Sci & Educ (2015) 24:661–698
DOI 10.1007/s11191-015-9743-0

Quod erat demonstrandum: Understanding and Explaining Equations in Physics Teacher Education

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Published online: 4 February 2015

Abstract In physics education, equations are commonly seen as calculation tools to solve problems or as concise descriptions of experimental regularities. In physical science, however, equations often play a much more important role associated with the formulation of theories to provide explanations for physical phenomena. In order to overcome this inconsistency, one crucial step is to improve physics teacher education. In this work, we describe the structure of a course that was given to physics teacher students at the end of their master's degree in two European universities. The course had two main goals: (1) To investigate the complex interplay between physics and mathematics from a historical and philosophical perspective and (2) To expand students' repertoire of explanations regarding possible ways to derive certain school-relevant equations. A qualitative analysis on a case study basis was conducted to investigate the learning outcomes of the course. Here, we focus on the comparative analysis of two students who had considerably different views of the math-physics interplay in the beginning of the course. Our general results point to important changes on some of the students' views on the role of mathematics in physics, an increase in the participants' awareness of the difficulties faced by learners to understand physics equations and a broadening in the students' repertoire to answer "Why?" questions formulated to equations. Based on this analysis, further implications for physics teacher education are derived.

1 Introduction

Students often regard physics equations as calculation tools into which numbers are "plugged in" and answers are "chugged out." In fact, this is a widespread approach found

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both in research and practice. This instrumental view has been identified in the literature as recursive plug-and-chug (Tuminaro and Redish 2007), plug-and-chug unstructured (Walsh et al. 2007), calculation framing (Bing and Redish 2009), technical dimension (Karam 2014), formulas epistemological beliefs (Hammer 1994), fixation with formulas (Schecker 1985), among others. Such view becomes evident in problem solving when students blindly search for equations without trying to make sense of the physical situation. Nonetheless, one important question that is seldom asked is the following: Why do students frequently perform this way? Here is one tentative answer based on plausible reasoning:

Fact (given)	Students often use plug-and-chug when solving physics problems.
Axiom (assumed)	Using mathematics (more specifically equations) to reason about the physical world is not an intrinsic/native human ability; it is a socially constructed one. Thus, one has to *learn how to do* that, usually in formal education.
Conclusion (proved)	Therefore, students perform this way (plug-and-chug) because they are *taught to do so*.

This illustrates a well-known vicious circle faced by physics educators. If we are taught physics in a particular way, it is likely that we will repeat it when we become teachers. An instruction tradition that focuses on "given-sought" tasks demanding mere calculational skills has already been identified in previous works. For instance, Dittmann et al. (1989) analyzed several German school textbooks and proposed the term "zerrechnete Physik" to represent the philosophy found in more than 80 percent of the proposed problems (tasks). The German word "zerrechnen" combines the verbs calculate (berechnen) and tear up (zerreißen), thus implying that physics is being destroyed by the plug-and-chug emphasis.[1]

Another important aspect related to the role of equations in physics education concerns their origin. In physics lessons, it is not unusual to find a naïve inductive approach in which equations are treated as descriptions of empirical regularities. The passage from experimental data to the algebraic expression is often presented as trivial, despite the major epistemological incompatibility of these two domains (Falk 1990). This highlights the importance of an instruction informed by essential characteristics of the *Nature of Physics*, especially regarding the role of mathematics. As in the case of treating equations as calculation tools, the routines already established for obtaining equations inductively (e.g., laboratory worksheets) also represent obstacles for a considerable change in the teaching practice.

It is possible to act in different directions to break the vicious circle that leads to the notion of equations as calculation tools and descriptions of experimental regularities in physics education. One crucial dimension is, undoubtedly, teacher education. In this paper, we describe a course given to physics teacher students that aimed at shifting focus from *calculate and describe* to *explain and understand,* concerning the role of equations in physics instruction. In the next two sections, the goals of the course are precisely stated and justified. Section 4 is dedicated to a detailed description of the course activities. Our research design is presented in Sect. 5, and a pre-post comparison (learning outcomes and

[1] However, if we seek to understand the reasons why this practice is so deeply penetrated in physics teaching culture, we have to look at the situation from the teachers' perspective. It is impossible to deny the practicality of this approach for the everyday life of a teacher. Numerical tasks that follow the "given-sought" scheme are very easy to be created and graded. This can be quite handy in the stressful routine of a school. Therefore, a deep epistemological conviction that physics is not about calculating with formulas seems to be a necessary condition for a different (not as handy) approach.

beliefs) of two students who have fundamentally different conceptions is analyzed in Sect. 6. In the last section, we discuss our main findings and derive implications for physics teacher education.

2 Understanding Equations (an Epistemological Motivation)

Fundamental ideas, principles and laws of physics are usually represented by equations. Newton's law of gravitation, Hamilton's principle of least action, Maxwell's field equations, Einstein's energy–mass equivalence and Schrödinger's wave equation are some among many other examples. In this sense, being able to understand equations is certainly an essential ability for a successful physicist. But what exactly does it mean to "understand an equation"? What are the required conditions that provide such personal "feeling of understanding"? And how do these subjective demands correlate with the didactical approaches chosen by physics instructors to explain the meaning of physics equations to their students?

Previous works on students' understanding of physics equations have adopted different frameworks for defining what "understanding equations" means. In one of the most influential among them, Sherin (2001) sustains that successful students can understand equations in a fundamental sense since they are able to address a rather sensorial meaning to certain formal operations. By videotaping group sessions of physics students solving problems in pairs and discussing their solutions (thinking out loud), Sherin reports several occasions in which students conceptualize physical situations in terms of equations. The analysis conducted by the author is synthesized in a set of 21 *symbolic forms*, each of them representing the association of a simple conceptual schema with a pattern of symbols in an equation. Some examples are the following: the physical notion of *balancing* (symbol pattern $\square = \square$), often found in equilibrium situations, the more general idea of *competing terms* ($\square \pm \square \pm \square \ldots$), commonly present in problems involving directed quantities like forces, and the overall *base* ± *change* ($\square \pm \triangle$) structure, used for instance in equations such as $x = x_0 + vt$ (uniform motion) and $p = p_0 + \rho gh$ (hydrostatic pressure). Among the instructional implications of this work, Sherin advocates in favor of a *modeling instruction*, in which students are encouraged to invent mathematical representations to model physical situations. Sherin's approach has developed in parallel to a concept introduced to the German mathematics education community by vom Hofe (1995). His "Grundvorstellungen" ("basic notions," for a lack of a better translation) have a lot in common with Sherin's conceptual schema. Finally, there is a third perspective that makes use of very similar assumptions, namely the conceptual metaphor approach put forward by Lakoff and Nunez (2000). While a comparative analysis of these frameworks is not part of this article, their parallel development can be considered an indicator for the relevance of this issue.

Another perspective of "understanding equations" is found in the work of Bagno et al. (2008). Based on the multiple task view regarding understanding (Perkins and Blythe 1994), the authors argue that students can demonstrate their understanding of a formula by performing several activities related to it, such as presenting an association map of the formula, describing its components with their own words, identifying special cases, applying it in problem solving, among others. This assumption led to the development of an instructional tool that allowed the identification of several learning difficulties and was successfully applied for promoting a better understanding of formulas both with high school students and in-service physics teachers. Similarly, in his very inspiring paper, Romer (1993)[2] proposes that understanding an equation is being able to *read* it, like reading and admiring a poem or a painting. Analyzing the equation that calculates the range of a projectile given its initial velocity and angle $\left(R = \frac{2v_0^2 \sin\theta \cos\theta}{g}\right)$, the author dedicates about five pages discussing interesting questions to be asked to the equation, such as "What if?" and "Why?"questions, idealizations, dimensional analysis, limiting cases, etc. He argues that those questions should be the focus of physics instruction instead of the usual "back of the book" numerical ones.

Other studies have investigated students' views on what it means to understand a physics equation. At university level, Domert et al. (2007) interviewed 20 students and categorized their answers in six *epistemological mindsets* that range from the superficial identification of descriptive aspects (e.g., recognize the symbols of the equation) to a more theoretical understanding (e.g., recognize the underlying physics of the equation or its structure). In high school level, Hechter (2010) asked the same question to 42 students and grouped their answers in the following categories: *Pragmatic position*—Know where and how to use the equation; *Mathematical evolution*—Know the process to obtain the equation (derivation); *Independent pieces*—Know what every term/variable means; *Literal/personal meaning*—being able to explain the equation fully to someone. This author emphasizes that understanding is a polysemic and subjective notion and that physics educators should make efforts to teach all "kinds of understanding."

Most of the works approaching this issue treat equations and formulas in a general way, and we prefer a more specific approach. From an epistemological perspective, an equation may have quite different roles. Thus, we propose an overall *epistemological categorization of equations*, which involve different views on their epistemological nature. Although this categorization is built on theoretical ground and not a result of the analysis of empirical data, we argue that it allows classifying one's conception about the origin of a particular equation, based on one's answers to questions like: Where does the equation come from? What does it represent? Why is it expressed in this way?

Our proposal is to classify the subjective epistemological status of equations into the following categories:

1. *Principles* (e.g., $\sum \vec{p} = 0$ and $\delta S = 0$)
2. *Definitions* (e.g., $\vec{v} = \mathrm{d}\vec{r}/\mathrm{d}t$ and $\vec{p} = m\vec{v}$)
3. *Empirical regularities* (e.g., Balmer's formula)
4. *Derivations* (e.g., $a = v^2/r$ and $E = mc^2$)

Thus, we argue that the "understanding equations" issue must be considered separately in each specific case (category).[3] Furthermore, being aware of this distinction seems crucial, since ways to explain a certain equation—and to structure lessons to teach it—may depend strongly on its epistemological character. *Principles* can be presented as plausible truths confirmed by observations and experiments; *Definitions* need to be somehow justified, concerning the needs to conceive particular physical quantities; *Empirical regularities* are normally clarified by performing experiments, whereas equations resulting from *Derivations* should be explained by formal deductions showing how they can be obtained from general principles and/or definitions.

[2] We thank one of the reviewers for raising our attention to this paper.

[3] A similar categorization is proposed by Romer (1993). The previously mentioned "range equation", for instance, belongs to category 4. We thank one of the reviewers for pointing that out to us.

It is important to stress, however, that this is not to be thought as a strict set of mutually exclusive categories, due to the intrinsic flexibility of theoretical constructs and the changes in the epistemological status of equations through history. Kepler's laws, for instance, were first identified as empirical regularities and later formally derived from Newton's law of gravitation. Moreover, one cannot ignore the essential changes inherent to every didactization process (Chevallard 1991; Duit et al. 2012). Therefore, it must be clear that the goal of this categorization is to draw attention to the subjective level (both teacher and student) and to its possible implications for the teaching and learning of equations and the physics insight condensed in them.

This work describes a course designed for pre-service teachers, in which the equations chosen for a detailed analysis (free fall: $s = gt^2/2$, centripetal acceleration: $a = v^2/r$, period of a simple pendulum: $T = 2\pi\sqrt{l/g}$ and Snell's law: $sin\ \theta_1/sin\ \theta_2 = v_1/v_2$) were deliberately selected because they could be classified in the last category of equations logically derived from definitions and principles. Taking the theoretical character of physical science into account, the focus is to highlight the essential explanatory role of mathematical reasoning in physics. To exemplify the pedagogical relevance of this issue, suppose that a high school student is studying the phenomenon of refraction and learns about Snell's law.[4] Although she can successfully use this equation to solve different kinds of problems, it should be natural (indeed desirable!) that at some point she would ask herself: *Why* does this formula have this particular structure? *Why* cannot it be a ratio of the angles, their cosines, tangents or something else?

Why questions reflect our human desire to formulate explanations for natural phenomena, and there are many ways to answer this particular one. One possibility is to say that "it comes from experiment." A physics teacher could perform an experiment with an optical bench, measure several angles and show that the relation is always valid. The hidden message behind this approach would be that physical formulas are obtained by inductive reasoning, i.e., by repeatedly performing experiments, finding regularities and expressing them mathematically. In this case, the equation would be seen as the *description of an empirical regularity*.

