SOLUZIONE ESAME 17/02

ES 1

I) $$f(z) = \frac{e^{iz}}{1+z^4}$$

$$1 + z^4 = 0 \Rightarrow z = z_k = e^{i\frac{\pi}{4}} e^{\frac{2\pi i k}{4}}, \quad k = 0,1,2,3$$

$$1+z^4 = (z-z_0)(z-z_1)(z-z_2)(z-z_3)$$

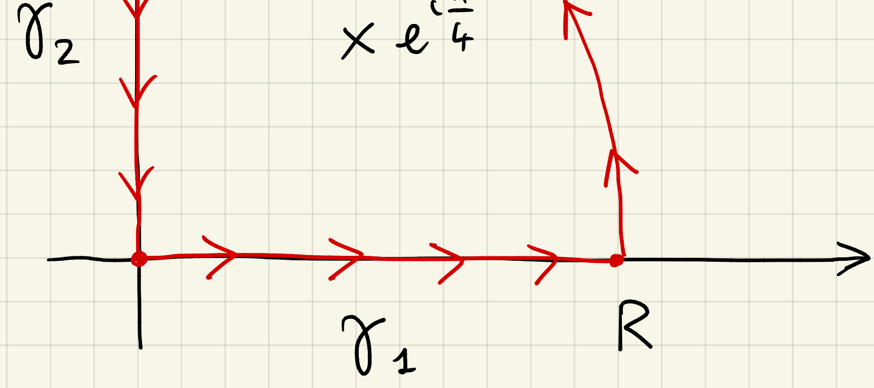
$\Rightarrow$ z_k sono poli di ordine 1.

e^{iz} è analitico su tutto $\mathbb{C}$ quindi non introduce nessuna altra singolarità.

$z = \infty$: infinite potenze positive di z da e^{iz}

$\Rightarrow$ è una singolarità essenziale

II)



$$\gamma_1 + \gamma_2 + \gamma_R = \gamma$$

$$\oint_\gamma dz\, f(z) = \left(\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_R} \right) dz\, f(z)$$

$$\int_{\gamma_1} dz\, f(z) \underset{z=x,\ x\in[0,R]}{=} \int_0^R dx\, \frac{e^{ix}}{1+x^4} = \int_0^R dx\, \frac{\cos x}{1+x^4} + i \int_0^R dx\, \frac{\sin x}{1+x^4}$$

$$\int_{\gamma_2} dz\, f(z) = -\int_{-\gamma_2} dz\, f(z) \underset{z=ix,\ x\in[0,R]}{=} -\int_0^R dx\, i\, \frac{e^{-x}}{1+(ix)^4}$$

$$\underset{i^4=1}{=} -i \int_0^R dx\, \frac{e^{-x}}{1+x^4}$$

$$\left| \int_{\gamma_R} dz\, f(z) \right| \leq \frac{\pi}{2} R \max_{\gamma_R} |f(z)| \leq \frac{\pi}{2} R \max_{\gamma_R} \frac{1}{|1+z^4|}$$

$$\left(|e^{iz}| = e^{-\mathrm{Im}(z)} \leq 1 \text{ su } \gamma_R \right)$$

$$\underset{R\to\infty}{=} \frac{\pi}{2} R \frac{1}{R^4} \left(1 + O\left(\frac{1}{R^4}\right) \right) \xrightarrow[R\to\infty]{} 0 .$$

$$\Rightarrow \lim_{R\to\infty} \oint_\gamma dz\, f(z) = \int_0^{+\infty} dx\, \frac{\cos x}{1+x^4} + i \int_0^{+\infty} dx\, \frac{\sin x - e^{-x}}{1+x^4}$$

Per il teorema dei residui:

$$\forall R \geq 1 \qquad \oint_\gamma dz f(z) = 2\pi i \, \mathrm{Res}_f\left(e^{i\frac{\pi}{4}}\right)$$

Dunque:
$$\int_0^\infty dx \frac{\cos x}{1+x^4} + i \int_0^\infty dx \frac{\sin x - e^{-x}}{1+x^4} = 2\pi i \cdot \mathrm{Res}_f\left(e^{i\frac{\pi}{4}}\right)$$

