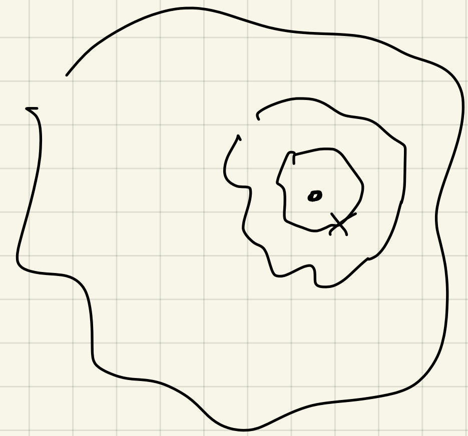
7 oct

$X$



$C^0_c(X)$

$d(f,g) = \sup_{x\in X} |f(x)-g(x)|$ $\quad X$

$f \leq g \iff f(x) \leq g(x) \quad \forall x$

$\Lambda : C^0_c(X) \to \mathbb{R}$ is positive if

$f \geq 0 \longrightarrow \Lambda f \geq 0$

Given $f \in C^0_c(X, [0,1])$

and $K$ compact $\subset X$, $(K \subset\subset X)$

we write $K \prec f$ if

$f|_K \equiv 1$

and given $V \subset X$ open with

$\operatorname{supp} f \subseteq V$ we write $f \prec V$

Th (Reisz Representation)

Th $X$ locally compact Hausdorff

$\Lambda : C_c^0(X) \longrightarrow \mathbb{R}$ $\Lambda \geq 0$

Then $\exists$ ~~a~~ a $\sigma$-algebra $\mathcal{M}$ containing Borel sets and a unique measure $\mu : \mathcal{M} \longrightarrow [0, +\infty]$ s.t.

1) $\Lambda f = \int_X f \, d\mu$ $\forall f \in C_c^0(X)$

2) $\mu(K) < +\infty$

3) $\forall E \in \mathcal{M}$

$$\mu(E) = \inf\{\mu(V) : E \subseteq V \text{ } V \text{ open}\}$$

4) We have

$$\mu(E) = \sup\{\mu(K) : K \subset\subset E\}$$

$\forall E$ open and $\forall E \in \mathcal{M}$ s.t. $\mu(E) < +\infty$

5) $\forall \; E \in \mathcal{M}$ with $\mu(E) = 0$ and $\forall \; A \subseteq E$ we have

$A \in \mathcal{M}$

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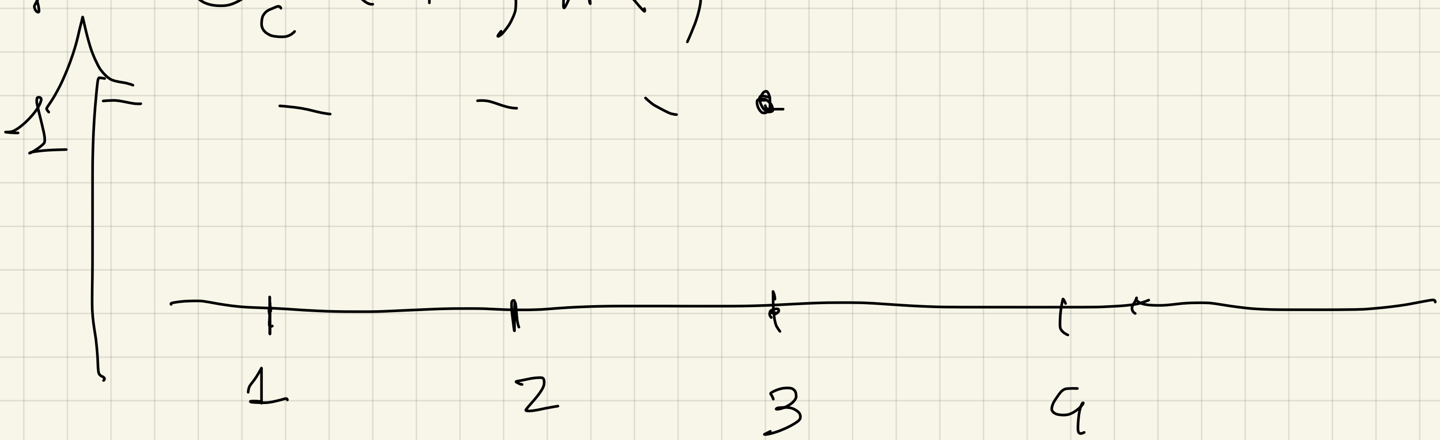
$X = \mathbb{N}$

