SISSA

Advanced Analysis - A

Academic year 2019-2020

Proposed problems

1. Let X be a separable Banach space and Y a subspace of X. Show that Y, endowed with the induced norm, is separable.

2. Let X be a Banach space and Y a finite-dimensional subspace of X. Show that Y is closed.

3. Let (M, d) be a compact metric space. Show that M is complete and separable.

4. Let (M, d) be a complete metric space and $\{A_n, n \in \mathbb{N}\}$ a countable family of open and dense subsets of M. Show that the set

$$A \doteq \bigcap_{n \in \mathbb{N}} A_n$$

is dense in M.

5. Let H be a real Hilbert space and $a \in H$ a nonzero vector. Show that for every $x \in H$, we have

$$\mathrm{dist}(x, \{a\}^{\perp}) = \frac{|(x, a)|}{\|a\|}.$$

6. Consider the Hilbert speace ℓ^∞ with its usual norm $\|\cdot\|_{\ell^\infty}$ and the sets $c_0 \doteq \{(a_n) \in \ell^\infty : a_n \to 0\}$ and $c \doteq \{(a_n) \in \ell^\infty : a_n \to a \in \mathbb{R}\}$. Show that c_0 and c are closed separable subspaces of ℓ^∞.

7. Consider the Hilbert space ℓ^2 and a real sequence (a_n) such that $a_n > 0$ for every $n \in \mathbb{N}$ and $a_n \to +\infty$. Show that the set

$$A \doteq \left\{ u \in \ell^2 : \sum_{n \in \mathbb{N}} a_n |u_n|^2 \leq 1 \right\}$$

is a precompact subset of ℓ^2.

8. Let H be a Hilbert space and C_1, C_2 two nonempty, closed and convex subsets such that $C_1 \subset C_2$. Given $x \in H$, call $P_{C_i}x$ the projection of x on C_i and $d(x, C_i)$ the distance of x from C_i $(i = 1, 2)$. Show that

$$\|P_{C_1}x - P_{C_2}x\|^2 \leq 2\left(d(x, C_1)^2 - d(x, C_2)^2\right), \quad \forall x \in H.$$

9. Let H be a complex Hilbert space and $T \in \mathcal{L}(H)$ an operator such that $\|T\| \leq 1$. Show that

(a) $Tx = x$ if and only if $(Tx, x) = \|x\|^2$;
(b) $ker(I - T) = ker(I - T^*)$.

10. Find a Banach space X and a subset $S \subseteq X$ such that S is strongly closed but not weakly closed.

11. Find a Banach space X, a bounded closed subset $S \subseteq X$ and a continuous function $f : S \to \mathbb{R}$ such that

$$\sup_{x \in S} f(x) = +\infty.$$

12. Let X be a Banach space and $K \subseteq X$ a compact subset. Show that any sequence in K which converges weakly, actually converges strongly.

13. Let (X, d) be a metric space. Given two subsets $A, B \subseteq X$, set

$$\operatorname{dist}(A, B) \doteq \inf\{d(x, y) : x \in A, y \in B\}.$$

a) Given $x \in X$ and positive numbers $0 < \rho < r$, show that there exists $\delta > 0$ such that
$$\operatorname{dist}(B(x, \rho), B(x, r)^c) \geq \delta.$$

b) Given a proper, nonempty, closed subset $C \subseteq X$, show that there exists a ball $B(x, r)$ in X such that $\operatorname{dist}(B(x, r), C) > 0$.

14. Let X, Y be Banach spaces and $T \in \mathcal{L}(X, Y)$ a compact operator. Let (x_n) be a sequence in X weakly converging to x in X. Show that the sequence (Tx_n) converges strongly to Tx in Y.

15. Let $\alpha > 0$ and consider the sewuence of functions given by

$$u_n(x) \doteq \min\{1, |x|^{-\alpha}\}\chi_{B(0,n)}(x), \quad n \in \mathbb{N},\ x \in \mathbb{R}^d.$$

Study the convergence of (u_n) in the strong and weak (weak* if $p = \infty$) topology of $L^p(\mathbb{R}^d)$ for $p \in [1, \infty]$.

16. Let H be a Hilbert space, $T \in \mathcal{L}(H)$ and T^* the adjoint of T.

(a) Show that $\|T^*T\| = \|TT^*\| = \|T\|^2$.

(b) Show that T^*T and TT^* are selfadjoint operators.

17. Let H be a Hilbert space and $\{M_k, k \in \mathbb{N}\}$ a countable collection of finite-dimensional subspaces of H. Call P_k the orthogonal projector on M_k ($k \in \mathbb{N}$) and set

$$P \doteq \sum_{k=1}^{\infty} 2^{-k} P_k.$$

Show that P is a compact operator in $\mathcal{L}(H)$.

18. Consider the sequence of functions given by

$$u_n(x, y) = \left(\cos\left(\frac{x}{n}\right) + \sin\left(\frac{x}{n}\right)\right)(1+e^{-ny^2}), \quad (x, y) \in I \doteq [-1, 1]\times[-1, 1], \quad n \in \mathbb{N}.$$

Study the convenrgence of (u_n) in the strong and weak topology of $L^p(I)$ (weak* if $p = \infty$).

19. Let H be a complex Hilber space, $T \in \mathcal{L}(H)$ and (x_n) a sequence in H weakly converging to $x \in H$. Show that the sequence (Tx_n) converges weakly to Tx.

