5 october

Complete metric spaces

<u>Def</u> Given X a set and $d : X \times X \to \mathbb{R}$

d is a distance metric if

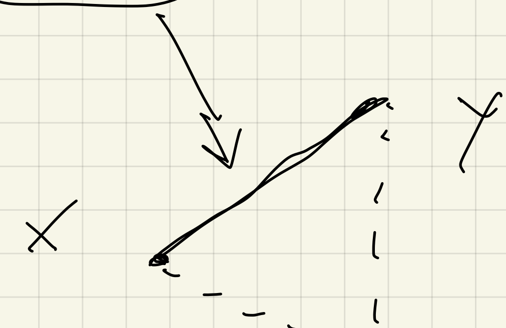
1) $d(x,y) \geq 0 \quad \forall x,y \in X$

2) $d(x,y) = 0 \iff x = y$

3) $d(x,y) = d(y,x) \quad \forall x,y \in X$

4) $d(x,y) \leq d(x,z) + d(z,y)$

$\forall x,y,z \in X$



A sequence $\{x_n\}$ in X is a Cauchy sequence if

$\forall \varepsilon > 0 \quad \exists N_\varepsilon \in \mathbb{R}$ st

$n, m \geq N_\varepsilon \implies d(x_n, x_m) < \varepsilon$.

Convergent sequences, that is sequences ~~s.t.~~ $\{x_n\}$ s.t. $\exists\, x \in X$ s.t. * $\lim_{n \to +\infty} x_n = x$, are always Cauchy sequences.

* $\forall\, \varepsilon > 0 \;\; \exists\, N_\varepsilon$ s.t.

$$n \geq N_\varepsilon \implies d(x, x_n) < \varepsilon.$$

(X, d) is complete when every Cauchy sequence is convergent.

<u>Def</u> A completion of (X, d) consists of a complete metric space $(\hat{X}, \hat{d})$ and an isometry $j : X \longrightarrow \hat{X}$ such that $j(X)$ is dense in $\hat{X}$.

Show that if (X, d_X) (Y, d_Y) are two complete metric spaces, if $Z \in X$ is dense in X and if $f : Z \longrightarrow Y$ is continuous, then there a unique continuous extension $\overline{f} : X \longrightarrow Y$ of f.

PDE

<u>Def</u> Given $\{X_\alpha\}_{\alpha \in A}$ ~ family of topological spaces we can consider

$$X = \prod_{\alpha \in A} X_\alpha = \{ (x_\alpha)_{\alpha \in A} : x_\alpha \in X_\alpha \}$$

there is a natural topology with base

$\prod_{\alpha \in A} U_\alpha$ with U_α open in X_α

and with only finitely many of them different from X_α, all $U_\alpha \neq \emptyset$

Tychonov Theorem

A subset of the form $\prod_{\alpha\in A} K_\alpha$

with $K_\alpha \subseteq X_\alpha$ is compact

if and only if each K_α is compact

in X_α.

Normal topological spaces

Def X is regular if $\forall x \in X$

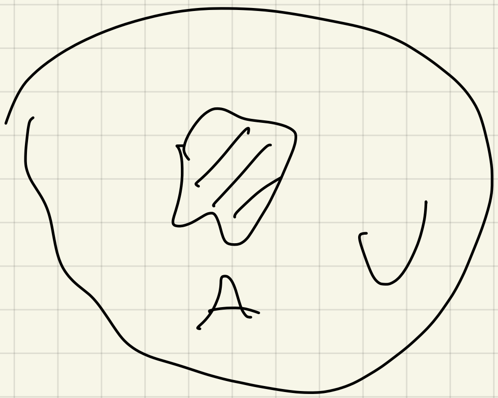
and for any closed $B \subseteq X$ with $x \notin B$

there are ~~a~~ ~~neiborhoods~~ neighborhood

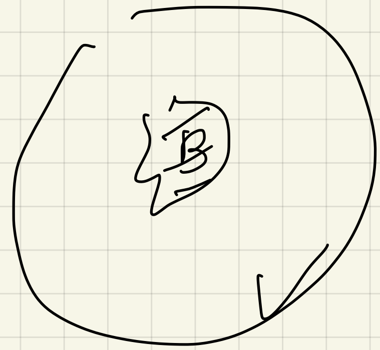
U of x and V of B s.t.

$$U \cap V = \emptyset$$

X is normal if given A and $B \subseteq X$ closed with $A \cap B$ there exist neighbor.. U of A and V of B s.t.