✓

III)
$$\mathrm{Res}_f\left(e^{i\frac{\pi}{4}}\right)$$
$$= \lim_{z \to e^{i\pi/4}} \left(z - e^{i\frac{\pi}{4}}\right) \frac{e^{iz}}{1+z^4}$$

$$= \lim_{z \to e^{i\pi/4}} \frac{e^{iz}}{\dfrac{z^4+1}{z-e^{i\pi/4}}}$$

della forma:
$$\frac{g(z)-g(z_0)}{z-z_0}$$

dove: $g(z) = z^4$, $z_0 = e^{i\frac{\pi}{4}}$

Pertanto nel limite tende alla derivata:

$$= \frac{e^{i\left(e^{i\frac{\pi}{4}}\right)}}{4z^3\big|_{z=e^{i\pi/4}}} = \frac{e^{i\left(e^{i\frac{\pi}{4}}\right)}}{4} e^{-3i\frac{\pi}{4}}$$

Notiamo:
$$e^{i\pi/4} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, \quad e^{-3i\frac{\pi}{4}} = -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$

$$\Rightarrow \operatorname{Res}_f\left(e^{i\frac{\pi}{4}}\right) = -\frac{e^{i\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)}}{4}\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)$$

$$= -\frac{e^{-\frac{1}{\sqrt{2}}}}{4}\left(\cos\frac{1}{\sqrt{2}} + i\sin\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)$$

$$= -\frac{e^{-\frac{1}{\sqrt{2}}}}{4}\frac{1}{\sqrt{2}}\left(\cos\frac{1}{\sqrt{2}} - \sin\frac{1}{\sqrt{2}} + i\left(\cos\frac{1}{\sqrt{2}}+\sin\frac{1}{\sqrt{2}}\right)\right)$$

Dunque:

$$\int_0^\infty dx\,\frac{\cos x}{1+x^4} + i\int_0^\infty dx\,\frac{\sin x - e^{-x}}{1+x^4} = 2\pi i \operatorname{Res}_f\left(e^{i\pi/4}\right)$$

$$= \frac{\pi e^{-\frac{1}{\sqrt{2}}}}{2\sqrt{2}}\left(\cos\left(\frac{1}{\sqrt{2}}\right)+\sin\left(\frac{1}{\sqrt{2}}\right) + i\left(\sin\left(\frac{1}{\sqrt{2}}\right)-\cos\left(\frac{1}{\sqrt{2}}\right)\right)\right)$$

Uguagliando le parti reali:

$$\int_0^\infty dx\,\frac{\cos x}{1+x^4} = \frac{\pi e^{-\frac{1}{\sqrt{2}}}}{2\sqrt{2}}\left(\cos\frac{1}{\sqrt{2}}+\sin\frac{1}{\sqrt{2}}\right)$$

e uguagliando le parti immaginarie:

$$\int_0^\infty dx\,\frac{\sin x - e^{-x}}{1+x^4} = \frac{\pi e^{-\frac{1}{\sqrt{2}}}}{2\sqrt{2}}\left(\sin\frac{1}{\sqrt{2}}-\cos\frac{1}{\sqrt{2}}\right)$$

Es 2

I) $\hat{f}(\omega)$ è una funzione in $L^2(\mathbb{R})$ che è $\neq 0$ solo per $\omega \in [-A, A]$. Se moltiplichiamo $\hat{f}(\omega)$ per una potenza positiva di ω, $\omega^k \hat{f}(\omega)$ è ancora a quadrato sommabile perché:

$$\int_{-\infty}^{+\infty} d\omega \, |\omega^k \hat{f}(\omega)|^2 \underset{\left(\substack{\hat{f}(\omega) \neq 0 \text{ solo} \\ \text{in } [-A, A]}\right)}{=} \int_{-A}^{+A} d\omega \, |\omega|^{2k} \, |\hat{f}(\omega)|^2$$

$$\leq A^{2k} \int_{-A}^{+A} d\omega \, |\hat{f}(\omega)|^2 = A^{2k} \|\hat{f}\|^2 < \infty$$

$$\Rightarrow \omega^k \hat{f}(\omega) \in L^2(\mathbb{R}), \; \forall k \geq 0$$

In particolare per k intero positivo, possiamo prendere l'antitrasformata e abbiamo:

$$\mathcal{F}^{-1}[\omega^k \hat{f}(\omega)](t) = i^k \frac{d^k}{dt^k} f(t)$$

L'(anti)trasformata di una funzione in $L^2(\mathbb{R})$ è ancora in $L^2(\mathbb{R})$. Quindi $f^{(K)}(t)$ è in $L^2(\mathbb{R})$, $\forall K > 0$.