$\Lambda : C^0_c(\mathbb{N}, \mathbb{R}) \longrightarrow \mathbb{R}$

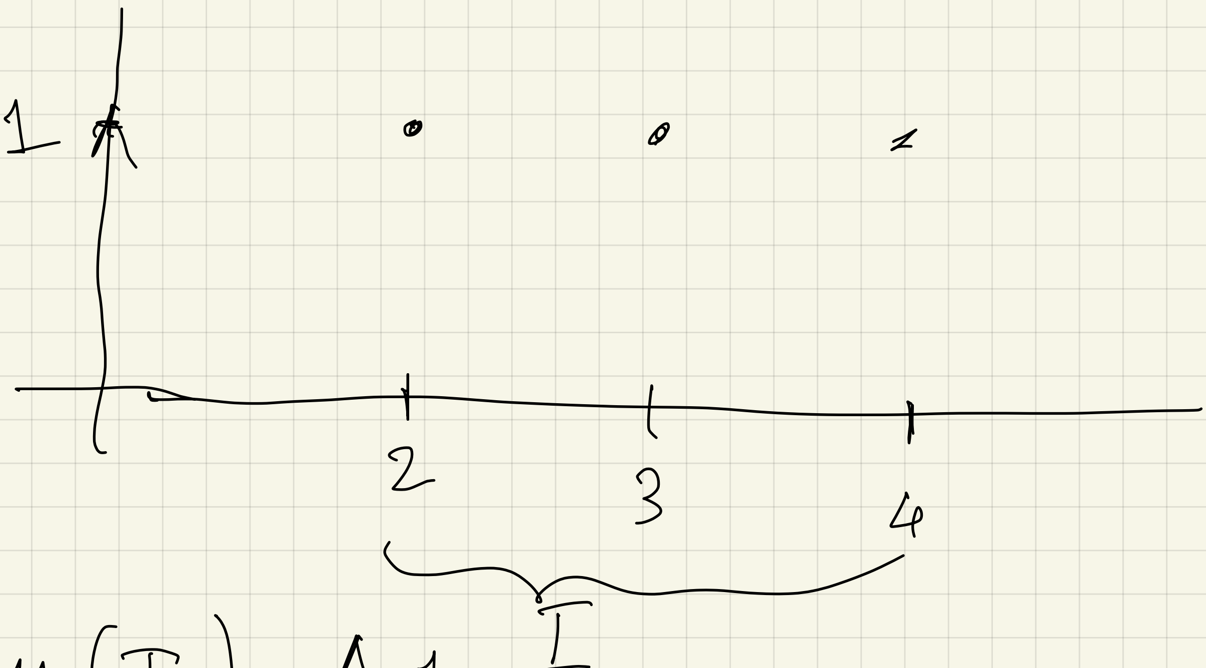



$e_n, \; e_3 \in C^0_c(\mathbb{N}, [0,1])$

$$e_3(n) = \begin{cases} 0 & n \neq 3 \\ 1 & n = 3 \end{cases}$$

$\{e_n\}$ are a basis

$$\mu(n) = \Lambda e_n$$




$$\mu(I) = \Lambda 1_I$$

$$1_I < I$$

Proof For any open $V$

$$\mu(V) := \sup\{\Lambda f : f \prec V\}$$

$$V_1 \subseteq V_2 \Rightarrow \mu(V_1) \leq \mu(V_2)$$

$\forall \; E \subseteq X$

$$\mu(E) := \inf\{\mu(V) : E \subseteq V\}$$

$$E_1 \subseteq E_2 \Rightarrow \mu(E_1) \leq \mu(E_2)$$

$\mathcal{M}_F$ is formed by the $E$'s

s.t. $\mu(E) < +\infty$ and

$$\mu(E) = \sup\{\mu(K) : K \subset\subset E\}$$

$\mathcal{M}$ is formed by the $E$'s

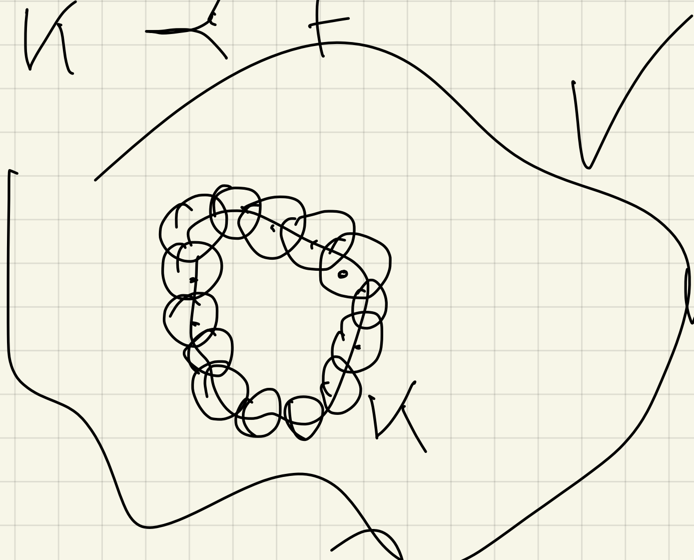
s.t. $K \cap E \in \mathcal{M}_F$

$\forall \; K \subset\subset X$.

I) $\mathcal{M}_F \ni K \quad \forall \; K \subset\subset X$

$\mu(K) < +\infty$ ?

$K \prec f$



$K \subset\subset \mathring{H}$