20. Given $x \in \mathbb{R}$, let $B(x, 1)$ be the open unit ball of center x in $\mathbb{R}$. Consider a sequence (x_n) in $\mathbb{R}$ and define the sequence of functions $u_n \doteq \chi_{B(x_n,1)}$, where χ denotes the characteristic function. Study the strong and weak convergence of the sequence (u_n) in the space $L^2(\mathbb{R})$ (that is to say establish if the sequence is converging in such topologies and, in affirmative case, find the limit), in the following cases:

(a) $x_n \to 0$;
(b) $|x_n| \to +\infty$.

21. Let H be a Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$ and D a subset of H such that $\mathrm{lsp}(D)$ is dense in H. Show that, given a bounded sequence (x_n) in H, such that $\langle x_n, y \rangle \to \langle x_n, y \rangle$ for any $y \in D$, then $x_n \rightharpoonup x$.

22. Let $I = [0, 1] \subseteq \mathbb{R}$ and consider the Hilbert space $X = L^2(I, \mathbb{R})$. Set

$$(Tu)(x) \doteq \int_0^x u(t)\, dt.$$

Show that $T \in \mathcal{L}(X)$ and find the adjoint T^* of T.

23. Consider the set $E \doteq \{e^n, n \in \mathbb{N}\}$ in ℓ^2 defined by

$$e^n(k) = \delta_{n,k}.$$

Show that E is a Hilbert basis in ℓ^2.

24. Let $\mathcal{U}$ be a bounded family in $L^1(\mathbb{R})$ and $\rho \in C_c^\infty(\mathbb{R})$. Show that the family $\{\rho \star u, u \in \mathcal{U}\}$ is equicontinuous.

25. Let H be a Hilbert space and $T \in \mathcal{L}(H)$. Show that T is compact if and only if the adjoint T^* is compact.

26. Let H be a Hilbert space on $\mathbb{C}$, $\{e_k, k \in \mathbb{N}\}$ an orthonormal system in H and (λ_k) an element of $\ell^1(\mathbb{C})$. Set

$$Tx \doteq \sum_{k=1}^\infty \lambda_k (x, e_k) e_k.$$

Show that T is a compact operator in $\mathcal{L}(H)$.

27. Consider the Hilbert space $E \doteq L^2(\mathbb{R}^n, \mathbb{C})$ and let $K \in L^2(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{C})$. Define

$$(T_K u)(x) \doteq \int_{\mathbb{R}^n} K(x, y) u(y)\, dy.$$

Show that $T_K \in \mathcal{L}(E)$ and that T_K is selfadjoint if and only if $K(x, y) = \overline{K(y, x)}$ for any pair $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$.

28. Let H be a Hilbert space and (u_n) an orthonormal sequence in H. Show that (u_n) converges weakly to zero.

29. Let $p \in [1, \infty[$ and $f \in L^p(\mathbb{R})$. Show that for every $\delta > 0$ we have

$$\mathrm{meas}\left(\{x : |f(x)| > \delta\}\right) \leq \delta^{-p} \|f\|_p^p.$$

30. Let $E \subseteq \mathbb{R}$ be a measurable set with finite measure, $p \in]1,\infty]$, (u_n) a sequence in $L^p(E)$ and $u \in L^p(E)$ such that $u_n \rightharpoonup u$ for $p < \infty$ or $\overset{*}{\rightharpoonup}$ if $p = \infty$. Prove that the sequence (u_n) is equiintegrable.

31. Let H be a real Hilbert space, $M \subseteq X$ a closed subspace and P the orthogonal projector on M. Show that P is selfadjoint.

32. Consider the space $X = C_0(\mathbb{R}^d) \doteq \overline{C_c(\mathbb{R}^d)}^{\|\cdot\|_\infty}$. Given $S \in X'$ define

$$U \doteq \left\{ O \subseteq \mathbb{R}^d \text{ open: } \langle S, u\rangle_{X',X} = 0 \ \forall u \in X \text{ with } \operatorname{supp} u \subseteq O \right\}.$$

Then introduce

$$N \doteq \bigcup_{O \in U} O \text{ (domain of nullity of } S); \qquad \operatorname{supp} S \doteq \mathbb{R}^d \setminus N \text{ (support of } S).$$

(i) Given $a \in \mathbb{R}^d$, set

$$T_a(u) = u(a).$$

Show that $T_a \in X'$ and find its norm and support.

(ii) Let (a_n) be a sequence in $\mathbb{R}^d$, consider the sequence $(S_n) = (T_{a_n})$ and the series

$$S = \sum_{n=1}^{\infty} 3^{-n} S_n.$$

(a) Show that $S \in X'$ and find its norm and support.

(b) Show that there exists a subsequence (S_{n_k}) weakly* converging in X'.

33. Consider the sequence of functions given by $u_n(t) = \sin(nt)$, with $n \in \mathbb{N}$ and $t \in I \doteq [0, 2\pi]$. Study the convergence of (u_n) in the uniform topology of $C(I)$, in the strong topology of $L^\infty(I)$ and in the weak* topology of $L^\infty(I)$ (that is to say establish if the sequence is converging in such topologies and, in affirmative case, find the limit).

34. Let $X = C_0(\mathbb{R}^2, \mathbb{R})$, endowed with the uniform norm, and (a_n) a sequence in $\mathbb{R}^+$. Set

$$\langle f_n, u\rangle \doteq \int_0^{2\pi} u(a_n \cos\theta, a_n \sin\theta)\, d\theta, \quad \forall n \in \mathbb{N},\ \forall u \in X.$$

Show that $f_n \in X'$ for every $n \in \mathbb{N}$ and find its norm and support.