II) $|(f^{(k)}, f^{(h)})|$

$$= \frac{1}{2\pi} |(\widehat{f^{(k)}}, \widehat{f^{(h)}})|$$

Parseval generalizzato

$$= \frac{1}{2\pi} |((-i\omega)^k \hat{f}, (-i\omega)^h \hat{f})|$$

$$= \frac{1}{2\pi} \left| \int_{-\infty}^{+\infty} d\omega \, (i\omega)^k (-i\omega)^h |\hat{f}(\omega)|^2 \right|$$

$$\leq \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \, |\omega|^{k+h} |\hat{f}(\omega)|^2$$

$$= \frac{1}{2\pi} \int_{-A}^{+A} d\omega \, |\omega|^{k+h} |\hat{f}(\omega)|^2$$

$\hat{f} \neq 0$ solo su $[-A, A]$

$$\leq A^{k+h} \frac{1}{2\pi} \int_{-A}^{+A} d\omega \, |\hat{f}(\omega)|^2$$

$$= A^{k+h} \frac{1}{2\pi} \|\hat{f}\|^2 = A^{k+h} \|f\| \quad ✓$$

Es 3

I) $\| D_{\lambda,p}[f] \|^2$

$$= \int_{-\infty}^{+\infty} dx\, \lambda^{2p} |f(\lambda x)|^2$$

$$\underset{\lambda x = y}{=} \int_{-\infty}^{+\infty} dy\, \frac{1}{\lambda}\, \lambda^{2p}\, |f(y)|^2$$

$$= \lambda^{2p-1} \| f \|^2$$

$$\Rightarrow \forall f \in L^2(\mathbb{R}),\ f \neq 0,\quad \frac{\| D_{\lambda,p}[f] \|}{\| f \|} = \lambda^{p-\frac{1}{2}}$$

$\Rightarrow$ $D_{\lambda,p}$ è limitato, $\| D_{\lambda,p} \| = \lambda^{p-\frac{1}{2}}$.

$$(g, D_{\lambda,p}[f]) = \int_{-\infty}^{+\infty} dx\, g(x)^* \lambda^p f(\lambda x)$$

$$\underset{y = \lambda x}{=} \int_{-\infty}^{+\infty} dy\, \frac{1}{\lambda} \lambda^p\, g(y/\lambda)^* f(y)$$

$$= \int_{-\infty}^{+\infty} dy\, \lambda^{p-1} g\left(\frac{y}{\lambda}\right)^* f(y) = \left((D_{\lambda,p})^\dagger [g], f \right)$$

Da cui: $(D_{\lambda,p})^\dagger [g](x) = \lambda^{p-1} g\left(\frac{x}{\lambda}\right)$

$$= \left(\frac{1}{\lambda}\right)^{1-p} g\left(\frac{1}{\lambda} x\right) = D_{1/\lambda,\, 1-p}[g](x)$$

Ovvero: $(D_{\lambda,p})^\dagger = D_{\frac{1}{\lambda},1-p}$

Per richiedere che sia unitario, richiediamo che $(D_{\lambda,p})^\dagger D_{\lambda,p} = \mathbb{1}$. Applichiamo a f:

$$(D_{\lambda,p})^\dagger[D_{\lambda,p}[f]](x)$$
$$= D_{\frac{1}{\lambda},1-p}[D_{\lambda,p}[f]](x)$$
$$= \left(\frac{1}{\lambda}\right)^{1-p} D_{\lambda,p}[f]\left(\frac{x}{\lambda}\right)$$
$$= \left(\frac{1}{\lambda}\right)^{1-p} \lambda^p f\left(\cancel{\lambda}\cdot\frac{x}{\cancel{\lambda}}\right) = \lambda^{2p-1} f(x)$$

Affinché sia $= f(x)$ serve che: $2p-1=0$

$\Longrightarrow$ $p_* = \frac{1}{2}$.

II) $D_{\lambda,p}[\mathcal{F}[f]](k)$
$$= \lambda^p \mathcal{F}[f](\lambda k)$$
$$= \lambda^p \int_{-\infty}^{+\infty} dx\, f(x) e^{i\lambda k x}$$
$$\underset{(\lambda x = y)}{=} \lambda^p \int_{-\infty}^{+\infty} dy \frac{1}{\lambda} f\left(\frac{y}{\lambda}\right) e^{iky}$$

$$= \int_{-\infty}^{+\infty} dy \left(\frac{1}{\lambda}\right)^{1-p} f\left(\frac{y}{\lambda}\right) e^{iky}$$

$$= \int_{-\infty}^{+\infty} dy \; D_{\frac{1}{\lambda}, 1-p}[f](y) \, e^{iky}$$

$$= \mathcal{F}\left[D_{\frac{1}{\lambda}, 1-p}[f]\right](k) \qquad , \forall f \in L^2(\mathbb{R})$$

$$\Rightarrow \quad D_{\lambda, p} \, \mathcal{F} = \mathcal{F} \, D_{\frac{1}{\lambda}, 1-p} \quad ✓$$