar{u}(k),\ k \geq k_0$ is

$$\begin{cases} \delta x(k+1) = A(k)\delta x(k) + B(k)\delta u(k) \\ \delta y(k) = C(k)\delta x(k) + D(k)\delta u(k) \end{cases}$$

# Linear Systems Obtained by Linearization (cont.)

## Important Special Case: Time-Invariant Systems

- Consider the nonlinear time-invariant system:

$$\begin{cases} x(k+1) = f(x(k), u(k)) \\ y(k) = g(x(k), u(k)) \end{cases}$$

- Moreover, consider an **equilibrium state** $\bar{x}$ obtained by the constant input sequence $u(k) = \bar{u},\ k \geq k_0$.
- Let us **perturb** the initial state and the nominal input sequence, thus getting a **perturbed state movement**:

$$x(k_0) = \bar{x}_0 + \delta x_0\,;\ u(k) = \bar{u} + \delta u(k) \implies x(k) = \bar{x} + \delta x(k)$$

- Hence:

$$\begin{aligned} x(k+1) = \bar{x} + \delta x(k+1) = f(\bar{x} + \delta x(k), \bar{u} + \delta u(k)) \\ \simeq f(\bar{x}, \bar{u}) + f_x(\bar{x}, \bar{u})\delta x(k) + f_u(\bar{x}, \bar{u})\delta u(k) \end{aligned}$$

# Linear Systems Obtained by Linearization (cont.)

- Since the equilibrium state $\bar{x}$ is the constant solution of the algebraic equation $\bar{x} = f(\bar{x}, \bar{u})$, it follows that

$$\begin{aligned} \delta x(k+1) \simeq \quad & f_x(\bar{x}, \bar{u})\delta x(k) + f_u(\bar{x}, \bar{u})\delta u(k) \\ & = A\delta x(k) + B\delta u(k) \end{aligned}$$

where $A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}$ are **constant matrices** defined as:

$$A = f_x(\bar{x}, \bar{u}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{x(k)=\bar{x}, u(k)=\bar{u}}$$

$$B = f_u(\bar{x}, \bar{u}) = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \dots & \frac{\partial f_1}{\partial u_m} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial u_1} & \dots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}_{x(k)=\bar{x}, u(k)=\bar{u}}$$

# Linear Systems Obtained by Linearization (cont.)

- Concerning the **perturbed output** one has:

$$\bar{y} = g(\bar{x}, \bar{u})\,; \quad y(k) = \bar{y} + \delta y(k)$$

Hence

$$\begin{aligned} y(k) = g(x(k), u(k)) &= g(\bar{x} + \delta x(k), \bar{u} + \delta u(k)) \\ &\simeq g(\bar{x}, \bar{u}) + g_x(\bar{x}, \bar{u})\delta x(k) + g_u(\bar{x}, \bar{u})\delta u(k) \end{aligned}$$

and then

$$\begin{aligned} \delta y(k) \simeq \quad & g_x(\bar{x}, \bar{u})\delta x(k) + g_u(\bar{x}, \bar{u})\delta u(k) \\ & = C\delta x(k) + D\delta u(k) \end{aligned}$$

where $C \in \mathbb{R}^{p\times n},\ D \in \mathbb{R}^{p\times m}$ are **constant matrices** defined as:

# Linear Systems Obtained by Linearization (cont.)

$$C = g_x(\bar{x}, \bar{u}) = \begin{bmatrix} \dfrac{\partial g_1}{\partial x_1} & \cdots & \dfrac{\partial g_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial g_p}{\partial x_1} & \cdots & \dfrac{\partial g_p}{\partial x_n} \end{bmatrix}_{x(k)=\bar{x}, u(k)=\bar{u}}$$

$$D = g_u(\bar{x}, \bar{u}) = \begin{bmatrix} \dfrac{\partial g_1}{\partial u_1} & \cdots & \dfrac{\partial g_1}{\partial u_m} \\ \vdots & & \vdots \\ \dfrac{\partial g_p}{\partial u_1} & \cdots & \dfrac{\partial g_p}{\partial u_m} \end{bmatrix}_{x(k)=\bar{x}, u(k)=\bar{u}}$$

