AG3 Exercises - first assignement

October 10, 2022

1. Prove that the map $\phi : \mathbb{A}^1 \to \mathbb{A}^3$ defined by $t \to (t, t^2, t^3)$ is a homeomorphism between $\mathbb{A}^1$ and its image, for the Zariski topology.

2. Let $X \subset \mathbb{A}^n$ be an affine Zariski closed subset, and let $p \in \mathbb{A}^n \setminus X$.

   Show that there exists $f \in K[x_1, \ldots, x_n]$ such that
   $$f(p) = 1, \quad f_{|X} = 0.$$

3. Show that polynomials $f \in K[x_1, \ldots, x_n]$ correspond to continuous maps $f : \mathbb{A}^n \to \mathbb{A}^1$ for the Zariski topology.

4. Let $\alpha = \langle x^2 + y^2 - 1, x - 1 \rangle$. Determine $X = V(\alpha)$ and $I(X)$. Show that $I(V(\alpha)) \neq \alpha$.

5. Let $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$. Let $\varphi : \mathbb{A}^n \to \mathbb{A}^m$ be defined by
   $$\varphi(p) = (f_1(p), \ldots, f_m(p)).$$
   Show that the graph
   $$\Gamma_\varphi := \{(p, q) \in \mathbb{A}^{n+m} \mid q = \varphi(p)\}$$
   is a Zariski closed subset.

6. Let $f : X \to Y$ be a continuous map between topological spaces. Prove that if X is irreducible, then $f(X)$ is irreducible.

7. Let $X \subseteq Y \subseteq \mathbb{A}^n$ two Zariski closed subsets. Prove that every irreducible component of X is contained in an irreducible component of Y.

8. Give an example of two affine irreducible closed subsets, which intersection is reducible.

9. Let X be a topological space and let $\{U_\alpha\}_{\alpha \in A}$ be an open covering such that
   $$U\alpha \cap U_\beta \neq \emptyset, \qquad \forall \alpha, \beta \in A,$$
   and such that U_α is irreducible for any $\alpha \in A$. Prove that X is irreducible.