$H$ compact

$K \subset V \subset \mathring{H}$

$V = \{ f > \frac{1}{2} \}$

$K \subset\subset V$

$\forall \quad g \prec V$ $\qquad V$ in $X$

$$g \leq 2f$$

$2f > 1$ in $V$

$$g \leq 2f \quad \text{in } V$$

$$g \leq 1 < 2f \quad \text{in } V$$

$$\Lambda g \leq 2 \Lambda f < +\infty$$

$\forall \quad g \prec V$

$$\mu(V) = \sup\{\Lambda g : \; g \prec V\}$$
$$\leq 2 \Lambda f < +\infty$$

$$K \Subset V$$

$$\mu(K) \leq \mu(V) < +\infty$$

(*) $\mu(E) = \sup\{\mu(K) : K \subset\subset E\}$

$\underline{\text{II}}$ $\cancel{V}$ open satisfies (*)

so $\mu(V) < +\infty \Rightarrow V \in \mathcal{M}_F$

$\alpha < \mu(V) = \sup\{\Lambda f : f \prec V\}$

$\exists$ $f$ with $\Lambda f > \alpha$

We want $\mu(\operatorname{supp} f) \geq \alpha$

for any $V_1 \supseteq \operatorname{supp} f$

$f \prec V_1 \Rightarrow \mu(V_1) \geq \Lambda f > \alpha$

$\Rightarrow$ $V_1 \supseteq \operatorname{supp} f \Rightarrow \mu(V_1) > \alpha$