Understanding an equation from that perspective would refer to being able to interpret the equation, to relate manipulations of the equation's parameter to phenomena in the real world, e.g., manipulating g in $s = gt^2/2$ would mean to change the acceleration caused by a gravitational force, e.g., by switching from free fall on earth to free fall on moon. This translation between the real world and the algebraic representation of a model of this world (and vice versa) seems to be indicative of some kind of "interpretative" understanding. As a process that links (mathematical) models and the real world, those translations play a major role in mathematical modeling as researched in the field of mathematics education (e.g., Lesh and Zawojewski 2007). Understanding $s = gt^2/2$ could also mean to develop a sense for how the expression $s = gt^2/2$ behaves, since this would reflect some (perhaps only implicit) insight into the mathematical structure of the equation, e.g., knowing that from $s_i = s(t_i)$ and $t_2 = 2t_1$, it follows $s_2 = 4s_1$ indicates at last some kind of "manipulative" understanding.

These ways to look at an equation and to make sense of it—the interpretative and the manipulative way—are essential in learning physics in a meaningful way. However, not in contradiction but in addition to this view, we go a step further by not asking *how* entities in the formula or real world behave, but *why* there are the entities we find and *why* they are

[4] The line of reasoning that follows could be equally applied for the other three equations approached in the course ($y = gt^2/2$, $a = v^2/r$, $T = 2\pi\sqrt{l/g}$).

related the way they are in this equation. From this point of view, an equation is more than a description of empirical regularities.

It is impossible to deny the importance of inductive reasoning in physics (Whewell 1858). However, this science is certainly more than "asking" nature for the fundamental rules. Much is achieved by abstracting, modeling, hypothesizing, i.e., the *deductive reasoning* is an essential trait of physical science. When Feynman used Fermat's least time principle to derive the law of refraction in his lectures, he explicitly emphasized that:

> In the further development of science, we want more than just a formula. First we have an observation, then we have numbers that we measure, then we have a law which summarizes all the numbers. **But the real glory of science is that we can find a way of thinking such that the law is evident** (Feynman et al. 1964, p. 26-3, our emphasis).

If the law is evident, it means that it can be logically deduced. Deriving formulas from physical principles and logical thought is similar[5] to proving mathematical theorems from a set of axioms. This style of reasoning is widely exemplified in Euclid's Elements and can also be found in several Physics' masterpieces, such as Galilei's Two New Sciences, Newton's *Principia* and Einstein's Special Relativity. In fact, Einstein clearly mentioned the similarity between geometry and theoretical physics by stating that:

> The theorist's method involves **his using as his foundation general postulates or principles from which he can deduce conclusions**. His work thus falls into two parts. He must first discover his principles and then **draw the conclusions that follow from them** (Einstein 1956, p. 110, our emphasis).

Hence, explaining in physics is strongly associated with establishing connections via mathematical reasoning. As Rivadulla (2005, p. 166) clearly puts it, "any physical construct can only receive a theoretical explanation when it can be deduced mathematically in the framework of another physical construct of higher level." In physics education, this kind of understanding is hardly assessed by the standard view of equations as calculation tools to solve quantitative problems or as regularities extracted from experiments. Considering the deductive character of physics, derivations should enhance the understanding about the origin of equations, allow students to penetrate into the inner structure of physics and avoid the rote memorization of meaningless expressions. Thus, highlighting this *explanatory* role of mathematical reasoning in physics is the first goal of the course designed to pre-service teachers. Achieving this goal, we believe, would not only enhance the content matter knowledge of our students, but also increase their awareness about the nature of physics and the role of mathematics. Behind this expectation lies the assumption of students' conceptions about equations being part of their views on the role of mathematics in physics (e.g., Angell et al. 2004; Krey 2012). Although we are aware of this body of research and borrow some of the instruments from this work to support our own investigation, we are concerned with more specific aspects, namely the view on equations.

3 Explanations Repertoire (a Didactical Motivation)

In the previous section, we presented an epistemological motivation for the course, i.e., to focus on how the derivation of equations from principles and definitions is associated with

[5] One has to be careful when speaking about the similarities between physics and mathematics because there are also important differences in the types of reasoning and the methods used by physicists and mathematicians. Feynman (1985, pp. 46–58) provides a good discussion about these differences when he distinguishes between the "Babylonian" and the "Greek" traditions of thinking when using mathematics.

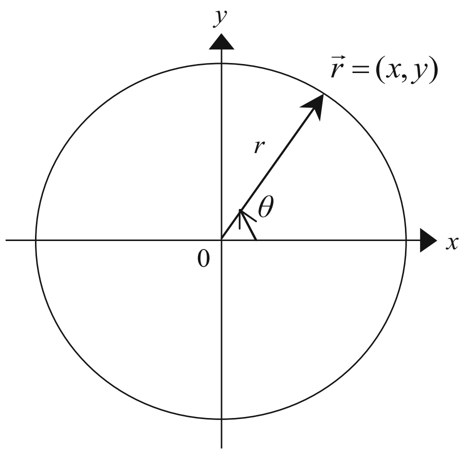


Fig. 1 Point mass in uniform circular motion

formulating theoretical explanations in physics. However, considered from a didactical perspective, this issue is far from being trivial. If, for instance, a physics teacher performs the derivation of an equation in a rigorous and straightforward manner, it is likely that the students will perceive it as an artificial and meaningless set of steps to be memorized and reproduced in the exams.

Let us illustrate this problem with a hypothetical situation. Suppose that a physics instructor wants to show her students why the acceleration in a uniform circular motion is equal to $a = v^2/r$. She believes that the reasoning is quite straightforward, since one just needs to derive the position vector with respect to time twice in order to obtain the acceleration. Considering a point mass moving in a circular trajectory, its position vector can be expressed in Cartesian coordinates as follows:

$$\vec{r} = r(\cos(\omega t)\hat{x} + \sin(\omega t)\hat{y}) \tag{1}$$

where r is the (fixed) radius, ω is the (constant) angular velocity $(\omega = \theta/t = v/r)$, $\hat{x}$ and $\hat{y}$ the unit vectors corresponding to the x and y axes, respectively (see Fig. 1).

The velocity is then obtained by taking the first derivative of position with respect to time[6]

$$\vec{v} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = -\omega r(\sin(\omega t)\hat{x} - \cos(\omega t)\hat{y}) \tag{2}$$

Finally, one more derivation gives the acceleration of the point mass

$$\vec{a} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = -\omega^2 r(\cos(\omega t)\hat{x} + \sin(\omega t)\hat{y}) \tag{3}$$

Substituting (1) in (3), we obtain

$$\vec{a} = -\omega^2 \vec{r} \tag{4}$$

[6] One could actually use physical arguments (inertia) to conclude that the velocity is perpendicular to the position. Then, starting from this "physical fact," it is possible to "prove" the derivation rules for the trigonometric functions sine and cosine [see Chapter 6 in Levi (2009)].

Thus, the absolute value of the acceleration is equal to $\omega^2 r$ and since $\omega = v/r$, the desired result ($a = v^2/r$) is obtained. In fact, Eq. (4) provides also the acceleration's direction, since the negative sign shows that it points in the opposite direction of the radius; therefore, it is often called the *centripetal* acceleration.

Undoubtedly, this is a very elegant derivation. It stresses the intrinsic mathematical nature of physical concepts (e.g., acceleration), and how they are strongly connected through mathematical reasoning. Nevertheless, the crucial question here is: can students follow this derivation and deeply understand each of its steps?

It is impossible, of course, to answer this question generally, but the average upper secondary-level student would face serious difficulties when trying to make sense of the performed steps. Typical questions are: Why can one represent the position this way? Why is the acceleration the second derivative of the position in respect to time? (Or even: What is a derivative and what does it mean?) Why is the derivative of sine equal to cosine? Why is the chain rule valid? This avalanche of questions reflects students' genuine attempts to understand the topic and should be encouraged in physics education. In this particular case, it seems that the overload of mathematical rules and representations can hinder a meaningful understanding of the physical situation.

Instead of simply telling the students that they are not ready yet to fully understand this topic, a well-prepared teacher should have at her disposal a broad range of possibilities to explain/derive this equation, an ability that will be addressed here as *explanations repertoire*. In fact, this is one of the most important components of what Shulman (1986) called pedagogical content knowledge (PCK). According to him:

> PCK includes the most useful forms of representation, the most powerful analogies, illustrations, examples, and demonstrations […] Since there are no single most powerful forms of representation, **the teacher must have at hand a veritable armamentarium of alternative forms of representations** (Shulman 1986, p. 9, our emphasis).

This reports us to the crucial role of multiple representations to learn and to do science (Prain and Tytler 2013). In fact, representations allow us to materialize abstract ideas. To represent ideas is to allow manipulation, which supports further thinking about and working with these abstract ideas. Representations in general are also used to communicate ideas and discuss them with fellow researchers or learners. Due to the nature of what we call physics, mathematical representations are crucial. Graphical representations, such as tables, diagrams, graphs, make information accessible and can be used to reduce cognitive load and increase the learner's or researcher's efficacy. As research has shown, graphical representations are perceived as helpful by most learners (Krey 2012), although learners sometimes have difficulties to read and interpret or even construct those graphical representations (Doorman 2005).

Algebraic representations (formulas or equations) as we consider them in this article are perceived as helpful mainly by successful learners (Krey 2012). Learners with difficulties in their performance in general show a dislike for algebraic representations. The question of what is cause and what is effect needs further research attention. What actually happens in a cognitive system when working with graphical and algebraic representations is still under research (e.g., Landy and Goldstone 2007a, b). However, we are not aware of any publication that would seriously neglect the value of those representations in general. While there is some debate about when and how mathematical concepts (and their algebraic representations) should be introduced to support learning in physics or science, there is no doubt that physics as we know it today is in need of mathematical descriptions in general and algebraic representations in particular. Thus, being able to express physical concepts in different semiotic modalities (Lemke 1998) and to transit between them is likely to be an essential ability for teaching physics.

As with any representation, there are intra-representational processes (identifying the constituting elements, their relation, etc.) and inter-representational processes (e.g., translating (aspects) into another representational form). The value of multiple representations is looked at in depth in Ainsworth (2006). Sherin (2001) has offered a list of symbolic forms that help identify constituting elements of algebraic representations that can be metaphorically/conceptually linked to non-mathematical experiences (Lakoff and Nunez 2000). This distinction can be helpful to identify and systematize problems when learning physics by means of algebraic representations. Another helpful distinction is based on the nature of processes related to the representational form. It is important to note that reading an equation, interpreting an equation and constructing an equation are fundamentally different processes that require different cognitive resources. This distinction again turns out to be helpful, for instance, when thinking about possible and suitable usage of an equation in a learning scenario, as we did with the participants of the seminar under consideration.

Besides the pedagogical relevance of a broad explanations repertoire, due to the fact that individuals experience different feelings of understanding when exposed to different representations, knowing how to represent and derive an equation in several ways is also related to a broader view of what it means to understand an equation. Quite often, different approaches involve starting from different physical assumptions, using various models and mathematical structures. If in the previous section we stressed that understanding an equation involves being able to derive it, now we wish to stress the importance of knowing how to do this in several ways. Similarly, after presenting an alternative derivation of the kinetic theory of gases, Pfeffer (1999) concludes:

> It is an instructive exercise, for both pupils and teachers, to show how the mathematical description of a physical system can be derived from different starting points. Apart from the novelty and the challenge of the intellectual exercise involved, **each approach gives different insights** and, when taken together, they serve to **deepen the students' appreciation** of the subject (Pfeffer 1999, p. 237, our emphasis).