Suppose $a_n \to 0+$ and study the convergence of the sequence (f_n) in the strong and weak* topology of X' (that is to say establish if the sequence converges in such topologies and, in the affirmative case, find the limit).

35. Given $\alpha \in \mathbb{R}$ and $R > 0$, consider the function u defined on $\mathbb{R}^d$ by

$$u(x) \ = \ \begin{cases} |x|^\alpha, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Establish for which $p \in [1, +\infty]$ we have $u \in L^p(B_{\mathbb{R}^d}(0,R))$.

36. Let $E \subseteq \mathbb{R}^d$ be a measurable set, $p, q \in [1, \infty[$ and $u \in L^p(E) \cap L^q(E)$. Given $\alpha \in [0,1]$, set

$$\frac{1}{r} \doteq \frac{1-\alpha}{p} + \frac{\alpha}{q}.$$

Show that $u \in L^r(E)$ and that

$$\|u\|_{L^r(E)} \leq \|u\|_{L^p(E)}^{1-\alpha} \cdot \|u\|_{L^q(E)}^{\alpha}.$$

37. Let $I \doteq [0,1]$ and consider the sequence of functions given by

$$u_n(t) = e^{-nt}, \quad t \in I, \quad n \in \mathbb{N}.$$

Study the convergence of the sequence (u_n) in the following spaces:
- (i) $C^0(I)$ endowed with the uniform topology;
- (ii) $L^1(I)$ endowed with the strong topology;
- (iii) $L^1(I)$ endowed with the weak topology;
- (iv) $L^\infty(I)$ endowed with the strong topology;
- (v) $L^\infty(I)$ endowed with the weak* topology.

38. Let $E \subseteq \mathbb{R}$ be a measurable set with finie measure and let $m \in L^\infty(E)$. Set

$$(Tu)(x) \doteq m(x) \cdot u(x) \ \text{ for a.e. } x \in E.$$

Given $p, q \in [1, \infty[$, with $p \geq q$, show that $T \in \mathcal{L}(L^p(E), L^q(E))$ and provide an estimate of its norm.

39. Let $X = C_c(\mathbb{R})$ and $T : X \to X$ be a linear application such that

$$\|Tu\|_{L^1} \leq \|u\|_{L^1}; \qquad \|Tu\|_{L^2} \leq \|u\|_{L^1} \quad \forall u \in X.$$

Given $r \in [1,2]$, show that there exists $\tilde{T} \in \mathcal{L}(L^1, L^r)$ such that $\|\tilde{T}\|_{\mathcal{L}(L^1,L^r)} \leq 1$ and $\tilde{T}|_X = T$.

40. Consider the sequence of functions given by

$$u_n(x,y) = \cos(nx)e^{-ny}, \quad (x,y) \in I \doteq [0,2\pi] \times [0,2\pi], \quad n \in \mathbb{N}.$$

- *(a)* Study the equicontinuity of (u_n) on I.
- *(b)* Study the convergence of (u_n) in the uniform topology of $C(I)$, in the strong topology of $L^\infty(I)$ and in the weak* topology of $L^\infty(I)$.

41. Let $X = C_0(\mathbb{R}^2, \mathbb{R})$ endowed with the uniform topology and consider the family of subsets of $\mathbb{R}^2$ given by

$$A_\alpha \doteq \left\{(x,y) \in \mathbb{R}^2 : y > \alpha|x|, \ x^2 + y^2 < \alpha^{-2}\right\}, \quad \alpha > 0.$$

Set

$$T_\alpha u \doteq \int_{A_\alpha} u(x,y)\, dxdy, \quad \alpha > 0.$$

- *(a)* Show that $T_\alpha \in X'$ for every $\alpha > 0$ and find its norm and support.
- *(b)* Study the convergence of the family $(T_\alpha)_{\alpha>0}$ in the strong and weak* topology of X' when $\alpha \to 0+$ and when $\alpha \to +\infty$.

42. Let $I = [0,1]$, $X = C(I, \mathbb{R})$ and $Y = L^2(I)$. Set

$$(Tu)(x) \doteq \int_{x^2}^{x} u(t)\, dt.$$

- *(a)* Show that $T \in \mathcal{L}(X)$ and establish if $T(B_1^X)$ is relatively compact in X.
- *(b)* Show that $T \in \mathcal{L}(Y)$ and establish if $T(B_1^Y)$ is relatively compact in Y.

43. Consider the sequence of functions given by

$$u_n(x,y) = \sin\left(\frac{n^2 x}{n+1}\right) e^{y/n}, \quad (x,y) \in I \doteq [0,2\pi] \times [0,2\pi], \quad n \in \mathbb{N}.$$

(a) Study the equicontinuity of (u_n) on I.
(b) Study the convergence of (u_n) in the uniform topology of $C(I)$; in the strong and in the weak* topology of $L^\infty(I)$.

44. Let $X = C_0(\mathbb{R}^2, \mathbb{R})$ endowed with the uniform norm and consider the family of subsets of $\mathbb{R}^2$ given by

$$A_\alpha \doteq \left\{(x,y) \in \mathbb{R}^2 : x > 0,\ y > \alpha|x|,\ x^2 + y^2 < \alpha^2\right\}, \quad \alpha > 0.$$

Set

$$T_\alpha u \doteq \frac{1}{\alpha^2} \int_{A_\alpha} u(x,y)\, dxdy, \quad \alpha > 0.$$

(a) Show that $T_\alpha \in X'$ for any $\alpha > 0$ and find its norm and support.
(b) Establish if the family $(T_\alpha)_{\alpha>0}$ converges in the strong and weak* topology of X' when $\alpha \to 0+$ and, in affirmative case, determine the limit T_0.
(c) Find norm and support of T_0.