$\mu(K) = \inf\{\underbrace{\mu(V)}_{>\alpha} : K \subseteq V\}$

$\Rightarrow \mu(K) \geq \alpha$

III

$$\mu\Big(\bigcup_{n\in\mathbb{N}} E_n\Big) \leq \sum_{n=1}^{\infty} \mu(E_n)$$ (1)

$$\mu(V_1 \cup V_2) \leq \mu(V_1) + \mu(V_2)$$

$$f \prec V_1 \cup V_2$$

$$h_j \prec V_j \qquad j = 1, 2$$

$$h_1(x) + h_2(x) = 1 \qquad \forall x \in \underbrace{\operatorname{supp} f}_{K}$$

$$f = h_1 f + h_2 f$$

$$f(x) = 1_K(x)\, f(x) = (h_1 + h_2)\, f$$

$$\Lambda f = \Lambda h_1 f + \Lambda h_2 f$$

$$h_j f \prec V_j$$

$$\Lambda(h_j f) \leq \mu(V_j)$$

$\forall \quad f \prec V_1 \cup V_2$

$\Lambda f = \Lambda h_1 f + \Lambda h_2 f \leq \mu(V_1) + \mu(V_2)$

$\mu(V_1 \cup V_2) \leq \mu(V_1) + \mu(V_2)$

$\mu(E_n) < +\infty \qquad \forall n$

$E_n \subseteq V_n$

$\mu(V_n) \leq \mu(E_n) + 2^{-n} \varepsilon$

$V = \bigcup_{n \in \mathbb{N}} V_n$ and $f \prec V$

$\operatorname{supp} f \subseteq \bigcup_{j=1}^{N} V_j \qquad f \prec \bigcup_{j=1}^{N} V_j$

$$\Lambda f \leq \mu\left(\bigcup_{j=1}^{N} V_j\right) \leq \sum_{j=1}^{N} \mu(V_j) \leq$$

$$\leq \sum_{j=1}^{N} \mu(E_j) + \underbrace{\sum_{j=1}^{N} 2^{-j} \varepsilon}_{< \varepsilon}$$

$$\mu(V) \leq \sum_{j=1}^{\infty} \mu(E_j) + \varepsilon$$

$$V \supseteq \bigcup_{n=1}^{\infty} E_n$$

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu(E_n) + \varepsilon$$

$$\forall \quad \varepsilon > 0$$

$$f \in C_c^0(X)$$

$$\Rightarrow \quad \Lambda f = \int f \, d\mu \quad \forall \quad f \in C_c^0(X, \mathbb{R})$$

$$\Lambda f \leq \int f \, d\mu \qquad \forall f$$

$$K = \operatorname{supp} f \qquad [a, b] \supseteq f(X)$$

$$y_0 < a < y_1 < \dots < y_n = b$$

$$0 < y_j - y_{j-1} < \varepsilon$$

$$E_j = f^{-1}((y_{j-1}, y_j]) \cap K$$

$$\bigcup_{j=1}^{n} E_j = K \qquad f < y_j + \varepsilon \text{ in } V_j$$

$$V_j \supseteq E_j \qquad \mu(V_j) \leq \mu(E_j) + \frac{\varepsilon}{n}$$

$$h_j \prec V_j \qquad \sum h_j = 1 \text{ in } K$$

$$f = \sum h_j f \qquad \nexists \qquad h_j \prec V_j$$

$$\Lambda f = \sum_{j=1}^{n} \Lambda \underset{V_j}{h_j f} \leq \sum_{j=1}^{n} (y_j + \varepsilon) \Lambda h_j$$

$$h_j f \leq h_j (y_j + \varepsilon)$$

$$\leq \sum_{j=1}^{n} (y_j + \varepsilon) \mu(V_j)$$

$$\leq \sum_{j=1}^{n} (y_j + \varepsilon) \left(\mu(E_j) + \frac{\varepsilon}{n}\right)$$

$$\leq \sum_{j=1}^{n} (y_j - \varepsilon) \mu(E_j) + 2\varepsilon \mu(K) +$$

$$+ \quad (b + \varepsilon) \varepsilon$$

$$\leq \sum_{j=1}^{n} \int_{E_j} f \, d\mu + 2\varepsilon\mu(K) + (b+\varepsilon)\varepsilon$$

$$\int_X f \, d\mu$$

$$\boxed{\Lambda f \leq \int_X f \, d\mu} + C\varepsilon$$