Hence, the second major goal of the course is to present and discuss different ways to explain and derive equations that are relevant for the high school physics curriculum ($s = gt^2/2$, $a = v^2/r$, $T = 2\pi\sqrt{l/g}$ and $sin\theta_1/sin\theta_2 = v_1/v_2$) with the purpose of expanding the students' *explanations repertoire*, as well as their understanding of the content matter. During the course, one meeting was dedicated to each of these equations and the different approaches enabled specific discussions concerning learning difficulties, mathematical prerequisites and school level. When looking for alternative ways to explain and derive these equations, historical sources such as Galileo's, Huygens' and Newton's original argumentations frequently provided more intuitive/concrete derivations, since they often relied in less abstract mathematics (mainly geometrical representations).

4 Course Description

The course is designed for future physics teacher students and its final version was conducted at the Technical University of Dresden, Germany. To become a teacher in Germany you have to study two subjects, one of which was physics for our students. The second subject could be almost any other subject, e.g., Mathematics, History, and English. The 13 students who participated in the course were in their seventh semester, roughly 1 year before they have taken their final exams. All of them had their Bachelor studies completed and passed all of their foundational and some of their advanced courses, including mathematics for physicists covering calculus, analytical geometry, vector analysis, etc. Students had also attended most of their pedagogical and science education courses and were studying to become high school teachers in their respective subjects. In general that means they are going to teach physics for pupils from lower and upper secondary level.

In this section, we describe the structure of the course and relate it with its two major goals, namely (1) *Epistemological*: recognize the explanatory role of mathematics in physics by means of derivation of equations and (2) *Didactical*: acquire a broad repertoire to explain (derive) relevant equations of secondary-level physics. The course was given in 2013 to physics teacher students of the universities of Helsinki and Dresden.[7] The six meetings (3 h each) were divided into a theoretical part (2 meetings of 3 h each)—in which historical and philosophical aspects of the relationship between physics and mathematics were approached—and a more practical one (4 meetings of 3 h each)—where the equations mentioned in Sect. 3 were presented and discussed.

In the theoretical part, reference texts[8] were previously recommended to the students. Divided into groups, the students were assigned with the task of summarizing parts of the texts and formulating questions related to them. A synthesis of core aspects was presented by the lecturer (first author) and followed by a group discussion based on the questions formulated by the students. Some of the main topics approached in these first two meetings were:

- Consequences of the historical (1700–1900) mathematization of physics: social, epistemological and ontological (Gingras 2001);
- Significance of some basic mathematical conceptions (e.g., multiplication, function, real and complex numbers) for physics (Bochner 1963; Falk 1990);
- Relation of mathematics to physics: mathematics as language, physical explanations through mathematical deduction, diverse mathematical formulations of the same phenomenon, Babylonian and Euclidean reasoning (Feynman 1985)

Considering the focus of this paper on understanding and explaining equations, we will provide a detailed description of the practical part, in which four equations typically taught in high school level were approached. In each meeting, the class was divided into groups that received different ways to explain (derive) the equation, based on different lines of argumentation, mathematical formalisms, physical models, etc. Each group had the task of presenting their explanation to the whole class, as if it were a high school lesson (micro teaching). After each presentation, the lecturer moderated a discussion with the whole group, concerning pros and cons of the explanation, such as typical difficulties faced by students, mathematical prerequisites and curricular constrains. Additionally, a brief overview of the historical development of the equation was presented. In the following we describe the explanations (derivations) received by the groups and stress some of the core issues discussed in the groups.

7 A first trial was conducted at the University of Hamburg (Dec 2012), a pilot version of the course was held at the University of Helsinki (April–May 2013) and the final version was given at the Technical University of Dresden (November–December 2013). In all cases the teacher students were in the end of their master's studies, i.e. all physics and mathematics courses have already been taken.

8 Gingras (2001), Bochner (1963), Falk (1990) and Feynman (1985).

4.1 Case Study 1: Free Fall

Due to its central historical importance, the free fall is a classical case study to discuss the relationship between physics and mathematics. Some usual questions associated with the equation $s = gt^2/2$ are: Where does the ½ come from? Why is s proportional to t^2? The four lines of reasoning approached in the course to obtain the formula were:

4.1.1 Inductive Way (from Experiment to Equation)

This is the classical inductive presentation found in many school textbooks. It depicts an experiment in which the position of a falling body is measured through time. The data are usually summarized in a table (position vs. time) and the relation $s \propto t^2$ (parabola) is obtained. High-resolution cameras enable very precise measurements. One of the main controversial points was the following: How can we be sure that s is (exactly) proportional to t^2? If we are looking for a mathematical expression that fits our data, why can it not be $s \propto t^{1.99}$ or say $s \propto t^{2.000001}$? This question motivated an intensive discussion among students, some defending that it is just a matter of making the instruments more precise and others arguing that the exact $s \propto t^2$ is a theoretical construct which is not to be found in the experimental data.[9]

4.1.2 Galileo's Reasoning

Another group received the task of presenting Galileo's line of reasoning found in his *Dialogues Concerning Two New Sciences* [1638, 1914]. In his own words:

> When I observe a stone initially at rest falling from an elevated position and continually acquiring new increments of speed, why should I not believe that such increases take place in a manner which is exceedingly simple and rather obvious to everybody? If now we examine the matter carefully we find no addition or increment more simple than that which repeats itself always in the same manner (Galilei 1914, p. 197).

Thus, by setting the assumption that speed varies constantly with time as a *principle*, Galileo was able to derive the relation $s \propto t^2$ (Ibid, Theorem II, p. 209). His presentation was, however, essentially geometrical and grounded in the previously known mean speed theorem (Ibid, Theorem I, p. 208). These aspects motivated discussions about the advantages of the geometrical approach and critical arguments against the inductive way to obtain the equation.

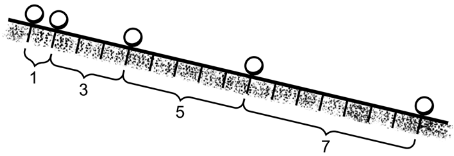
4.1.3 Law of Odd Numbers

This presentation involves a different way to formulate the free fall law. It goes as follows: "The distance travelled by a falling body during equal intervals of time stand to one another in the same ratio as the series of odd numbers beginning from unity" (Ibid, Corollary I, Theorem II, p. 210). In other words, the distances traversed by falling bodies in equal times obey the following sequence 1, 3, 5, 7, 9… (see Fig. 2). This is quite a comprehensible formulation of the free fall law (Wagenschein 1968) and depicts an astonishing regularity, which motivates discussions about the possibility of describing natural laws through mathematics.[10] This group was also responsible for reflecting on

[9] The same kind of critical argument can be made for all equations discussed in the course.

[10] See, for instance, a hypothetical dialogue between Galileo and Aristotle at http://hipstwiki.wikifoundry.com/page/moving+bodies.

Fig. 2 Ball falling down an inclined plane and the series of odd numbers



Galileo's strategy to slow down the fall by observing balls moving down inclined planes. Several students manifested positive opinions about the pedagogical potential of this approach.

4.1.4 Differential and Integral Calculus

The last group presented the traditional and abstract way of deriving the free fall law using calculus. The line of reasoning is quite straightforward: Considering that the acceleration is the second derivative of position in respect to time and the fact that it is constant in free fall, finding the $s(t)$ function is a matter of solving a very simple differential equation

$$\frac{d^2 s}{\mathrm{d}t^2} = g$$

whose solution for $g = \text{constant}$ is

$$s(t) = s_0 + v_0 t + \frac{gt^2}{2}$$

where $v_0 = 0$ (body abandoned) and $s_0 = 0$ (frame of reference).

The discussion after this presentation was mainly focused on the issue of mathematical prerequisites. Several participants argued that since high school students do not learn to solve differential equations in their mathematics lessons, this approach would be impossible at this level. Nevertheless, prerequisite arguments cannot be the only relevant ones, especially if we consider the wide-ranging character of the calculus approach. Moreover, the historical development of calculus sustains the idea that motion is an extremely appropriate context to learn the basic concepts of derivative and integral.

The free fall is an exemplar for analyzing many aspects of the relationship between mathematics and physics. During the lessons, this case study stimulated discussions about the interplay between theory and experiment (e.g., Does the exact 2 come from the data? Is the inclined plane experiment equivalent to the free fall?), the geometrical character of Galileo's original work, the simplicity assumptions in his argumentations (e.g., constant speed change in time) and the eventual mathematical essence of nature (How come the rolling ball can count odd numbers?). Furthermore, the modern calculus derivation highlights the refinement of our language and shows how we can derive the free fall equation as natural consequence of the definition of acceleration.

4.2 Case Study 2: Centripetal Acceleration

The fact that a uniform circular motion is accelerated already poses a major difficulty for students and is well documented in the literature (e.g., Viennot 1979; Clement 1982; Hestenes et al. 1992; Galili and Bar 1992). But the centripetal acceleration equation

($a = v^2/r$) carries a deeper mystery: Where does the v^2 come from? There are numerous ways to obtain this equation and the ones approached in the course were:

4.2.1 Vector Difference—Similar Triangles

This derivation is commonly found in high school textbooks. A material point moves in uniform circular motion. The position ($\vec{r}_1$ and $\vec{r}_2$) and velocity ($\vec{v}_1$ and $\vec{v}_2$) vectors are separated by an infinitesimal time interval (see Fig. 3a). This assumption enables one to make the approximation that the point's displacement is a line segment and the desired equation is obtained by triangle similarity (ΔOAB–ΔBCD—see Fig. 3b).

$$\frac{\mathrm{d}r}{r} = \frac{\mathrm{d}v}{v} \therefore \frac{v\mathrm{d}t}{r} = \frac{\mathrm{d}v}{v} \therefore \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{v^2}{r} \therefore a = \frac{v^2}{r}$$

Although the mathematics needed for this derivation is quite simple, its physical interpretation is far from being trivial. The biggest problem seems to be the conciliation of two very abstract ideas, namely vector subtraction ($\vec{v}_2 - \vec{v}_1$) and infinitesimal increase ($\mathrm{d}\theta$, $\mathrm{d}r$, $\mathrm{d}v$). This is especially complicated when one tries to draw infinitesimal triangles that exist only in our minds. During the course, students reported that they became more aware of many of these difficulties after thinking carefully about this derivation and facing the challenge of explaining it to the whole class.

4.2.2 Analogical Circles

Velocity is the rate of change of position. Acceleration is the rate of change of velocity. In a uniform circular motion, velocity is always perpendicular to position, whereas acceleration is always perpendicular to velocity. These similarities lead to an astonishingly simple derivation of the centripetal acceleration equation (Wedemeyer 1993). The whole argument is based on the fact the three vectors involved ($\vec{r}$, $\vec{v}$, $\vec{a}$) take the same time to complete one revolution (see Fig. 4).

The period of the left circle (Fig. 4a) is $T = 2\pi r/v$ and, by analogy, the one of the right circle (Fig. 4b) is $T = 2\pi v/a$. Since they are equal:

$$\frac{2\pi r}{v} = \frac{2\pi v}{a} \therefore a = \frac{v^2}{r}$$

If one represents the analogy between the vectors mathematically, there is actually no need to consider revolution periods:

$$\frac{v}{r} = \frac{a}{v} \therefore a = \frac{v^2}{r}$$

This derivation has the advantage of being extremely short and stresses the formal analogy between velocity and acceleration. However, the downside is that circle Fig. 4b can be rather abstract for students, since it is represented in velocity space.