45. Let $I = [0,1]$, $X = C(I, \mathbb{R})$ and $\alpha(x) \doteq \min\{1, 2x\}$. Set

$$(Tu)(x) \doteq \int_0^{\alpha(x)} |u(t)|^2\, dt.$$

Establish if $T \in \mathcal{L}(X)$ and if $T(B_1^X)$ is relatively compact in X.

46. Consider the following family of Cauchy problems:

$$\begin{cases} y' = \frac{1}{1+ty} & t > 0 \\ y(0) = 1 + \frac{1}{n} & n \in \mathbb{N}. \end{cases}$$

(a) Show that for every $n \in \mathbb{N}$ there exists a solution $y_n(\cdot)$ defined on the whole $\mathbb{R}^+$.
(b) Show that the sequence (y_n) amdits a subsequence uniformly converging on each compact subinterval of $\mathbb{R}^+$.

47. Consider the sequence of functions given by

$$u_n(x,y) = \sin\left(\frac{nx}{n+1}\right)(1 + e^{-n|y|}), \quad (x,y) \in I \doteq [-1,1] \times [-1,1], \quad n \in \mathbb{N}.$$

(a) Study the equicontinuity of (u_n) on I.
(b) Study the convergence of (u_n) in the uniform topology of $C(I)$, in the strong topology of $L^\infty(I)$ and in the weak* topology of $L^\infty(I)$.

48. Let $I = [0,1]$, $X = C^0(I)$ and $m \in X$. Set

$$(T_m u)(x) \doteq m(x)u(x), \quad u \in X,\ x \in I.$$

Show that $T_m \in \mathcal{L}(X)$ and that it is compact if and only if $m(x) = 0$ for every $x \in I$.

49. Let $\overline{B}$ the closed unit ball in $\mathbb{R}$, endowed with the euclidean norm $\|\cdot\|$. Define

$$u_n(x) \doteq |\sin(\|x\|)|^{\frac{1}{n}} \qquad n \in \mathbb{N}.$$

Study the equicontinuity of the family $\{u_n, n \in \mathbb{N}\}$ on $\overline{B}$.

50. Consider the sequence of functions given by

$$u_n(x,y) = \frac{e^{-\frac{ny}{n+1}}}{(1+e^{-nx^2})}, \qquad (x,y) \in I \doteq [-1,1] \times [-1,1], \qquad n \in \mathbb{N}.$$

Study the convergence of (u_n) in the uniform topology of $C(I)$, in the strong topology and in the weak* topology of $L^\infty(I)$.

51. Let $\varphi \in C_c(\mathbb{R})$ and (a_n) a sequence in $\mathbb{R}$. Define

$$u_n(x) \doteq \varphi(x - a_n), \qquad x \in \mathbb{R}, \quad n \in \mathbb{N}.$$

a) Show that $u_n \in L^p(\mathbb{R})$ for every $p \in [1, \infty]$.

b) Study the relative compactness of the sequence (u_n) in the strong and in the weak topology of L^p (weak* if $p = \infty$). That is to say: establish if and for which $p \in [1, \infty]$ there exists a converging subsequence in such topologies.

52. Let $I = [0,1] \subset \mathbb{R}$ and $B \doteq \{u \in C^1(I) : \|u'\|_{L^2(I)} \leq 1\}$.

a) Show that B is an equicontinuous family.

b) Given a sequence (u_n) in $\{u \in B : u(0) = 0, u(1) = 1\}$, show that there exist $u \in C^0(I)$ and a subsequence (u_{n_k}) which converges uniformly to u.

c) Show by a counterexample that property **b)** does not hold in B.

53. Let $\overline{B}$ the closed unit ball in $\mathbb{R}^d$, endowed with the euclidean norm $\|\cdot\|$. Set

$$u_n(x) \doteq e^{-n\|x\|} \qquad n \in \mathbb{N}.$$

Study the equicontinuity of the family $\{u_n, n \in \mathbb{N}\}$ on $\overline{B}$.

54. Consider the sequence of functions given by

$$u_n(x,y) = \min\left\{n, |x|^{-\frac{1}{2}}\right\} \sin\left(\frac{ny}{n+1}\right), \qquad (x,y) \in I \doteq [-1,1] \times [-1,1], \qquad n \in \mathbb{N}.$$

Study the convergence of (u_n) in the strong and weak topology (weak* if $p = \infty$) of $L^p(I)$.

55. Let $\varphi \in C_c(\mathbb{R})$ with $\operatorname{supp}\varphi \subseteq [-1,1]$, $\varphi \geq 0$ e $\int_{\mathbb{R}} \varphi \, dt = 1$. Consider the Dirac sequence given by

$$\rho_n(t) \doteq n\varphi(nt) \quad \forall t \in \mathbb{R} \quad \forall n \in \mathbb{N}$$

and let (a_n) be a sequence in $\mathbb{R}$. Set

$$u_n(t) \doteq \rho_n(x - a_n) \qquad \forall t \in \mathbb{R} \quad \forall n \in \mathbb{N}.$$

a) Show that $u_n \in L^p(\mathbb{R})$ for every $p \in [1,\infty]$ and for every $n \in \mathbb{N}$.

b) Considering the cases $a_n = n$ and $a_n = n^{-2}$, study the convergence of the sequence (u_n) in the strong and weak topology of L^p (weak* if $p = \infty$).