**Summing up**: the linear time-invariant system obtained by linearization around a given equilibrium state $\bar{x}$ obtained by the constant input sequence $u(k) = \bar{u},\ k \geq k_0$ is

$$\begin{cases} \delta x(k+1) = A\delta x(k) + B\delta u(k) \\ \delta y(k) = C\delta x(k) + D\delta u(k) \end{cases}$$

Consider the nonlinear discrete-time system:

$$\begin{cases} x_1(k+1) = x_1(k) + \alpha(1-\beta x_1(k))x_1(k) - \gamma x_1(k)x_2(k) + u(k) \\ x_2(k+1) = x_2(k) - \delta x_2(k) + \eta x_1(k)x_2(k) \\ y(k) = x_2(k) \end{cases}$$

Imposing the constant input sequence $\bar{u}(k) = 0$ the following equilibrium states are obtained:

$$\bar{x}_{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} ; \ \bar{x}_{(2)} = \begin{bmatrix} \frac{1}{\beta} \\ 0 \end{bmatrix} ; \ \bar{x}_{(3)} = \begin{bmatrix} \frac{\delta}{\eta} \\ \frac{\alpha}{\gamma}\left(1 - \frac{\beta\delta}{\eta}\right) \end{bmatrix}$$

# Linear Systems Obtained by Linearization: Example (cont.)

The general expression for matrix $A$ of the linearized system is:

$$f_x(\bar{x},\bar{u}) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{bmatrix}_{\bar{x},\bar{u}} =$$

$$= \begin{bmatrix} (1+\alpha-2\alpha\beta x_1-\gamma x_2) & -\gamma x_1 \\ \eta x_2 & 1-\delta+\eta x_1 \end{bmatrix}_{\bar{x},\bar{u}}$$

Substituting the expressions of the specific equilibrium states one gets:

$$\bar{x}_{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad \bar{A}_{(1)} = \begin{bmatrix} (1+\alpha) & 0 \\ 0 & 1-\delta \end{bmatrix}$$

# Linear Systems Obtained by Linearization: Example (cont.)

$$\bar{x}_{(2)} = \begin{bmatrix} \frac{1}{\beta} \\ 0 \end{bmatrix} \implies \bar{A}_{(2)} = \begin{bmatrix} (1-\alpha) & -\frac{\gamma}{\beta} \\ 0 & 1-\delta+\frac{\eta}{\beta} \end{bmatrix}$$

$$\bar{x}_{(3)} = \begin{bmatrix} \frac{\delta}{\eta} \\ \frac{\alpha}{\gamma}\left(1-\frac{\beta\delta}{\eta}\right) \end{bmatrix} \implies \bar{A}_{(3)} = \begin{bmatrix} \left(1-\frac{\alpha\beta\delta}{\eta}\right) & -\frac{\gamma\delta}{\eta} \\ \frac{\alpha\eta}{\gamma}\left(1-\frac{\beta\delta}{\eta}\right) & 1 \end{bmatrix}$$

# Linear Systems Obtained by Linearization: Example (cont.)

Finally, the other matrices $B$, $C$, and $D$ of the linearized systems are given by (their values do not depend on the specific equilibrium states):

$$f_u(\bar{x}, \bar{u}) = \left[ \begin{array}{c} \dfrac{\partial f_1}{\partial u} \\ \dfrac{\partial f_2}{\partial u} \end{array} \right]_{\bar{x}, \bar{u}} = \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] = \bar{B}$$

$$g_x(\bar{x}, \bar{u}) = \left[ \begin{array}{cc} \dfrac{\partial g}{\partial x_1} & \dfrac{\partial g}{\partial x_2} \end{array} \right]_{\bar{x}, \bar{u}} = \left[ \begin{array}{cc} 0 & 1 \end{array} \right]_{\bar{x}, \bar{u}} = \left[ \begin{array}{cc} 0 & 1 \end{array} \right] = \bar{C}$$

$$g_u(\bar{x}, \bar{u}) = \left. \frac{\partial g}{\partial u} \right|_{\bar{x}, \bar{u}} = 0_{\bar{x}, \bar{u}} = 0 = \bar{D}$$

267MI –Fall 2022

Lecture 1

Generalities: Systems and Models

END