4.2.3 Moon's Constant Fall Toward the Earth (Gamow)

The two previous presentations treated acceleration formally as velocity change. The next ones shall improve this aspect by deriving the equation using concrete/physical situations based on the notion of force. The approach received by this group involves considering the

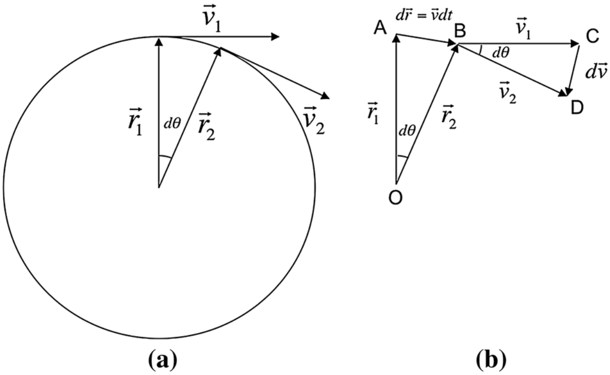


Fig. 3 Point mass in uniform circular motion

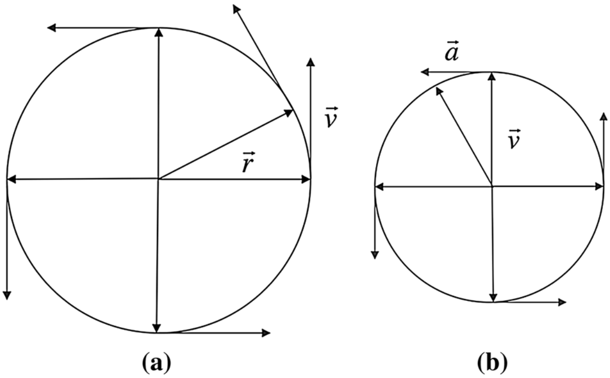
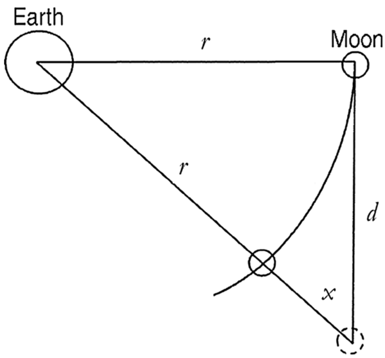


Fig. 4 Point mass in uniform circular motion

Fig. 5 Moon falls toward the Earth (adapted from Corrao 2012)



Moon's orbit and analyzing its movement if the Earth were to vanish (Gamow 1962; Corrao 2012).

Using the Pythagorean theorem in the right triangle of Fig. 5:

$$(r + x)^2 = d^2 + r^2$$

where d is the distance traversed by the Moon if the Earth were not there ($d = v\Delta t$) and x the distance the Moon has *fallen* toward the Earth $\left(x = \frac{1}{2}a\Delta t^2\right)$. Substituting and rearranging the terms

$$ra\Delta t^2 + \frac{1}{4}a^2\Delta t^4 = v^2\Delta t^2$$

dividing both sides by Δt^2

$$ra + \frac{1}{4}a^2\Delta t^2 = v^2$$

Here comes a crucial argument that was quite controversial among students. The smaller the time interval the better is the approximation (from straight lines to the curve). Thus, assuming $\Delta t \to 0$ the second term on the left vanishes and the desired equation $(a = v^2/r)$ is obtained.

4.2.4 *Infinite Collisions (Newton)*

We are quite familiar with describing rectilinear motions, but how to model curvilinear ones? As it is usually the case in physics (and mathematics) we start from what we know. Newton modeled circular motion by considering elastic collisions inside a circle (Newton and Henry 2000). Depending on the speed and angle with which the particle collides, regular polygons are obtained. The circle is achieved through an *ad infinitum* argument (see Fig. 6a). Due to geometrical arguments, it is possible to justify why the force (acceleration) acting on the particle during each collision points to the center (centripetal) (Fig. 6b).

Applying conservation of linear momentum, we can show that the magnitude of the change in velocity after each collision is $\Delta v = 2v\cos\theta$. After n collisions, the sum of the fractional changes is $\Sigma\Delta v = 2nv\cos\,\theta$. Since $n\varphi = 2\pi$ and $\varphi + 2\theta = \pi$ (Fig. 6c)

$$\Sigma\Delta v = 2\frac{2\pi}{\varphi}v\cos\left(\frac{\pi}{2} - \frac{\varphi}{2}\right) = \frac{4\pi}{\varphi}v\sin\left(\frac{\varphi}{2}\right)$$

The greater the number of collisions, the closer we get to the circle ($\varphi \to 0$ and $\sin\left(\frac{\varphi}{2}\right) \cong \frac{\varphi}{2}$). Thus, the sum of the changes in velocity after one complete circle is

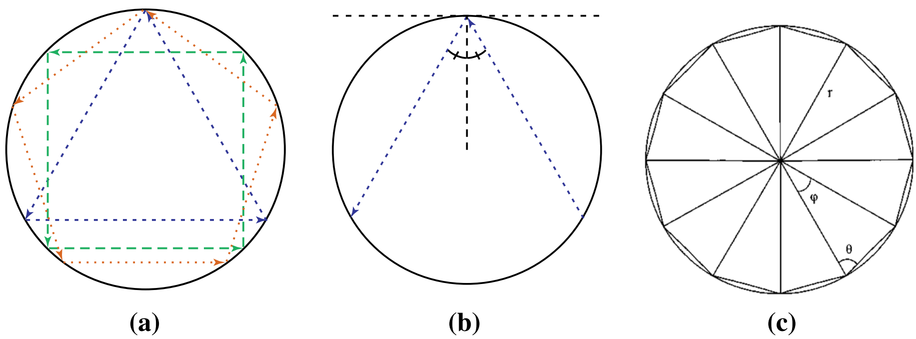


Fig. 6 Collisions inside a *circle*. The more collisions, the closer to a circular trajectory. *Source*: **a**, **b** http://www.imsc.res.in/~savery/notes/poly-a.pdf and **c** Newton and Henry (2000)

$\Sigma\Delta v = 2\pi v$. From the total amount of change in velocity, we can obtain an average acceleration, which is our desired expression

$$a = \frac{\Delta v}{\Delta t} = \frac{\Sigma\Delta v}{T} = \frac{2\pi v}{\left(\frac{2\pi r}{v}\right)} = \frac{v^2}{r}$$

Here no infinitesimal argument was needed. Using symmetry arguments, it is possible to show that the *instantaneous* acceleration is also $a = v^2/r$ (Newton and Henry 2000). Students appreciated the "concreteness" of this derivation, but criticized its mathematical complexity.

The centripetal acceleration equation is often a mystery to the students. In this case study, it is possible to highlight the difficulties involved in describing curvilinear motions using the formalism developed for analyzing rectilinear ones. The challenge of combining infinitesimal and vector reasoning becomes evident for the teacher students, which increases their awareness of students' difficulties. Substituting numbers in $a = v^2/r$ is undoubtedly easier than explaining why a is proportional to v^2. Another interesting aspect is to compare the derivations that treat acceleration abstractly as velocity change with the more concrete ones that relate acceleration to force.

4.3 Case Study 3: Simple Pendulum

The simple pendulum is a classical in physics teaching. Among many epistemological lessons that can be learned in this case study (Matthews 2000; Matthews et al. 2005), our focus was on the fact that this pendulum does not exist in nature. Furthermore, the formula of the pendulum's period ($T = 2\pi\sqrt{l/g}$) is full of obscurities. Why is the period proportional to $\sqrt{l/g}$? Where does the 2π come from?

4.3.1 Projection of a Circular Motion

The most common representation of harmonic oscillations found in school textbooks is the projection of a circular motion (Fig. 7). This is a didactical strategy designed to surmount

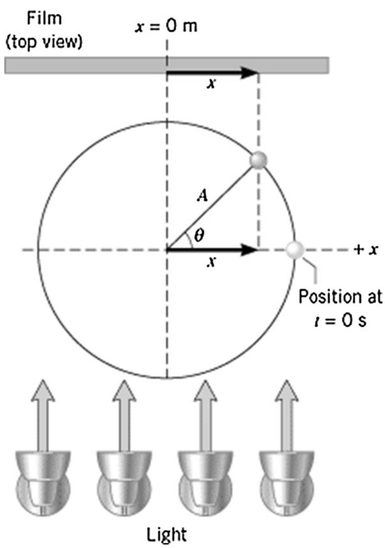


Fig. 7 Harmonic oscillation as projection of a circular motion. *Source*: http://demo.webassign.net/ebooks/cj6demo/pc/c10/read/main/c10x10_2.htm

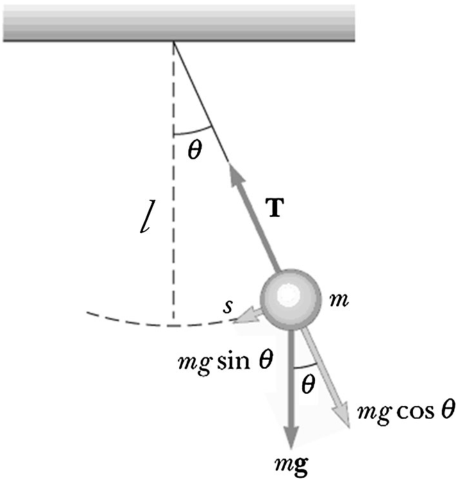


Fig. 8 Forces acting on the bob and its components. *Source*: http://static.sewanee.edu/physics/PHYSICS101/Pendulum.html

the formal problem of solving differential equations. It enables one to write position, velocity and acceleration as functions of time:

$$x = A\cos\left(\frac{2\pi}{T}t\right)$$

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{2\pi}{T}A\sin\left(\frac{2\pi}{T}t\right)$$

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{4\pi^2}{T^2}A\cos\left(\frac{2\pi}{T}t\right)$$

Harmonic oscillations are the ones in which the restoring force is proportional to the deformation (Hooke's law). The classical example is the spring-mass system (spring constant k and mass m), whose period can be determined as follows:

$$F = ma \therefore -kx = ma \therefore -kA\cos\left(\frac{2\pi}{T}t\right) = -m\frac{4\pi^2}{T^2}A\cos\left(\frac{2\pi}{T}t\right)$$

$$T = 2\pi\sqrt{\frac{m}{k}}$$

The "simple" pendulum is a bit more complicated than the spring-mass system. By decomposing the forces acting on the bob it is possible to show that the resultant is the tangential component of the weight ($-mg\sin\theta$) as depicted in Fig. 8. A crucial assumption needed to treat the simple pendulum as a harmonic oscillator is to consider only small amplitudes, so that the approximation $\sin\theta \approx \theta$ can be made.

$$F = ma \therefore -mg\sin\theta = ma$$

for small angles, $\sin\theta \approx \theta$ and $\mathrm{s} \approx \mathrm{x} \approx \theta \mathrm{l}$

$$-mg\theta = ma \therefore -mg\frac{x}{l} = ma \therefore -mg\frac{\mathrm{A}\cos\left(\frac{2\pi}{T}t\right)}{l} = -m\frac{4\pi^2}{T^2}\mathrm{A}\cos\left(\frac{2\pi}{T}t\right)$$

$$T = 2\pi\sqrt{\frac{l}{g}}$$

This is certainly a long and arduous path to the equation. Among the positive aspects mentioned by the students is the fact that 2π comes automatically from the projection of a circular motion and that the derivation makes evident that the expression $T = 2\pi\sqrt{l/g}$ is only valid (or at least it is a good approximation) for small amplitudes.

4.3.2 Differential Equation

Seeing Newton's second law as a differential equation makes several dynamic problems quite straightforward.

$$F = ma \therefore -mg\sin\theta = ml\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$$

The small angle approximation ($\sin\theta \approx \theta$) can now be justified from a pragmatic (mathematical) point of view, i.e., it makes the differential equation linear

$$-mg\theta = ml\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$$

whose solution [for $\theta(0) = \theta_0$ and $\frac{\mathrm{d}\theta}{\mathrm{d}t}(0) = 0$] is

$$\theta(t) = \theta_0\cos\left(\sqrt{\frac{g}{l}}t\right)$$

Thus, the periodic character of this motion comes directly from the solution. The period of the function is

$$T = 2\pi\sqrt{\frac{l}{g}}$$

This derivation has the advantage that it is based on the fundamental characteristic of harmonic motion, namely the linear restoring force. The solution may be both elegant and straightforward, but the amount of mathematical knowledge needed seems to be above reasonable for high school level. A possible strategy also discussed in the course is the numerical approach. By solving the equation numerically, students gain an important conceptual understanding about the meaning of differential equations.