56. Let $I = [0,1] \subset \mathbb{R}$, $X = C^0(I)$ and $Y = L^1(I)$. Set

$$Tu(x) \doteq \int_0^x xyu(y)\,dy.$$

a) Show that $T \in \mathcal{L}(X)$ and $T \in \mathcal{L}(Y)$.
b) Establish if T is compact in $\mathcal{L}(X)$ and in $\mathcal{L}(Y)$, explaining the reasons.

57. Let $(\rho_n)_{n\in\mathbb{N}}$ be a regularizing sequence in $\mathbb{R}$. Study the equiintegrability of the following families:
a) $f_n = \rho_n, \quad n \in \mathbb{N}$;
b) $g_n = \rho_n' \ , \quad n \in \mathbb{N}$;
c) $h_n \doteq \rho_1 \star \rho_n, \quad n \in \mathbb{N}$.

58. Let $\alpha > 0$ and consider the sequence of functions given by
$$u_n(x) \doteq \min\{1, |x|^{-\alpha}\}\chi_{B(0,n)}(x), \quad n \in \mathbb{N},\ x \in \mathbb{R}^d.$$
Study the strong and weak (weak* if $p = \infty$) convergence of (u_n) in the spaces $L^p(\mathbb{R}^d)$ for $p \in [1, \infty]$.

59. Let $Q \doteq [-1, 1]^3 \subseteq \mathbb{R}^3$ and set
$$f(x_1, x_2, x_3) = \begin{cases} \left(x_1\, x_2^2\, x_3^3\right)^{-1}, & x_1\, x_2\, x_3 \neq 0 \\ 0, & x_1\, x_2\, x_3 = 0. \end{cases}$$
- Establish for which $p \in [1, \infty]$ we have $f \in L^p(\mathbb{R}^3)$;
- establish for which $p \in [1, \infty]$ we have $f \in L^p(Q)$;
- establish for which $p \in [1, \infty]$ we have $f \in L^p(\mathbb{R}^3 \setminus Q)$.

60. Let $E \subseteq \mathbb{R}$ be a measurable set, $p_i \in [1, \infty]$, $f_i \in L^{p_i}(E)$ for $i = 1, \dots, n$, and $r \in [1, \infty]$ given by
$$\frac{1}{r} \doteq \sum_{i=1}^{n} \frac{1}{p_i}.$$
Show that
$$\prod_{i=1}^{n} f_i \in L^r(E)$$
and that the following inequality holds:
$$\left\| \prod_{i=1}^{n} f_i \right\|_{L^r(E)} \leq \prod_{i=1}^{n} \|f_i\|_{L^{p_i}(E)}.$$

61. Let $(\rho_n)_{n\in\mathbb{N}}$ be a regularizing family in $\mathbb{R}$ and $f \in C^0(\mathbb{R})$. Set
$$f_n(x) \doteq (\rho_n \star f)(x), \quad x \in \mathbb{R}.$$
Show that the definition is well posed and that the sequence (f_n) converges uniformly to f on any compact subset $K \subseteq \mathbb{R}$.

62. Let $I = [-1, 1] \subseteq \mathbb{R}$ and (u_n) a sequence in $C^2(\mathbb{R})$ such that
(a) u_n is convex on $\mathbb{R}$ for every $n \in \mathbb{N}$;
(b) There exists $K \geq 0$ such that $|u_n(0)| + |u_n'(t)| \leq K$ for every $t \in I$ and for every $n \in \mathbb{N}$.

(1) Show that the sequence (u_n') is relatively compact in $L^1(I)$.
(2) Show that there exists a subsequence (u_{n_k}) and a map $u \in C^0(I)$ such that (u_{n_k}) converges uniformly to u on I.

63. Let $I = [0,1] \subseteq \mathbb{R}$ and $\{e_n, n \in \mathbb{N}\}$ a Hilber basis $L^2(I)$. Set

$$(e_m \otimes e_n)(x,y) \doteq e_m(x)e_n(y); \quad m,n \in \mathbb{N},\ (x,y) \in I \times I.$$

Show that the family $\{e_m \otimes e_n;\ m,n \in \mathbb{N}\}$ is a Hilbert basis in $L^2(I \times I)$.

64. Let $I \doteq [\,-1,1] \subseteq \mathbb{R}$ and consider the sequence of functions given by:

$$u_n(t) = e^{-n} \cdot e^{nt^2}; \quad t \in I, \quad n \in \mathbb{N}.$$

Study the convergence of the sequence (u_n) in the following spaces:

(i) $C^0(I)$ with uniform topology;
(ii) $L^1(I)$ with strong topology;
(iii) $L^1(I)$ with weak topology;
(iv) $L^\infty(I)$ with strong topology;
(v) $L^\infty(I)$ with weak* topology.

65. For every $n \in \mathbb{N}$ set

$$f_n(x) = \sin\left(\frac{x}{n}\right); \quad g_n(x) = \sin\left(n^2 x\right); \quad h_n(x) = \sin\left(\frac{nx}{n+1}\right); \quad x \in [0,2\pi].$$

Study the equicontinuity of the sequences $\{f_n, n \in \mathbb{N}\}$, $\{g_n, n \in \mathbb{N}\}$ e $\{h_n, n \in \mathbb{N}\}$ on $[0,2\pi]$.

66. Let $I = [0,1] \subset \mathbb{R}$ and, for every $n \in \mathbb{N}$, consider the subintervals of the form

$$I_n^m \doteq \left[\frac{m}{n}, \frac{m+1}{n}\right[, \quad m = 0,1,\dots,n-1.$$

Then set

$$u_n(t) \doteq (-1)^m \text{ for } t \in I_n^m.$$

Study the strong and weak convergence of the sequence (u_n) in $L^2(I)$.

