12 ottobre

Teor (Weierstrass approx. th)

$\mathbb{R}[x]$ is dense in $C^0([0,1],\mathbb{R})$

$$d(f,g) = \sup_{x\in[0,1]} |f(x)-g(x)|$$

Pf

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \quad *$$

$y = 1-x$

$$1 = \sum_{k=0}^{n} r_k(x)$$

$$r_k(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

$$x\,\partial_x (x+y)^n = n\,x\,(x+y)^{n-1} =$$

$$= \sum_{k=0}^{n} \binom{n}{k} k\, x^k y^{n-k}$$

$y = 1-x$

$$n\,x = \sum_{k=0}^{n} k\, r_k(x)$$

$$x^2 \partial_x^2 (\cancel{x}+y)^n = x^2 \partial_x \left[n (x+y)^{n-1} \right]$$

$$= x^2 \cancel{\{} n(n-1) (x+y)^{n-2} =$$

$$= x^2 \partial_x^2 \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} =$$

$$= \sum_{k=0}^{n} \binom{n}{k} k(k-1) x^k y^{n-k}$$

$y = 1 - x$

$$\begin{cases} n(n-1)x^2 = \sum_{k=0}^{n} k(k-1) \, r_k(x) \\ n x = \sum_{k=0}^{n} k \, r_k(x) \\ 1 = \sum_{k=0}^{n} r_k(x) \end{cases}$$

$$\sum_{k=0}^{n} (k - n x)^2 r_k(x) = n x (1 - x)$$

$$n^2 x^2 \sum_{k=0}^{n} r_k(x) - 2 n x \sum_{k=0}^{n} k \, r_k(x)$$

$$+ \sum_{k=0}^{n} k^2 \, r_k(x) =$$

$$= n^2 x^2 - 2 n^2 x^2 + \sum_{k=0}^{n} k(k-1)\, r_k(x) + \sum_{k=0}^{n} k\, r_k(x)$$

$$= - n^2 x^2 + n(n-1)\, x^2 + n x$$

$$= n x (1-x)$$

$f \in C^0([0,1])$ is uniformly continuous

$\forall\, \varepsilon > 0 \quad \exists\, \delta > 0$ s.t.

$\forall\, I \subseteq [0,1]$ with $|I| < \delta$

$$\Rightarrow \quad \operatorname{osc}_I f \doteq \underbrace{\sup f(I) - \inf f(I)}_{\geq 0} < \varepsilon$$

$$\left| f(x) - \sum_{k=0}^{n} f\left(\tfrac{k}{n}\right) r_k(x) \right|$$

$$\leq \sum_{k=0}^{n} \left| \left(f(x) - f\left(\tfrac{k}{n}\right) \right) \right| r_k(x)$$

$$\leq \sum_{|x-\frac{k}{n}|<\delta} \left| f(x) - f\left(\frac{k}{n}\right) \right| r_k(x)$$

$$+ \sum_{|x-\frac{k}{n}|\geq\delta} \left| f(x) - f\left(\frac{k}{n}\right) \right| r_k(x)$$

$$=: \mathrm{I} + \mathrm{II}$$

$$\mathrm{I} < \varepsilon \sum_{k=0}^{n} r_k(x) = \varepsilon$$

$$\mathrm{II} = \sum_{|x-\frac{k}{n}|\geq\delta} \left| f(x) - f\left(\frac{k}{n}\right) \right| r_k(x)$$

$$|f(x)| \leq M \qquad \forall x \in [0,1].$$

$$\Rightarrow \left| f(x) - f\left(\frac{k}{n}\right) \right| \leq 2M$$

$$\mathrm{II} \leq \frac{2M}{\delta} \sum_{|x-\frac{k}{n}|\geq\delta} r_k(x) \left(x - \frac{k}{n}\right)^2$$

$$\sum_{|x-\frac{k}{n}|\geq\delta} \left(x-\frac{k}{n}\right)^2 r_k(x) \leq$$

$$\leq \frac{1}{n^2}\sum_{k=0}^{n} (nx-k)^2\, r_k(x)$$

$$= \frac{1}{n^2}\, n\, x\,(1-x) = \frac{1}{n}\, x(1-x)$$

$$\leq \frac{1}{4}\,\frac{1}{n} \xrightarrow[n\to+\infty]{} 0$$

$$\mathrm{II} \leq \frac{2M}{\delta}\,\frac{1}{4}\,\frac{1}{n} < \varepsilon$$

for $n \gg 1$

So for $n \gg 1$

$$\left| f(x) - \sum_{k=0}^{n} f\left(\frac{k}{n}\right) r_k(x) \right| < 2\varepsilon$$

$$\forall x \in [0,1]$$

__Remark__

$$X = \begin{cases} 1 & \text{probability } x \\ 0 & 1-x \end{cases}$$

$$E[X] = x$$

$$\mathrm{Var}[X] = \underline{x\,(1-x)}$$

X

$$E[X] = m$$

$$\mathrm{Var}[X] = \sigma^2$$

$$S_n = \frac{X_1 + \cdots + X_n}{n}$$

$$P\left[\,|S_n - m| \geq \delta\,\right] \leq \frac{\sigma^2}{n\,\delta^2}$$

$$\sum_{|m - \frac{k}{n}| \geq \delta} r_k(x) \leq \frac{x\,(1-x)}{n\,\delta^2}$$

Theorem (Ascoli Arzelà) X compact metric space $S \subseteq C^0(X, \mathbb{R})$

$f, g : X \longrightarrow \mathbb{R}$

$\sup_{x \in X} |f(x) - g(x)|$

Then $\overline{S}$ is compact $\Longleftrightarrow$ iff the following are true

1) S is bounded

2) S is equicontinuous

$\Longleftarrow$

S is equicontinuous if

$\forall \varepsilon > 0 \quad \forall x_0 \in X \quad \exists$

$\delta = \delta(x_0, \varepsilon) > 0$

s.t. $\quad \underset{X}{\mathrm{dist}}(x, x_0) < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

$\forall f \in S.$

Proof $\Leftarrow$

In X there exists a sequence $\{x_n\}$ which is dense in X and in particular is such that

$\forall \varepsilon > 0 \quad \exists \quad n(\varepsilon) \quad$ st.