4.3.3 Conic Pendulum

A conic pendulum (Fig. 9) is apparently more complicated than the simple pendulum, but in this case, it offers an easier derivation of the equation (Gindikin 2007; Mahajan 2010).

We start by applying Newton's second law to the bob in order to calculate the period of the conic pendulum (r is the circle's radius and $F = T\sin\theta = \frac{mg}{\cos\theta}\sin\theta$)

$$mg\tan\theta = \frac{mv^2}{r} \therefore v = \sqrt{gr\tan\theta}$$

Since $T = 2\pi r/v$

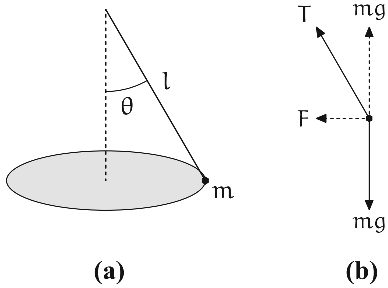


Fig. 9 Conic pendulum (Mahajan 2010)

$$T = 2\pi\sqrt{\frac{l\cos\theta}{g}}$$

For small angles ($\cos\,\theta\,\approx\,1$), the desired equation is obtained. One way to interpret this is to realize that for small angles the conic pendulum is a good approximation to the simple pendulum. Moreover, the mysterious 2π comes relatively easy out of the derivation.

4.3.4 Inclined Planes

The fundamental idea that we start from rectilinear motions to think about curvilinear ones is again highlighted in this derivation. Furthermore, it gives us an idea about how Galileo, Huygens, Newton and others studied pendulum motion. The derivation consists in modeling the pendulum motion as two inclined planes (Pitucco 1980). In Fig. 10 we notice that the pendulum's period can be estimated by calculating the time needed by the bob (ball) to slide down the first inclined plane and then multiplying the result by 4. Once again, the smaller the angle θ, the better the approximation.

Considering that the acceleration in an inclined plane is $g\sin\,\theta$ and assuming the small angle approximation $\sin\,\theta\,\approx\,\theta,$ we obtain

$$s \approx l\theta \approx {}^1\!/_2(g\theta)t^2$$

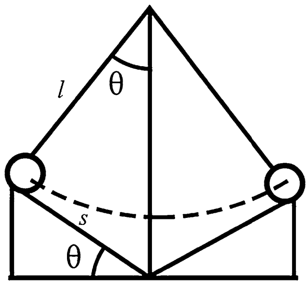
$$t = \sqrt{2}\sqrt{l/g}$$

$$T = 4t = 4\sqrt{2}\sqrt{l/g}$$

In only two lines, we got the desired $\sqrt{l/g}$, but instead of $2\pi(\approx 6,28)$ we obtained $4\sqrt{2}(\approx 5,66)$. This is a great opportunity to discuss how mathematical precision goes beyond our intuition. The biggest problem is that the inclined plane angle, which is tangent to the pendulum motion, varies during the descent and this variation must be taken into account mathematically. This was indeed a major challenge for Galileo, who was not able to obtain the correct expression for the period of the simple pendulum (Matthews 2000).

The formula of the simple pendulum period ($T = 2\pi\sqrt{l/g}$) contains elements that are so far away from the concrete world that it would be a fallacy to argue that it comes out of the experiment. These demonstrations highlight the ideal character of this pendulum and emphasize the fact that it does *not* exist in nature. For different reasons, the approximation $\sin\,\theta\,\approx\,\theta$ has to be made in all the derivations. In three of them, the periodic character of the pendulum motion is given, whereas in the modern calculus derivation, it is obtained in

Fig. 10 Simple pendulum as inclined planes (Pitucco 1980)



the solution of a differential equation, which once again stresses the greater explanatory power of this mathematical structure.

4.4 Case Study 4: Refraction Law

The refraction law $\left(\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2} = \frac{n_2}{n_1} = \mathrm{const}\right)$ has an intriguing history that is seldom explored in physics education (Barth 1992; Mihas 2008). The standard inductive approach—measuring both incidence and refraction angles and showing that the ratio of their sines is constant—is both extremely artificial and unsatisfying. In reality, the sine function would rarely appear spontaneously. In fact, despite his great talent to look for mathematical regularities in experimental data, not even Kepler was able to see the sine function (Schiffer and Bowden 1984). Moreover, the "from the data to the formula" approach is actually a description and not an *explanation* of this law. This case study offers many possibilities to approach core issues of the mathematical nature of physics and is also a great opportunity to show how different models can be thought out to understand the same phenomenon.

4.4.1 Huygens' Elementary Waves Principle

The classical textbook derivation of this law uses Huygens' principle of elementary waves (Fig. 11a—Huygens' original). The central argument is that the wavefront will maintain its

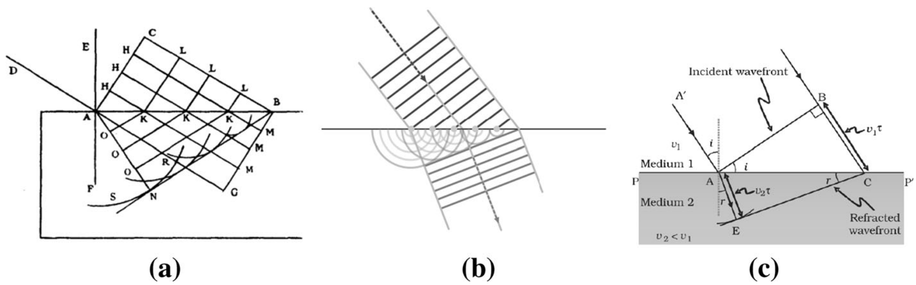


Fig. 11 Deriving the refraction law from Huygens' principle. **a** *Source*: Treatise on Light (Huygens, 1690)—English translation at http://www.gutenberg.org/files/14725/14725-h/14725-h.htm; **b** *Source*: http://en.wikipedia.org/wiki/Huygens%E2%80%93Fresnel_principle#mediaviewer/File:Refraction_-_Huygens-Fresnel_principle.svg; **c** *Source*: http://www.physics365.com/shared/images/huygen%20ppl_refraction%20on%20denser%20medium.png

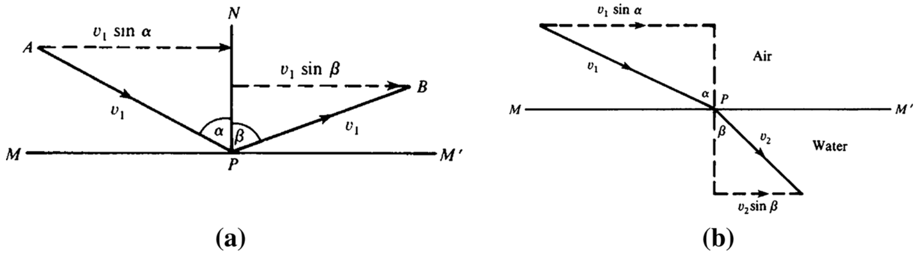


Fig. 12 Reflection (**a**) and refraction (**b**) explained with linear momentum conservation (Schiffer and Bowden 1984)

form when coming from Medium 1 to 2 (Fig. 11b). This implies that the time needed for the wave to traverse the distance BC in Medium 1 is the same as the time to traverse the distance AE in Medium 2 (Fig. 11c). From this equality, the law is derived as follows:

$$\frac{BC}{v_1} = \frac{AE}{v_2} \therefore \frac{AC \sin\theta_1}{v_1} = \frac{AC \sin\theta_2}{v_2} \therefore \frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$

It is clear that the wave model of light is needed to this derivation. However, the connection between the model and the phenomena (e.g., refraction with a laser pointer) is rather counterintuitive. One critical aspect discussed is that the path traced by the laser pointer seems to be a line (1D), whereas the wavefront is necessarily two-dimensional.

4.4.2 Newton's Corpuscular Model

The problem raised by the previous derivation could be eliminated if we model light as moving particles. Assuming that the force acting on a particle is perpendicular to the surface, the horizontal component of the linear momentum will remain constant. This assumption provides an explanation for the observed phenomena of reflection and refraction (Fig. 12).[11]

For reflection (Fig. 12a), the equality of the horizontal components yields

$$v_1 \sin\alpha = v_1 \sin\beta \therefore \alpha = \beta$$

which is in accordance with the law of reflection. For refraction (Fig. 12b), we have

$$v_1 \sin\alpha = v_2 \sin\beta \therefore \frac{\sin\alpha}{\sin\beta} = \frac{v_2}{v_1}$$

Even though the ratio between the sines is constant, this result predicts something different when compared to the one obtained with the wave model, namely that the speed of light in the water is *greater* than in the air ($\alpha > \beta \to v_2 > v_1$). Here we have a great opportunity to conduct a critical and informed discussion about the relationship between theory and experiment. Some students of the course were intrigued by this derivation because they were convinced that the corpuscular model could explain both reflection and refraction.

[11] More details in Schiffer and Bowden (1984), chapter 3.

4.4.3 Fermat's Least Time Principle

But after all, is light made of waves or particles? "Who cares!", thought Fermat. The most important is that light is very smart; it always chooses the quickest path. With this simple and general principle it is possible to derive the entire geometric optics. Connecting the refraction law with Fermat's principle is, however, quite demanding from a mathematical point of view.[12] The basic idea is to build an expression that calculates the total time for light to come from point A (in Medium 1) to point B (in Medium 2) as a function of the position where the light changes medium and then minimize this function. The essential variables for the derivation are depicted in Fig. 13.

The total time ($t = t_1 + t_2$) is

$$t = \frac{d_1}{v_1} + \frac{d_2}{v_2} = \frac{\sqrt{h_1^2 + x^2}}{v_1} + \frac{\sqrt{h_2^2 + (q-x)^2}}{v_2}$$

In order to minimize the function, we take its first derivative

$$t'(x) = \frac{x}{v_1\sqrt{h_1^2 + x^2}} - \frac{q-x}{v_2\sqrt{h_2^2 + (q-x)^2}} = \frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2}$$

and set the derivative equal to 0 to obtain the desired equation

$$\frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} = 0 \therefore \frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$

Besides the mathematical difficulty of this derivation, its essential idea motivated quite a fruitful discussion with the students. One of them asked: "How can the light know which is the quickest path even before taking the path? Does it possess some kind of omniscient consciousness?" Others refused to accept this derivation as an explanation to the phenomenon, since there is no model involved in it.

4.4.4 Mechanic Analog (Polya)

The last presentation highlights how reasoning by mathematical analogies is quite fruitful for physics. It is extracted from Polya's masterpiece *Mathematics and Plausible Reasoning* (1954, Vol. 1, pp. 142–155). Assuming Fermat's least time principle, the basic idea is to solve an analogical (mechanical) problem (14b) and infer the solution of the optical one (Fig. 14a).

Figure 14b represents a ring X that can slide without friction along a horizontal and rigid rod l. Two strings (XAP and XBQ) are attached to the ring, each of them passes through a pulley and carries a weight (p and q, respectively). Considering the general mechanical principle that the total potential energy of a system tends to a minimum (i.e., bodies tend to stay closer to the surface of the Earth), we arrive to the conclusion that

$$AP \cdot p + BQ \cdot q$$

must be a maximum, and therefore,

$$AX \cdot p + XB \cdot q$$

[12] An alternative geometric derivation—which is far less (mathematically) demanding and was also presented by the students—is found in *Feynman Lectures* (Feynman et al. 1964, p. 26-3).