67. Let D be te unit disk in $\mathbb{C}$. Study the equicontinuity of the following families of functions in $C(D)$:

(i) $\{f_a(z) = e^{iaz}, a \in \mathbb{R}\}$;
(ii) $\{f_a(z) = e^{i\frac{z}{a}}, a \in \mathbb{R}\ a \neq 0\}$;
(iii) $\{f_a(z) = e^{iaz}, a \in \mathbb{R}, |a| > 1\}$;
(iv) $\{f_a(z) = e^{iaz}, a \in \mathbb{R}, |a| < 1\}$.

68. Let $X = C([0,1],\mathbb{R})$ and (a_n) a sequence in $[0,1]$. Set

$$\langle f_n, u\rangle \doteq u(a_n),\ \forall n \in \mathbb{N},\ \forall u \in X.$$

Show that $f_n \in X'$ for every $n \in \mathbb{N}$ and that there exists a subsequence (f_{n_k}) which converges in the topology $\sigma(X', X)$.

69. Study the equicontinuity of the following families in $C(I)$ ($I \subseteq \mathbb{R}$)).

(i) $\{f_a(x) = e^{ax}, a \in \mathbb{R}\},\ I = \mathbb{R}$;
(ii) $\{f_a(x) = a(1-x)^2, a \in \mathbb{R}^+\}, I = [-1,1]$;
(iii) $\{f_a(x) = x^{-a}, a \in \mathbb{R}^+, \}, I =]1,+\infty[$;
(iv) $\{f_a(x) = x^{-a}, a \in \mathbb{R}^+, \}, I =]0,+\infty[$.

70. Let $p \in [1, \infty[$. Consider the space $X = L^p([0,1])$ and set

$$(Tu)(x) = \int_0^x u(t)\,dt.$$

(i) Show that $T \in \mathcal{L}(X)$ and that $\|T\|_{\mathcal{L}(X)} \leq \left(p^{\frac{1}{p}}\right)^{-1}$.

(ii) Given a sequence (u_n) in X weakly converging to u in X, show that the sequence (Tu_n) converges strongly to Tu in X.

71. Let $C > 0$, $p \in [1, \infty[$, $\alpha \in]0,1[$ and $B \doteq \{x \in \mathbb{R}^d : \|x\| \leq 1\}$. Consider the set

$$U \doteq \{u \in C(B) : u(0) = 0, |u(x) - u(y)| \leq C|x-y|^\alpha \ \forall x, y \in B\}.$$

Show that U is relatively compact in $L^p(B)$.

72. Let $I \doteq [0,1]$ and (u_n) a sequence in $C^1([0,1])$ such that

$$|u_n(0)| + \int_I |u_n'(t)|\,dt \leq 1 \quad \forall n \in \mathbb{N}.$$

Show that there exist a subsequence (u_{n_k}) and a map $u \in L^1(I)$ such that $u_{n_k} \to u$ strongly in $L^1(I)$.

73. Let $E \subseteq \mathbb{R}^d$ be ameasurable set such that $0 < m(E) < +\infty$. For every $p \in [1, +\infty[$ and for every $f \in L^p(E)$ set

$$N_p[f] \doteq \left(\frac{1}{m(E)} \int_E |f(x)|^p\right)^{\frac{1}{p}}.$$

Show that $N_p[\cdot]$ is a norm on $L^p(E)$ and that, if $1 \leq p \leq q < +\infty$, we have

$$N_p[f] \leq N_q[f] \qquad \forall f \in L^q(E).$$

74. Let X be a Banach space and set $\mathcal{K}(X) \doteq \{T \in \mathcal{L}(X) : T \text{ is compact}\}$. Show that $\mathcal{K}(X)$ is closed in $\mathcal{L}(X)$.

75. Let $X = C_0(\mathbb{R}^2)$ and (a_n) a sequence in $\mathbb{R}^+$. For every $n \in \mathbb{N}$ and for every $u \in X$ set

$$T_n(u) = \int_{-a_n}^{+a_n} u(x, nx)\,dx.$$

Show that $T_n \in X'$ for every $n \in \mathbb{N}$ a find its norm and support. Study the convergence of the sequence (T_n) in the strong and weak* topology of X' in the cases $a_n = 1 + n^2$ and $a_n = e^{-\frac{1}{n}}$.

76. Let $I = [0,1]$ and H an equicontinuous subset of $C^0(I)$. Show that $\overline{H}$ is equicontinuous.

77. Let $I = [0,1]$, $B_r = B(0,r)$ the ball in $\mathbb{R}^d$ of center zero and radius r, $p \in [1, \infty]$, $X_p \doteq L^p(B_1)$ and $Y \doteq C^0(I, \mathbb{R})$. Given $u \in X_p$ and $t \in I$, set

$$(Tu)(t) \doteq \int_{B_t} u(y)\,dy.$$

Show that $T \in \mathcal{L}(X_p, Y)$ for every p and establish for which p it is compact.

78. For $(x,y) \in I \doteq [-1,1] \times [-1,1]$, consider the sequence of functions given by

$$u_n(x,y) = \left(\cos\left(\frac{nx^2}{n+1}\right)\sin(nx)\right)(1+e^{-ny^2}), \quad n \in \mathbb{N}.$$

Study the convergence of (u_n) in the strong and weak topology (weak* if $p=\infty$) of $L^p(I)$.