$\forall \quad x \in X \qquad \exists \quad k \leq n(\varepsilon)$

with $\quad d_X(x, x_k) < \varepsilon$

$\varepsilon_n \searrow 0$

$$\varepsilon_n \qquad D_X(x_{n1}, \varepsilon_n) \cup \dots \cup D_X(x_{nm_n}, \varepsilon_n)$$

$$= X$$

$x_{11} \dots x_{1m_1}, x_{21} \dots x_{2m_2}, \dots$

$x_1, x_2, \ldots$ X

$\{f_n\}$ in S

$f_{1n}(x_1) = f_n(x_1)$

$f_{11}(x_1), f_{12}(x_1), \ldots$

$f_{21}(x_2)$ $f_{22}(x_2)$ $\ldots$

$\vdots$ $\vdots$

$f_{n1}(x_n)$ $\ldots$

$f_{nn}(x)$

$$\lim_{n \to +\infty} f_n(x_j) = f(x_j)$$

$\{f_n(x)\}$ converges $\forall\, x \in X$

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_n(x_j)| + |f_m(x_j) - f_m(x)| + |f_n(x_j) - f_m(x_j)|$$

$\varepsilon > 0$ $\quad \delta = \delta(x, \varepsilon)$ and consider

the $j < n(\delta)$ s.t. $d_X(x, x_j) < \delta$

$$|f_n(x) - f_m(x)| \leq 2\varepsilon + \underbrace{|f_n(x_j) - f_m(x_j)|}_{<\varepsilon}$$

$\exists$ N st for $n, m > N$

we get

$$|f_n(x) - f_m(x)| \leq 3\varepsilon$$

$$\Longrightarrow \lim_{n \to +\infty} f_n(x) =: f(x)$$

We need to show now that $f_n \longrightarrow f$ in $C^0(X, \mathbb{R})$

Top vector spaces on the field

$K = \mathbb{R}, \mathbb{C}$

<u>Def</u> A vector space X on K with a topology τ is a topological vector space if

$$X \times X \longrightarrow X$$
$$(x, y) \longrightarrow x + y \quad \text{is continuous}$$

and

$$K \times X \longrightarrow X$$
$$(\lambda, x) \longrightarrow \lambda x \quad \text{is continuous}$$

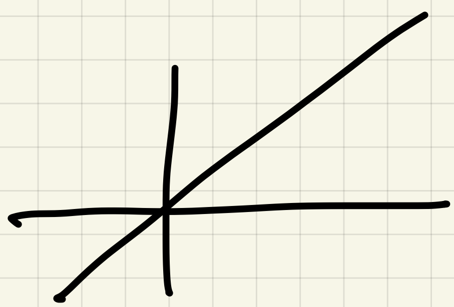
We ask also that X be Hausdorff.

Observation A set $U \subseteq X$ is a neighborhood of $x_0 \in X$ iff $\exists$ a neighborhood V of 0 s.t.

$$U = x_0 + V$$

Def A subset Ω of X is

1) balanced if $\forall x \in \Omega$ and for any $\lambda \in K$ with $|\lambda| \leq 1$ we have $\lambda x \in \Omega$
2) absorbing if $\forall x \in X$ $\exists \lambda$ s.t. $\lambda \Omega \ni x$



Lemma $\forall$ neigh. U of 0 $\exists$ a neigh. V of 0 with $V \subseteq U$ and V balanced

Pf given U, by continuity of

$$K \times X \longrightarrow X$$

$$D_K(0,\delta) \times \tilde{V} \qquad U \ni 0$$

$$D_K(0,\delta)\tilde{V} \subseteq U$$

$$|\lambda| \leq \delta \quad \text{and} \quad \forall x \in \tilde{V}$$

$$\lambda x \in U$$

$$V = \bigcup_{0<|\lambda|\leq \delta} \lambda \tilde{V} \subseteq U$$

V is the correct set $0<|\mu| \leq 1$

$$\mu V \subseteq \bigcup_{0<|\lambda|\leq \delta} \mu \lambda \tilde{V} \subseteq \bigcup_{0<|\lambda|\leq \delta} \lambda \tilde{V} = V$$

<u>Lemma</u> If in the defintion of t.v.s. we replace X Housdorff with $\{0\}$ closed then we obtain an equivalent defintion.

<u>Proof</u> With the new defintion, to show that X is Housdorff it ~~is~~ is enough to consider

$$0 \quad \text{and} \quad x \neq 0$$

Since $\{x\}$ is closed $\exists$ U neigh. of

0 s.t. $U \not\ni x$.

$\exists$ V neigh of 0 ~~absorbing~~ balanced and

$$V - V \subseteq V + V \subseteq U \not\ni x$$

$$V \cap (x + V) = \phi$$

$v \in V$ $\qquad v = x + v_1 \qquad v_1 \in V$

$x = v - v_1$

$\Longrightarrow$ $x \in V - V$ not true