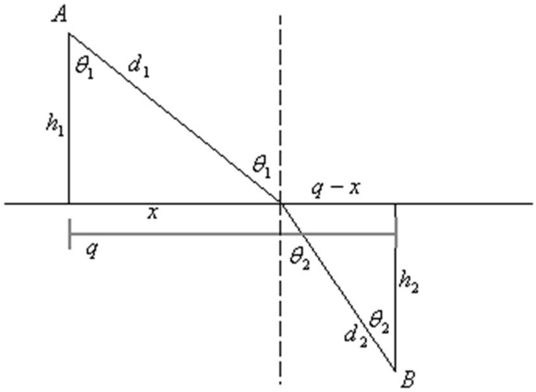


Fig. 13 Deriving the refraction law from Fermat's principle. *Source*: http://optics.mbhs.edu/snells-law/image3.png

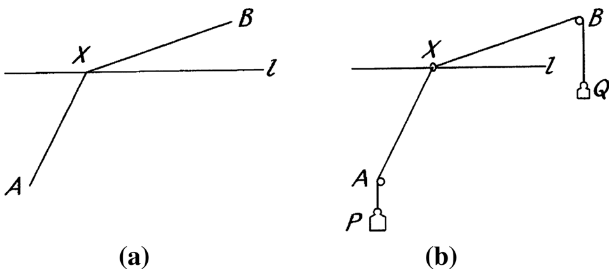


Fig. 14 Derivation from an analogical mechanical problem (Polya 1954)

must be a minimum. When compared to the optical problem that assumes the total time ($AX/v_1 + XB/v_2$) to be minimum, the formal analogy between these problems becomes evident ($p \to 1/v_1$ and $q \to 1/v_2$). But how does the analogy help us to solve the problem?

The mechanical context provides an extra condition—Polya calls it a "surplus of knowledge"—namely that the ring X must be in equilibrium, and therefore, the horizontal components of the wires' tensions must be equal. This leads to our desired result

$$p \cdot \sin\theta_1 = q \cdot \sin\theta_2$$

which by analogy yields

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$

This case study is extremely rich in opportunities to conduct epistemological discussions. The pure inductive (from experiment to formula) derivation can be questioned: Is it an explanation or a mere description? This leads to the explanatory role of models (wave or particle?). Interestingly, in this case the two models predict different experimental results and the real world (measurement) can decide which one is appropriate. But one can also forget the models and assume a general principle (Fermat's) to obtain Snell's law. Furthermore, it is possible to model the optical situation by analogy with a mechanical one and encourage students to appreciate the physics' unifying character as well as the wide-ranging power of mathematical analogies. How many learning opportunities are lost in a physics lesson that merely "derives" Snell's law from an experiment or just mentions it in order to solve numerical problems!

5 Research Design

Having described the course's goals and structure, we now concentrate on how it is perceived by physics teacher students at the end of their university studies and what kind of development in their thinking it might be able to trigger. Since this is the first course of its kind that we know of, it is not the time yet to ask what specific feature of the course has provoked what specific learning outcome for what student, or what (hidden) messages might have led learners to understand or interpret the course content in a perhaps unintended way. Furthermore, we cannot perform an overall evaluation of the course since there is no other course to compare with. Instead, we decided to conduct a comparative analysis on two students—pseudonymously called Michael and James—that we identified to be extremely different in their views on mathematics and physics. In our view, the analysis of these two contrasting cases will provide preliminary information about the effectiveness of the course and contribute to the not too rich body of research on students' views on the role of mathematics (here equations) in physics.

Our case study research design demands a qualitative analysis. In order to be able investigate how the course affected these two participants' conceptions and knowledge, we need to consider a wide range of information provided by them. An overview of the research design and the methods used can be found in Fig. 15. The methods and instruments will be described in more detail in the following paragraphs.

Our design allows us to gather process data as well as results of the learning process. We decided to apply a typical pre-post-design, using questionnaires that helped us to get information on our students' views about mathematics in physics and more specifically on their understanding of equations. The pretest included the following questions/tasks (summarized wording):

- How interested are you in experimental and theoretical physics? (5-level Likert item);
- What is mathematics? What is physics? Why and what for is mathematics used in physics? (open-ended questions, cf. Krey 2012);
- What does it mean to understand a physics equation? (open-ended question);
- 4-level Likert items about the role and status of equations in physics (e.g., "In physics, equations are derived from experiments." or "The main purpose of equations in physics is to calculate." (extracted from Krey 2012);
- How to explain the meaning the centripetal acceleration equation ($a = v^2/r$) in a physics lesson? The students are asked to mention important aspects and describe their explanation to a student, to explain why they consider the chosen aspects to be important and how they can help students to understand why the equation structure looks the way it does. (open-ended question);
- Do you feel like you fully understand the following equations? Why, why not? (e.g., the four equations approached in the course);
- Define the epistemological status (cf. categorization on Sect. 2) and importance of 16 physics equations (multiple choice);
- Find a metaphor for the relationship between mathematics and physics ("mathematics is for physics as …" "physics is for mathematics like …"

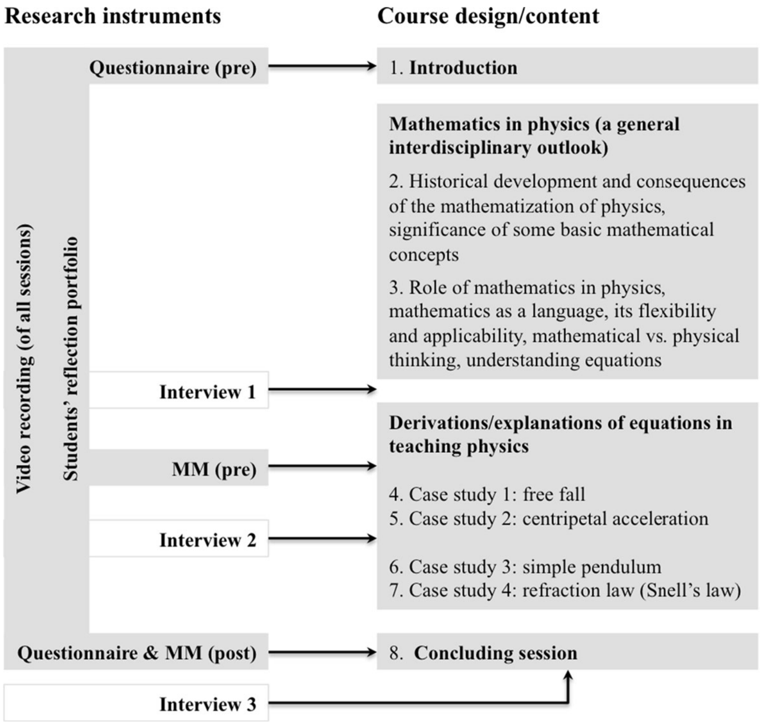


Fig. 15 Course program and research instruments

Before going through the case studies in lessons 4–7 (see Fig. 15), we also asked students to draw a mind map (MM) around the equations under consideration. So they are also part of our pretest data.

Students were asked to document their impressions about the course in a reflection portfolio and all sessions were video recorded to allow permanent access to what has happened during the seminar, e.g., for analyzing spontaneous discussions and the preparation of interviews. Interviews have been conducted with four learners after the more general part (lessons 1 and 2), after the 2nd case study (lesson 4) and at the end of the course (after lesson 8). Interviewees have been chosen to be contrasting based on the pretest described above, so that we could study for distinct cases more in depth, how the course has been perceived by different learners and what impact the course had.

For the following, we did not systematically make use of the video data. The comparative analysis offered below is based on data taken from the questionnaire and interviews. In a first (content analytical) step, relevant information has been identified in the interviews and questionnaire by the two authors. Important passages of the interviews have been transcribed and paraphrased. This summary of the data from both sources has been cross-evaluated by the other author and was followed by an argument-based explication of meaning. This allowed us to give a detailed description of the students' statements, but also to weight the statements in relation to the explicated core view. This process of description and interpretation was one of collaborative work, in which we validated our results through constant communication and controversial discussion.

6 Comparative Analysis

We will concentrate the comparative analysis of two students (Michael and James) on three aspects approached in the questionnaires (pre and post). These aspects and the corresponding questions are:

1. Mathematics, physics and the interplay
   (a) What is mathematics?
   (b) What is physics?
   (c) Why and what for is mathematics used in physics?
   (d) Complete the sentences metaphorically:
      (d.1) Mathematics is for physics like …
      (d.2) Physics is for mathematics like …

2. Explaining $a = v^2/r$ in a physics lesson
   You are the physics teacher of a class (upper secondary level) and prepare a lesson on circular motion. In one particular moment of this lesson, the centripetal acceleration equation ($a = v^2/r$) will be taught. Considering your didactical choices to present and explain this equation, answer the following questions:
   (a) What do you consider very important/essential?
   (b) How would you proceed?
   (c) Justify your answers to items a) and b).
   (d) During this lesson one of your students asks the following question: "Why is the equation like this?" Please sketch different ways to answer the student's question (the more the better). Note: The question is related to the equation's structure. The student wants to know, for instance, why the acceleration is directly proportional to v^2 or indirectly proportional to r, etc.

3. Understanding equations
   (a) What does it mean to you to understand a physics equation? When do you have the feeling that you really understood a physics equation/formula?
   (b) Given the equations below [among them the ones approached in the seminar] answer, for each one, the following questions:
      (b.1) Do you feel that you totally understand this equation?
      (b.2) **Why?** Try to justify your answer (both **yes and no**) as detailed as possible.

Since both Michael and James were asked to elaborate on their answers during the interviews and to reflect on possible changes in the pre-post comparison (Interview 3), we will complement our analysis with their justifications, especially when they clearly refer to some episode that occurred during the course. As it will be clear in the following, Michael and James have considerably different views regarding the role of mathematics (particularly equations) in physics. This can be concluded from the whole set of data we collected.

6.1 Michael

6.1.1 Biographical Information

Michael is 23 years old. He is studying to become a physics and history teacher, and took advanced courses in Physics and German in high school. In a 1–5 scale, he finds experimental physics interesting (4), whereas theoretical physics less interesting (2). He chose physics as one of his teaching disciplines because he believes a science teacher can easily get a job and he is more interested in physics than in biology or chemistry.

6.1.2 Physics, Mathematics and the Interplay (Pre)

In his pre-questionnaire, Michael assigns a rather positive and phenomenological function for physics when he defines it as "the study of nature." For him, this science "describes natural phenomena," makes them "useful for humans" and offers knowledge that "protects us from danger." Only after emphasizing its utilization and application for human welfare he also points to a human desire to know/understand, when he states that "physics offers infinite knowledge about inner processes of the universe." As to mathematics, he regards it initially as a "useful tool" to "categorize, structure and measure the world" and stresses that "there is no modern science without mathematics." However, a rather negative view is perceived in many of Michael's answers, e.g., when he states that "mathematics brings students to despair and is responsible for many family conflicts." Overall, we perceive that he ascribes a negative emotional connotation to mathematics, usually mentioning its authoritative character. This posture was confirmed not only in his written material, but also in the seminar videos and interviews.

Concerning the role of mathematics in physics, Michael states that the former enables a "precise description of phenomena" and allows to "standardize or generalize measurement data." For him, measurement data are the starting point of theoretical work, since they are close to the actual phenomenon. When asked to express the relationship between physics and mathematics metaphorically, Michael describes the importance of physics for mathematics by saying that the former is for the latter like "the annoying little brother, who does not deserve much attention, but is meant to be dominated by means of axioms, equations and theories." Michael clearly perceives a dominating role of mathematics and is emotionally attached to his statements. This goes well with what he writes to describe the role of mathematics in physics. In his words, mathematics is for physics "the eternal twin, a foothold to study nature, without which it is impossible to get along." Moreover, mathematics is like "a cage in which any form and beauty is imprisoned due to its precision and exactness." This seems to imply that for him mathematics has some sort of imposed (unjustified or even threatening) authority.