79. Let (a_n) and (b_n) sequence in $\mathbb{R}^+$ and set $R_n \doteq [-a_n, a_n] \times [-b_n, b_n] \subseteq \mathbb{R}^2$ and

$$u_n(x,y) \doteq \chi_{R_n}(x,y), \quad (x,y) \in \mathbb{R}^2.$$

Study the convergence of (u_n) in the strong and weak topology of $L^1(\mathbb{R}^2)$ and in the strong and weak* topology of $L^\infty(\mathbb{R}^2)$ in the following cases:

1. $a_n = n$, $b_n = n^{-1}$;
2. $a_n = n$, $b_n = n^{-\frac{1}{2}}$;
3. $a_n = \frac{n}{n+1}$, $b_n = n^{-1}$;
4. $a_n = \frac{n}{n+1}$, $b_n = \frac{n}{n+1}$.

80. Let $X = C_0(\mathbb{R}^2, \mathbb{R})$, endowed with the uniform norm, and $(a_n), (b_n)$ sequences in $\mathbb{R}^+$. Define

$$\langle f_n, u\rangle \doteq \int_0^{2\pi} u(a_n\cos\theta, b_n\sin\theta)\,d\theta, \quad n \in \mathbb{N},\ u \in X.$$

Show that $f_n \in X'$ for every $n \in \mathbb{N}$ and find its norm and support.

Suppose $a_n \to 1, b_n \to 0$ and study the convergence of the sequence (f_n) in the strong and weak* topology of X'.

81. Let $I = [0,1]$, $M > 0$ and (u_n) a sequence in $C^1(I)$ such that

1. $\int_I |u_n(t)|^2 \leq M \ \forall n \in \mathbb{N}$;
2. $u_n'(t) + t \geq 0 \ \forall t \in I, \forall n \in \mathbb{N}$.

Show that the sequence (u_n) is relatively compact in $L^1(I)$.

82. Let (x_n) be a sequence in a Hilbert space H endowed with the inner product $\langle\cdot,\cdot\rangle$. Show that, if the sequence $(\langle x_n, y\rangle)$ converges for every $y \in H$, then the sequence (x_n) converges weakly.

83. Let $I = [0,1]$ and call X the Banach space $C(I)$, endowed with the uniform norm. Introduce the space

$$Y \doteq \{u \in X, u \text{ differentiable on } I \text{ with } u' \in X\}$$

and set

$$\|u\|_Y \doteq \|u\|_\infty + \|u'\|_\infty,\ u \in Y.$$

Prove that $(Y, \|\cdot\|_Y)$ is a Banach space.

Let α be a nonzero element of X and set

$$(Tu)(x) \doteq \alpha(x)u'(x) \quad u \in Y,\ x \in I.$$

(i) Prove that $T \in \mathcal{L}(Y,X)$ and find its norm.
(ii) Establish if T is compact and justify the answer.

84. Let H be a Hilbert space. For $T \in \mathcal{L}(H)$ denote by $R(T)$ and $N(T)$, respectively, the range and the kernel of T. Calling T^* the adjoint of T, prove that $N(T) = (R(T^*))^\perp$ and $\overline{(R(T))} = (N(T^*))^\perp$.

85. Let $B_r = B(0,r)$ be the ball in $\mathbb{R}^d$ of center zero and radius r and $X \doteq C_0(\mathbb{R})$. Let m be a map in $C(\mathbb{R})$, with $m(x) \geq 0$ for every $x \in \mathbb{R}$, and, for every $t > 0$, set

$$T_t(u) \doteq t^{-d} \int_{B_t} m(y)u(y)\,dy.$$

Prove that $T_t \in X'$ for every $t > 0$ and find its norm and support. Study the convergence of T_t as $t \to 0+$ in the strong and weak* topology of X'.

86. Let $I = [0,1]$ and (u_n), (v_n) be two bounded sequences in $L^2(I)$. Assume in addition that the maps $I \ni x \mapsto u_n(x)$ and $I \ni x \mapsto v_n(x)$ are continuous and monotone non decreasing for every $n \in \mathbb{N}$; then define

$$f_n(x,y) \doteq u_n(x)v_n(y), \quad (x,y) \in Q \doteq I \times I.$$

Prove that f_n lies in $L^2(Q)$ for every $n \in \mathbb{N}$ and that the sequence (f_n) is relatively compact in $L^1(Q)$.

87. Let $I = [0,1]$, $Q \doteq I \times I$ and (a_n), (b_n) sequences in $]0,1]$. Define the family of sets $R_n \doteq [0,a_n] \times [0,b_n] \subseteq Q$ and set

$$u_n(x,y) \doteq (1+\sin(nx))(1+e^{-ny})\chi_{R_n}(x,y), \quad (x,y) \in Q.$$

Study the convergence of (u_n) in the strong and weak topology of $L^1(Q)$ and in the strong and weak* topology of $L^\infty(Q)$ in the following cases:

1. $a_n = n^{-2}$, $b_n = 1 - n^{-1}$;
2. $a_n = 1 - n^{-2}$, $b_n = 1 - n^{-1}$.

88. Let H be a complex Hilbert space with inner product $(\cdot,\cdot)$. Prove that we have

$$4(x,y) = (\|x+y\|^2 - \|x-y\|^2) - i(\|x+iy\|^2 - \|x-iy\|^2) \quad \forall x,y \in H.$$

89. Let $I = [0,1]$ and call X the Banach space $C(I)$, endowed with the uniform norm. Let $g \in C(I \times I)$ and set

$$(Tu)(x) \doteq \int_I g(x,y)u(y)\,dy \quad u \in X,\ x \in I.$$

(i) Prove that $T \in \mathcal{L}(X)$ and estimate its norm.
(ii) Establish if T is compact and justify the answer.
(iii) Compute the norm of T in the case $g(x,y) = e^{x+y}$.