$$U \cap V = \phi$$

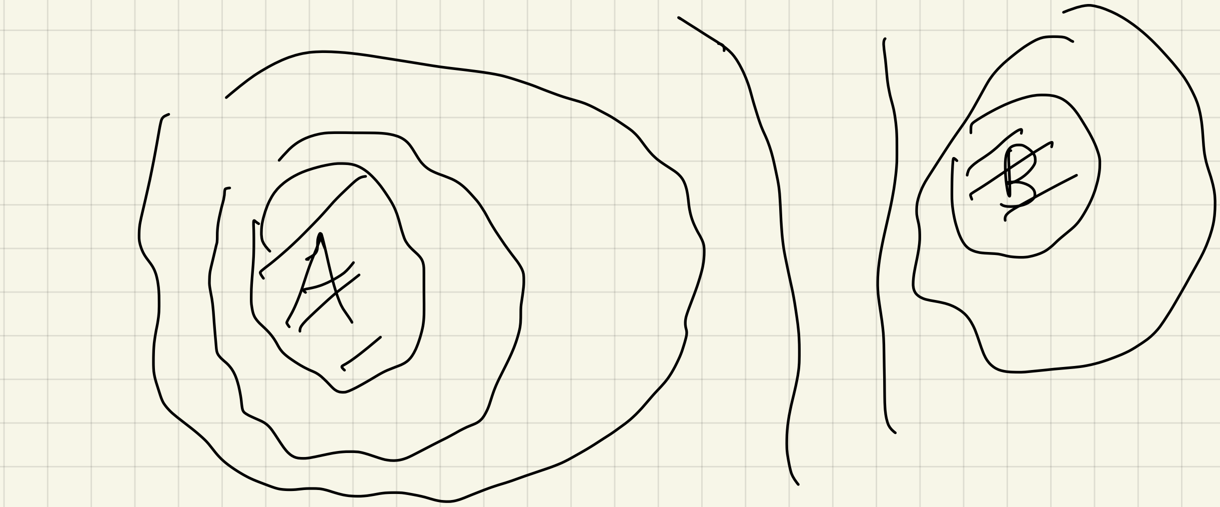


Every metric space is normal

Every compact Hausdorff space is normal

Thm (Uryshon's Lemma) let X be normal and let A and B be closed and disjoint. Then, for any $[a,b] \subset \mathbb{R}$ there exists $f \in C^0(X, [a,b])$ s.t.

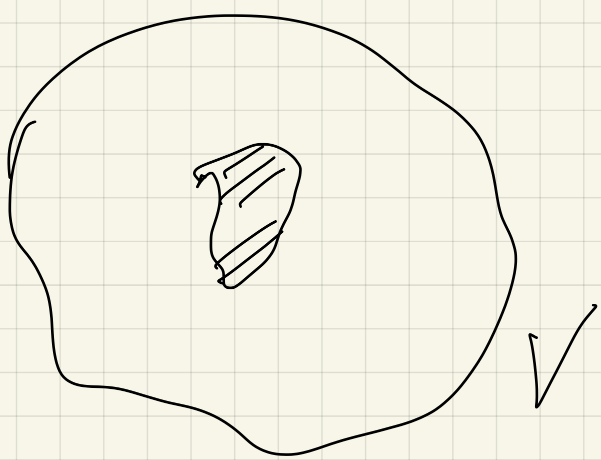
$$f|_A = a \qquad \text{and} \qquad f|_B = b$$



<u>Corollary</u> X locally compact Haus.

$K \subseteq X$ compact and V open $\subset X$

with $V \supset K$



$$\exists \; f \in C^0(X, [0,1])$$

s.t. $f|_K = 1$,

$\operatorname{supp} f \subseteq V$ and so $f|_{V \setminus K} = 0$

Corollary

X, K

$$K \subset V_1 \cup \dots \cup V_n$$

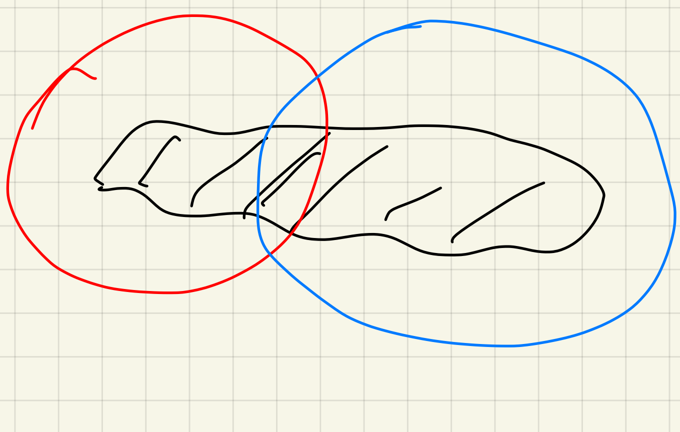
open sets

Then there exists $h_1, \dots, h_n \in C^0(X, [0,1])$

s.t. $\operatorname{supp} h_j \subseteq V_j$ and such that

$\forall\, x \in K$

$$h_1(x) + \dots + h_n(x) = 1$$



Theorem (Weierstrass approx theorem) $\forall [a,b] \subseteq \mathbb{R}$

$\mathbb{R}[x]$ is dense

in $C^0([a,b], \mathbb{R})$ for the topology

of uniform convergence.