6.1.3 Physics, Mathematics and the Interplay (Post)

Considerable differences were found in Michael's post-questionnaire. The most significant is the absence of negative connotations regarding mathematics. When expressing his view on the role of mathematics in physics he now writes that the former is useful in "describing, stating more precisely and generalizing natural phenomena," that it "supports a better understanding of different theories," "expands our knowledge" and "provides a universal language to better comprehend […] nature." The metaphors used to describe the interplay also changed considerably: mathematics is for physics "like a language that categorizes natural phenomena and provides a general construct that allows to generate new knowledge" and physics is for mathematics like "a possibility to confirm theories and concepts in the real world."

Michael is aware of the fact that his view has changed significantly. In Interview 3 he states: "A lot has changed here." (7 min:20 s) and later he also jokes "I have been enlightened" (15 min:25 s). On the other hand he is not brain washed and states that he still holds his initial view "a little" (8 min:20 s). However, during the seminar, he made experiences and gained insights that he did not have before. "To deal with mathematics [short pause]—it can be fun. This is something I realized" (8 min:30 s).

The physical reality and the phenomena are the core of his view of physics and experimental data still remain crucial in his theorizing. However, the line of thought sometimes changes direction, e.g., when he states that "mathematics can help to explain a phenomenon" and in regards to Snell's law "Mathematics provides a construct that helps to conclusively describe the situation" (10 min:00 s). There are also signs that points toward a view of mathematics in physics as a structuring instrument: "To summarize different aspects of nature in the language of mathematics and to better connect ideas to generate new knowledge" (10 min:45 s).

Considering the pre-post comparison and the reasons given by Michael in the interviews, it seems that the course had an impact on his general conceptions about mathematics, physics and their interplay. Moreover, Michael himself is aware of major shifts in his perception of the role of mathematics in physics. Most remarkable is the fact that, at least directly after the invention, there is not only a cognitive change perceivable, but also an emotional one.

6.1.4 Explaining $a = v^2/r$ in a Physics Lesson (Pre)

When asked to outline a lesson in which the equation for the centripetal acceleration ($a = v^2/r$) should be explained, the emphasis on description and application is rather evident Michael's plan. For him, it is important to "explain the meaning of each of the equation's symbols," situate "contexts of application" (like Carousel, curves) and "discuss the use of the formula." The focus on description and application is again expressed in the four steps of the proposed teaching sequence: (1) Identify and explain the meaning of each of the equation's symbols; (2) Make hypotheses concerning the change in one magnitude caused by the change in others; (3) Describe the equation with the help of pictorial representations and examples; (4) Discuss applications in everyday life. When asked to answer one student's question (Why is the equation like this?) Michael presents the following deduction: Since $a = v/t$ and $t = s/v$, then $a = v^2/s$. Substituting s for r we have $a = v^2/r$.

The confirmative role that Michael addresses to formal deductions is evident in this derivation, in which a series of manipulations is performed without an attempt to make sense of the steps taken. This leads him to accept a desired conclusion based on inappropriate and even contradictory assumptions.

6.1.5 Explaining $a = v^2/r$ in a Physics Lesson (Post)

A very different teaching proposal designed to explain the centripetal acceleration formula is found in Michael's posttest. Already its goals have significantly changed. He now says that students should be able to "explain why v^2, why $1/r$, why the acceleration points to the center and understand one derivation of the formula." His new lesson plan is the following: (1) Present a case in which the centripetal acceleration appears (e.g., the orbital motion of the moon around the Earth); (2) Sketch the problem pictorially (What would happen if the Earth was not there?); (3) Derive the equation using the Pythagorean theorem [see Sect. 4.2.3]; (4) a is directed to the center: visualization through the direction of the gravitational force pointed to the Earth. He justifies his choices by saying that "it seems important to show students where this equation comes from. With a visual sketch of the problem it becomes clear what equations are needed to derive this new equation. Additionally, one can present other derivations that are more geometrical. Thus, pictorial representations and language go hand in hand, which is more meaningful according to learning psychology." The hypothetic student question (Why is the equation like this?) is now answered with the schematic representations of three different deductions (The ones presented in Sects. 4.2.1, 4.2.2 and 4.2.3).

6.1.6 Understanding Equations (Pre)

When asked to express the necessary conditions for him to be satisfied with his understanding of a particular formula in the first questionnaire, Michael says he "must know where the formula comes from, how it was originated and be able to explain it with everyday life examples." Concerning specific cases, he argues, for instance, that his understanding of the formula $T = 2\pi\sqrt{l/g}$ is quite weak because "the fact that a square root appears (and in it the ratio of two other magnitudes) makes it almost impossible to understand this equation fully. The factor 2π contributes for adding an extra level of difficulty."

6.1.7 Understanding Equations (Post)

A significant change is also noticed in Michael's description of what it means to understand an equation. In the post-questionnaire he now clearly states that an equation is fully understood when he has "no problem in deriving it in many different ways" and is able to "explain and understand the origin of its terms." The positive role attributed to the derivation of the formula is again highlighted when he states that "it is only possible to really grasp an equation when one fully understands its derivation." This substantial modification is confirmed in his new answer to the item 3 of the Likert items "In physics, the deduction of a particular formula only confirms that it is ok to use it" (strong agreement (4) in the Pretest to slight disagreement (2) in the Posttest).

In the interviews, Michael repeatedly addressed a crucial role to one particular derivation of the centripetal acceleration equation (Sect. 4.2.3) when he justifies most of the differences in his answers in the pre-post comparison. However, he does not seem to have had such a meaningful experience with the other equations approached in the course. When asked in the post-questionnaire if he has a deep feeling of understanding for these equations, he answers:

$s = gt^2/2$	No! It is still difficult for me to understand where the ½ comes from. On the other hand, the t^2 is clear, since Galileo's experiments have proven it.
$a = v^2/r$	Yes! I can derive the formula and clearly explain where the v^2 and $1/r$ come from.

$T = 2\pi\sqrt{l/g}$ No! Although I know why 2π is used, I still need to understand better the origin of $\sqrt{l/g}$. It does not seem to come straight out of the experiments. I lack some concepts and ideas to be able to explain this factor clearly.

$\frac{sin\theta_1}{sin\theta_2} = \frac{v_1}{v_2}$ No! Perhaps the least time principle is the best way to arrive at this equation, but it does not satisfy me 100 % because it seems too arbitrary and a matter of luck. […] I still have some open questions regarding the derivation of this equation.

From an instructor's perspective, these answers could sound somehow disappointing. However, the most important learning effect experienced by Michael during the course was arguably an extension of his personal demands for feeling satisfied with the understanding of a formula. In his words:

> *We often asked ourselves: What is actually behind this formula? The most important thing was to* ***learn to ask these questions*** [our emphasis]*; where does this and that come from? Derivations are very important for this purpose […] My favorite formula is now the centripetal acceleration: I know where it comes from, I know what is behind it and I can derive it in a meaningful way. And that is when we really understand a formula. […] A lot is happening when we derive a formula – considering examples, special cases* etc. *(Int 3 - 20 min:55 s)*

6.1.8 Course Evaluation

When asked to summarize and evaluate the course, Michael states in the final interview "I enjoyed […] being pushed to ask myself: What's behind this formulae?" (47 min:30 s) and hopes that such a course will be established for future physics teacher students. In his written feedback, he emphasizes that "realizing that mathematics is not a bad thing was a great benefit for me." He only wished that the question whether there are "ways to do physics without mathematics" would have been discussed in more depth.

In sum, the analysis of diverse data sources shows that the course had a considerable effect in Michael's conceptions, both on his emotional relationship with mathematics and on the role of mathematics in physics education. More specifically, after the course he ascribes a central role for derivations of equations, not only to his personal understanding of equations but also for physics teaching at school.

6.2 James

James is 23 years old and studies to become a mathematics and physics high school teacher. He likes teaching and passing on knowledge, mathematics is his favorite subject and physics seemed to be the natural choice for a second subject. In a 1–5 scale, he finds theoretical physics highly interesting (5), which comes as no big surprise, but also experimental physics is rated as interesting (4).

6.2.1 Physics, Mathematics and the Interplay (Pre)

In his pre-questionnaire, James draws an epistemologically elaborated picture of what physics is. To him, it is the "study of processes in nature" and seems to be a promising source of knowledge to find answers to deep existential questions (e.g., "physics asks questions about the past and future of life"). According to him, in physics "experiments are used as a fundamental method" and he is conscious that theories are evolving and cannot be considered to be final truths. Theories are valid only as long as they do not contradict experiments.

Mathematics is perceived as instrumental ("a tool for all natural sciences") and "a language […] that can be understood everywhere." Mathematics for James is "logical throughout" and "man-made," but can be used to make "absolute statements about right and wrong."

In physics, mathematics serves as a "tool to express relations in a simplified way," shapes a "universal look of physics" and also affects the way knowledge is generated by "allowing to make predictions testable in experiments." Again, it can be assumed, that James holds a rather developed epistemological belief system. Although he does focus on the experimental method, he also considers mathematics to be an essential part of physics. This is shown in his metaphors likening the role of mathematics for physics with the role of "words for an author" or "notes for a composer." However, at the same time, he leaves no doubt about the tool character of mathematics for physics, when he says that "physics is for mathematics like a piece of furniture for a screw." Thus, despite the essential role he ascribes to mathematics in physics, he is well aware that physics is much more than only mathematics.

6.2.2 Physics, Mathematics and the Interplay (Post)

In his post-questionnaire, James still considers mathematics to be "a very important tool"; however, now he emphasizes that this tool character is "not [meant to imply] that it would be dispensable." The tool character in his sense only refers to the fact, that mathematics in physics is helpful to "make exact and precise statements." While the enormous help mathematics offers has been mentioned in James' statements at the beginning of the course, he now explicitly ascribes an essential role of mathematics in physics even more strongly. However, the idea was already implicit in his metaphors given at the beginning of the course. James also sticks to the idea of considering mathematics as a language for physics. While at the beginning of the course he emphasized the universal look of physics shaped by mathematics, he now points out that from his perspective, mathematics is "not the only possible language for physics." In accordance with what he wrote in the pretest, but still a bit more elaborated and with more emphasis on the role of derivations, he (still) considers mathematics a way "to derive the laws of physics or hypotheses, that then are tested in experiments." Those experiments remain the final criterion for the validity of any statement in physics for James.

Compared with Michael, James shows a more mathematical approach. He does not start with phenomena that need to be explained and are the starting point for knowledge production in physics, he is rather concerned with laws and principles that are modeled or constructed and stated in the language of physics to be (then) confronted with the real world via experiments. This view offers a clear contrasting case to Michael. James also differs from Michael in so far, as his thinking and attitude toward the role of mathematics did not develop revolutionary as in Michael's case, but rather slight adjustments were perceived.

6.2.3 Explaining $a = v^2/r$ in a Physics Lesson (Pre)

When asked at the beginning of the seminar to outline a teaching sequence in which the equation for the centripetal acceleration ($a = v^2/r$) should be explained, James identifies the "direction of the acceleration to be central" for understanding, since here we have the special case of a "change of direction of the velocity, without a change of its absolute value." He considers this aspect to be relevant, but his justification is somewhat indirect. The essence of acceleration, he argues, is that it is "caused by a force" and force is an "important concept of investigation in mechanics. So for understanding centripetal force one must comprehend the characteristics of the centripetal acceleration." While this does reflect positively on his interlinked content knowledge, it is not exactly a deep reason.

In his teaching sequence, James suggests two alternatives depending on the level of the course. For a basic level: show a circular motion → introduce the concept of centripetal force → relate it to centripetal acceleration. For mathematically high performing learners, he suggests: perform an experiment with uniform circular motion → derive the equation $a = v^2/r$ based on the experiment. In both cases, he tends to favor an inductive approach, although the second highly depends on what he means by "deriving the equation," since this could go beyond and toward a structural/deductive approach. As a justification, James claims a derivation to be "aesthetic" and also argues that a derivation would "foster the advancement of the learners mathematical thinking." This seems to transcend or at least add to the inductive approach identified above, even though no concrete mathematical derivation is provided.