90. Let $X = C_0(\mathbb{R}^2)$ and, for every $n \in \mathbb{N}$, consider the set

$$R_n \doteq]-n,n[\times]-n^{-1},n^{-1}[\subseteq \mathbb{R}^2.$$

Given $u \in X$ and $n \in \mathbb{N}$ set

$$(T_n u)(x) = \frac{1}{n} \int_{R^n} e^{-(x^2+y^2)} u(x,y)\,dxdy.$$

Prove that $T_n \in X'$ for every $n \in \mathbb{N}$ and find its norm and support. Study the convergence of the sequence (T_n) in the strong and weak* topology of X'.

91. Let $Q = [0,1]^d \subseteq \mathbb{R}$ and consider (u_n), (v_n), two relatively compact sequences in $L^2(Q)$. Define

$$f_n(x) \doteq u_n(x)v_n(x), \quad x \in Q, \ n \in \mathbb{N}.$$

Prove that f_n lies in $L^1(Q)$ for every $n \in \mathbb{N}$ and that the sequence (f_n) is relatively compact in $L^1(Q)$.

92. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be map of class C^1 such that $\varphi(0) = 0$ and $1 \le \varphi'(t) \le 2$ for every $t > 0$. Let $I = [0,1]$ and (u_n) a sequence in $L^1(\mathbb{R})$.

(i) Prove that the sequence (v_n) defined by $v_n(t) \doteq u_n(\varphi(t))$ for $t \in I$ and $n \in \mathbb{N}$ lies in $L^1(I)$.

(ii) Assuming that $u_n \to u$ strongly in $L^1(\mathbb{R})$, study the convergence of (v_n) in the strong and weak convergence of $L^1(I)$.

(iii) Assuming that $u_n \rightharpoonup u$ weakly in $L^1(\mathbb{R})$, study the convergence of (v_n) in the strong and weak convergence of $L^1(I)$.

93. Let $Q = [0,1] \times [0,1]$ and X the Banach space $C^0(Q)$, endowed with the uniform norm. Set

$$(T_n u) \doteq \int_0^1 n\, e^{-nx}\, u(x, x^2)\, dx, \quad u \in X.$$

Prove that $T_n \in X'$ for every $n \in \mathbb{N}$ and find its norm and support. Study the convergence of (T_n) in the strong and weak* topology of X'.

94. Let H be a Hilbert space, $T \in \mathcal{L}(H)$ and (T_n) a sequence in $\mathcal{L}(H)$.

(i) Prove that $T_n \to T$ if and only if $T_n^* \to T^*$.

(ii) Prove that the sequence $(T_n x)$ converges weakly to Tx for every $x \in H$ if and only if the sequence $(T_n^* x)$ converges weakly to $T^* x$ for every $x \in H$.

95. Let $I = [0,1] \subseteq \mathbb{R}$ and $X = C^0(I)$. Given a map $m \in L^2(I)$, set

$$Tu(x) \doteq \int_0^{x^2} m(y)u(y)\, dy.$$

Prove that $T \in \mathcal{L}(X)$ and establish if T is compact in $\mathcal{L}(X)$, justifying the answer.

96. Let $Q = [0,1]^d \subseteq \mathbb{R}$. Consider two relatively compact families U and V in $C^0(Q)$ and define

$$F \doteq \{f : f(x) = \sin(u(x) \cdot v(x)),\ x \in Q, u \in U, v \in V\}.$$

Prove that F is a relatively compact family in $C^0(Q)$.

97. Let $I = [0,1] \subseteq \mathbb{R}$, $p > 1$ and $X = L^\infty(I)$. Given a map $m \in L^p(I)$, set

$$Tu(x) \doteq \int_0^x m(y)u(y)\, dy.$$

Prove that $T \in \mathcal{L}(X)$ and establish if T is compact in $\mathcal{L}(X)$, justifying the answer.

98. Let X be the Banach space $C_0(\mathbb{R}^2)$, endowed with the uniform norm, and let (g_n) be a sequence in $C_b(\mathbb{R}^2)$ such that

$$0 \le g_n(x,y) \le (1 + x^2 + y^2)^{-1} \quad \forall (x,y) \in \mathbb{R}^2, \forall n \in \mathbb{N}$$

and

$$g_n \longrightarrow g \quad \text{in} \quad C_b(\mathbb{R}^2).$$

Set

$$(T_n u) \doteq \int_{\mathbb{R}} g_n(x,x)u(x,x)\,dx, \qquad u \in X.$$

Prove that $T_n \in X'$ for every $n \in \mathbb{N}$ and find its norm and support. Study the convergence of (T_n) in the strong and weak* topology of X'.

99. Let $f \in L^2(\mathbb{R})$ and set

$$(Tu)(x) \doteq \int_{\mathbb{R}} f(x-y)u(y)\,dy.$$

Establish for which indices $p, q \in [1, +\infty]$ we have $T \in \mathcal{L}(L^p(\mathbb{R}), L^q(\mathbb{R}))$.

100. Let $I = [0,1] \subseteq \mathbb{R}$, $X = C^0(I)$ and $Y = L^1(I)$. Set

$$Tu(x) \doteq \int_0^x xyu(y)\,dy.$$

a) Prove that $T \in \mathcal{L}(X)$ and $T \in \mathcal{L}(Y)$.
b) Establish if T is compact in $\mathcal{L}(X)$ and in $\mathcal{L}(Y)$, justifying the answer.