In order to answer the student's question regarding the equation's structure, he suggests

- to use a "mathematical derivation based on an experiment"
- to conduct experiments to validate the relations $a \sim v^2$ and $a \sim 1/r$
- to argue using plausibility considerations based on everyday experience (e.g., cars cornering).

Again, it is clear that for James a derivation is related with an experiment, which confirms an inductive approach. It is evident that the experiment is the starting point in his teaching sequence, which needs further explanation. Overall, this seems to contradict his initial conception of mathematics as providing relations to be confronted with experiments (deductive approach).

6.2.4 *Explaining $a = v^2/r$ in a Physics Lesson (Post)*

A significant change is noted in James' post-questionnaire. While he still considers important that students understand "in what direction the centripetal acceleration is pointing," some new (rather specific) aspects are mentioned. For him, students should perceive "that the velocity depends on the radius of the circular motion ($v(r) = \omega r$) and therefore for a constant angular velocity ω the relation $a \sim r$ (not $a \sim \frac{1}{r}$) is valid." He now suggests to start with an experiment to investigate/find the relations $a \sim v^2$ and $a \sim r$ for constant ω. After that he would perform a mathematical derivation of the equation based on the assumptions of the approach presented in Sect. 4.2.3).

His rather ambiguous justification found in the pretest is now much more explicit, when he states "more important than the formula itself is understanding circular motions. What forces are acting and in which direction." For this purpose, he considers his suggested lesson plan as viable, since it allows relating with everyday experiences. To answer the question why the equation looks the way it does, he still considers the "execution of experiments to validate the proportionalities" as a valid approach. However, he now also offers three specific ways of deriving the equation (all presented in Sect. 4, except for Sect. 4.2.2).

James has clearly gained more specific and detailed information (explanations repertoire), adding to his already solid content knowledge. Furthermore the emphasis on a theoretical derivation of the equation is present in his lesson plan. He seems to be fascinated by details that have not been as clear to him before the seminar, but need to be put in perspective when introducing the equation to novices, e.g., in his final interview, he states that circular motion $v(r) = \omega r$ appeared to him more central at the end of the course because this was one of his insights during a group discussion that was still present in his mind (33 min:05 s). When asked for a reason why he did not stick to two teaching approaches depending on students' levels, he argues that this derivation, which was new to him, is understandable "for most students." He also argues that it is not enough to present the experiment and find proportionalities, since the experiment is based on assumptions about relations between quantities, while the mathematical derivation allows one to understand the final form of the equation (35 min:40 s). Interestingly enough, James argues that answering the question why the equation looks the way it does requires the use of at least one mathematical derivation. However, on the other hand, he neglects the necessity of a derivation for "understanding" an equation. To keep his argumentation consistent he proposes a distinction of understanding an equation with or without "understanding the physics behind it" (43 min:30 s). While this does reflect his need and awareness to be consistent, we cannot help to regard it as an ad hoc hypothesis. James' view on what it means to understand physics equations is discussed in the following.

6.2.5 Understanding Equations (Pre)

According to James, formulas represent meaningful relations between observables and are helpful for understanding physics. Nevertheless, he is aware that understanding formulas is not the same as knowing them by heart and also requires some prior knowledge ("I don't have to know a formula by heart, but if I see it, I need to know what quantities are represented by the symbols and to what domain of physics they belong and what there meaning is."). For him, the ultimate test is "to be able to explain an equation to someone else." When asked about his understanding of specific equations ($s(t) = s_0 + v_0 t + at^2/2$, $T = 2\pi\sqrt{l/g}$, $E = mc^2$), he focuses on performing calculations with them. For all three equations under consideration, he states an understanding of the formula in the sense that they allow the calculation of the quantity on the left-hand side. For the uniformly accelerated motion he adds that it is plausible to find the quantities that are included in the equation. For the simple pendulum equation he states that it is surprising that only the l and g are part of the equation. In both cases, he does not see a problem with any kind of factors or the way the symbols are arranged. While James without doubt is able to interpret the equations and to use them for any kind of calculation, it does not seem like there is an awareness or need for any other aspect of understanding. Only with the third equation he states explicitly that he understands the "mathematical statement" but the physical content is difficult to grasp.

6.2.6 Understanding Equations (Post)

While James apparently did not change his opinion significantly, small changes are still detectable. His view on understanding formulas expressed in the post-questionnaire is not much different from before the intervention. He still emphasizes the necessity to know "what is meant by each symbol of a quantity, how they are related to another" according to the equation and the "ability to verbalize the equation."

In the posttest, James was asked to specifically judge his understanding of the four equations covered in the course. As before, he considers all of them as understood. However, in his reasoning, he now refers to derivations in three of four cases as follows:

$s = gt^2/2$	Yes! The equation is a special case of the general $s(t)$ expression of uniformly accelerated motions, which can be derived by integrating the equations of motion.
$a = v^2/r$	Yes! If the relation $v = \omega r$ is known, then the equation is connected with everyday life experiences.
$T = 2\pi\sqrt{l/g}$	Yes! At first glance it is surprising that the equation does not contain the mass of the bob. But one can derive it from the small angle approximation.
$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$	Yes! If one accepts Fermat's principle, then the law is a natural consequence.

6.2.7 Course Evaluation

James has demonstrated, along the course, a solid content matter knowledge. For him, the course was beneficial from this perspective. He especially liked the philosophical and historical insights, the active discussions and the introduction of alternative derivations for the equations approached. However, he complains that the course did not provide specific didactic strategies to approach the derivations in physics lessons as well as ways to overcome students' learning difficulties. He concludes that the seminar was very relevant for his future career because it helped him to be "more aware of the role of physics equations and their derivations and to better depict positive influence of mathematics in physics."

In sum, although James' conceptions about mathematics, physics and their interplay were not significantly changed after the course, it is possible to notice considerable changes on his views about how to teach physics equations at school. More specifically, a central role of derivations is found in his posttest and interviews. Furthermore, the broadening of his explanations repertoire is quite evident in the pre-post comparison.

7 Conclusions and Perspectives

In this paper, we described the structure of an innovative course given to physics teacher students that was motivated by the need to shift focus from *calculate and describe* to *explain and understand,* concerning the role of equations in physics instruction. We argued, theoretically, that in order to overcome the established plug-and-chug tradition in physics education, teachers should both recognize the explanatory/deductive role of mathematics (equations) in physics and possess a broad repertoire to answer "Why?" questions addressed to equations. Moreover, we presented a categorization of equations according to their perceived epistemological status and stressed the relationship between understanding and deriving the ones belonging to the last category (Sect. 2). The main activities of the course were dedicated to the presentation (by the students) of different ways to derive school-relevant equations and the discussion of their relevance for physics education at high school level.

Overall, we received quite a positive feedback from all the participants of the course in Dresden, concerning their perceptions of the course's contribution to their future work as physics teachers. In the last meeting, the students were asked to fill out an anonymous evaluation form. Here are their answers to the question "Please estimate the relevance of this course for your preparation to become a physics teacher and justify your answer" (translated from German):

1. Very useful because some interesting ways to derive equations were approached.
2. I will be able to approach physical formulas and their derivations more consciously and positively in my lessons.
3. High. Deriving formulas is always a difficult issue since it is not very popular among students. After attending the seminar I have more possibilities to choose and have a clearer view of students' difficulties.
4. Extremely high! As a teacher one should dedicate more time for these detailed questions instead of being overloaded with content knowledge.
5. I estimate the relevance is about 80 %. I learned important formulas for high school and their derivations, and was confronted with students' possible learning difficulties. However, concrete strategies to overcome these difficulties were not approached in the course.
6. From the examples studied, we have learned that besides the academic standard textbook derivation there are 1–2 other derivations that seem more appropriate. Especially the geometrical reasoning one can provide better understanding of several topics. This knowledge is certainly extremely useful for the physics teaching profession.
7. Very relevant; sometimes it was not at all clear to me that students could encounter difficulties with these topics. Since the boundaries between "derivable" and "derivable to some extent, but certain aspects not trivially derivable" blur, the seminar is very, very important!
8. I wish we had this kind of seminar more often and sooner, because it really drew my attention to the structure of formulas. It is a pity—it is actually almost sad—that our teacher training program is so focused on content and that our didactic seminars are so far away from the practice.
9. Very relevant—finally something concrete related to general connections, derivations and students difficulties! That is why we should have this kind of seminar more often.
10. It has a high relevance, because one learns different derivations of formulas, becomes aware of students' difficulties and is better prepared to explain to students why mathematics is so important for physics.

We notice two common aspects in their answers. First, the students recognize the importance of expanding their explanations repertoire. Second, they also highlight that the course contributed to a greater awareness of students' difficulties to understand equations. In general, it seems that the biggest gain is indeed this "awareness" and a predisposition to ask "unusual" questions to equations. Taking into account the fact that all the students had already finished their physics and mathematics studies when the course began, we can infer that the subject specific courses are not suitable to fulfill these goals. Specifically, when teacher students take the same courses as science majors, which is the case in our study, this is actually not a surprise. In fact, it seems clear that the awareness of students' difficulties and the explanations repertoire belong to another kind of knowledge (PCK, cf.

Shulman 1986) that is specifically important for teachers, but not necessarily relevant for science majors, which reinforces the importance of similar courses in physics teacher education.

Although our written data do provide some promising results concerning the effectiveness of the course, a deeper analysis that includes video and audio information (e.g., interviews and recordings of the lessons) is necessary to understand the reasons behind the apparent learning outcomes of the students. This led us to focus on a comparative analysis between two students who presented quite opposite conceptions in their pretests. Even though another research design would be necessary to draw generalizable conclusions about the learning gains provided by the course, this deliberate selection was an attempt to cover a broad spectrum of learners. While the less mathematically oriented Michael experienced some profound changes in his views on the role of mathematics in physics, the more mathematical thinker James appreciated some additions to his content knowledge and explanations repertoire. Both students show considerable changes in the pre-post comparison when asked to plan a lesson to explain the centripetal acceleration equation. For different reasons, Michael and James regarded the course as useful for their future career.

We knew of no other course given to physics teacher students with similar goals, so we decided to start by conducting an exploratory study. In sum, four important lessons can be learned from this experience. First, is it very effective to focus on specific cases (equations) that the students regard as relevant for their future practice. The quality and intensity of the discussions in lessons 4–7 (cf. Fig. 15) were considerably superior when compared to the two, more general, lessons about historical and philosophical aspects of the relationship between physics and mathematics. Second, the use of derivations extracted from historical sources (e.g., Galileo, Newton, Huygens) was very effective in offering more didactically attractive ways to explain the origin of these equations, since they usually demand less abstract (usually geometrical) mathematics. Third, it is indispensible to engage students in the struggle of understanding the derivations and to give them enough time (in our case 1 week) to prepare their explanations to the whole class (micro teaching). This provides an authentic learning experience and makes them aware of the difficulties that can be faced by their students. Last but not least, it became clear that the main reason for having different derivations is not to choose “the most appropriate one”, but to realize that each of them contributes to broadening our understanding of the equation.

There is certainly a long road to be travelled until we manage to eliminate the plug-and-chug tradition of physics education. Among many other actions, it is necessary to create better problems to substitute the “given-sought” tasks, invest systematic research efforts to map students’ most common difficulties to grasp the importance of mathematical thinking in physics, develop appropriate teaching and assessment strategies to approach deductive reasoning, etc. In order to contribute to this effort, we designed, applied and evaluated an original course to physics teacher students. We hope to have offered an initial proposal to be further improved and to have increased the interest of physics educators in integrating similar approaches into teacher education.

Acknowledgments We would like to thank all the students that attended the course for their engagement and the reviewers for their valuable suggestions. Many thanks also to Gesche Pospiech (Dresden), Ismo Koponen (Helsinki) and Dietmar Höttecke (Hamburg) for the opportunity of giving the course in their universities. This work was supported by the Alexander von Humboldt Foundation (Postdoctoral Fellowship to RK—BRA 1146348 STP).

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