12 ottobre $z = x + iy$

Utilizzando $z = x + iy$, esprimere

$z^n = (x+iy)^n$ in coordinate cartesiane

$$z^n = (x+iy)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \, i^{n-k} \, y^{n-k} =$$

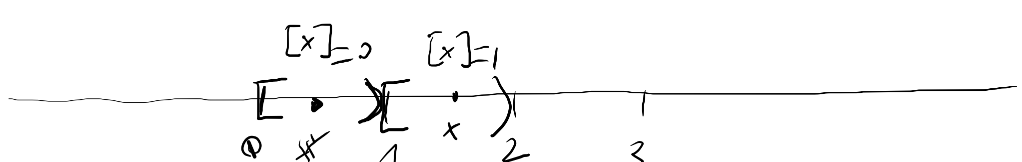
$$= \sum_{k=0}^{n} \binom{n}{k} i^k y^k x^{n-k} =$$

$$= \sum_{\substack{k=0 \\ k \text{ pari}}}^{n} \binom{n}{k} i^k y^k x^{n-k} + \sum_{\substack{k=0 \\ k \text{ dispari}}}^{n} \binom{n}{k} i^k y^k x^{n-k}$$

$$= \sum_{0 \le j \le \left[\frac{n}{2}\right]} \binom{n}{2j} i^{2j} y^{2j} x^{n-2j} + \sum_{\substack{k=0 \\ k \text{ dispari}}}^{n} \binom{n}{k} i^k y^k x^{n-k}$$

$k = 2j$ $\quad 0 \le k \le n \quad 0 \le 2j \le n \quad 0 \le j \le \frac{n}{2}$

$[x] : \mathbb{R} \to \mathbb{Z}$ $\qquad [x] \le x < [x] + 1$



$$= \sum_{j=0}^{\left[\frac{n}{2}\right]} \binom{n}{2j} i^{2j} y^{2j} x^{n-2j} + \sum_{\substack{k=1 \\ k \text{ dyou}}}^{n} \binom{n}{k} i^{k} y^{k} x^{n-k}$$

$$k = 2j+1 \qquad 1 \le k \le n$$

$$= \sum_{j=0}^{\left[\frac{n}{2}\right]} \binom{n}{2j} (i^2)^j y^{2j} x^{n-2j} + \sum_{j=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2j+1} i \, (i^2)^j y^{2j+1} x^{n-2j-1}$$

$$1 \le 2j+1 \le n \qquad 0 \le 2j \le n-1$$

$$0 \le j \le \left[\frac{n-1}{2}\right]$$

$$= \underbrace{\sum_{j=0}^{\left[\frac{n}{2}\right]} \binom{n}{2j} (-1)^j y^{2j} x^{n-2j}}_{\operatorname{Re} z^n} + i \underbrace{\sum_{j=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2j+1} (-1)^j y^{2j+1} x^{n-2j-1}}_{\operatorname{Im} z^n}$$

$$= z^n$$

Radici n-esime di un numero complesso w_0

Sono le soluzioni di $z^n = w_0$

$$w_0 = \rho_0 \left(\cos(\alpha_0) + i \sin(\alpha_0) \right) \qquad \rho_0 \geq 0$$

$$z = r\left(\cos(\vartheta) + i \sin(\vartheta) \right)$$

$$\cancel{r^n}\left(\cos(n\vartheta) + i \sin(n\vartheta) \right) = \cancel{\rho_0}\left(\cos(\alpha_0) + i \sin(\alpha_0) \right)$$

$$r^n = \rho_0 \qquad r = \sqrt[n]{\rho_0} = \rho_0^{\frac{1}{n}}$$

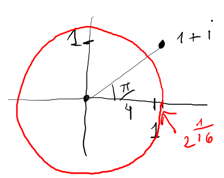
$$\cos(n\vartheta) + i \sin(n\vartheta) = \cos(\alpha_0) + i \sin(\alpha_0)$$

$$\begin{cases} \cos(n\vartheta) = \cos(\alpha_0) \\ \sin(n\vartheta) = \sin(\alpha_0) \end{cases} \iff n\vartheta = \alpha_0 + 2\pi k \qquad k \in \mathbb{Z}$$

$$\vartheta = \frac{\alpha_0}{n} + \frac{2\pi}{n} k \qquad k \in \mathbb{Z}$$

Per $k = 0, \dots, n-1$ si ottengono numeri complessi distinti

Esempio $\quad (1+i)^{\frac{1}{8}} \qquad z^8 = 1+i = \sqrt{2}\,\frac{1+i}{\sqrt{2}}$



$$|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$$

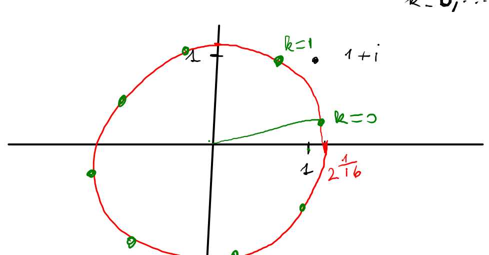
$$z^8 = \sqrt{2}\left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4} \right) \qquad z = r(\cos\vartheta + i \sin\vartheta)$$

$$r^8\left(\cos(8\vartheta) + i \sin(8\vartheta) \right) = 2^{\frac{1}{2}}\left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4} \right)$$

$$r^8 = 2^{\frac{1}{2}} \qquad r = 2^{\frac{1}{16}} \qquad 1 < 2^{1/16} < 2^{\frac{1}{2}}$$

$$\begin{cases} \cos(8\vartheta) = \cos(\frac{\pi}{4}) \\ \sin(8\vartheta) = \sin(\frac{\pi}{4}) \end{cases} \iff \vartheta = \frac{\pi}{32} + \frac{2\pi}{8} k \qquad k \in \mathbb{Z}$$

$k = 0, \dots, 7$



$z^6 - |z|^4 + |z|^2 = 1$

$z = r(\cos\vartheta + i\sin\vartheta)$ $\qquad |z| = r$

$r^6\left(\cos(6\vartheta) + i\sin(6\vartheta)\right) - r^4 + r^2 = 1$

$r^6\cos(6\vartheta) + i\,r^6\sin(6\vartheta) - r^4 + r^2 = 1$

$$\begin{cases} r^6\cos(6\vartheta) - r^4 + r^2 = 1 \\ r^6\sin(6\vartheta) = 0 \end{cases}$$

$r^6\sin(6\vartheta) = 0$

- $r = 0$ non dà soluzioni del sistema
- $\sin(6\vartheta) = 0$

$\sin(6\vartheta) = 0$

- $\cos(6\vartheta) = -1$
- $\cos(6\vartheta) = 1$

$\cos(6\vartheta) = -1$ $\qquad r^6\cos(6\vartheta) - r^4 + r^2 = 1$

$-r^6 - r^4 + r^2 = 1 \iff r^2 = 1 + r^4 + r^6$

non ha soluzioni $r > 0$.

Se $r > 1$ $\quad r^2 < r^6 \implies r^2 < 1 + r^4 + r^6$

$0 < r < 1$ $\quad r^2 < 1 \implies r^2 < 1 + r^4 + r^6$

$r = 1$ $\quad r^2 = 1$ $\quad 1 + r^4 + r^6 = 3$

Quindi, in tutti i casi abbiamo $r^2 < 1 + r^4 + r^6$ $\quad \forall\, r \geq 0.$

Conclusione. Per $\cos 6\vartheta = -1$ non ci sono soluzioni

$$r^6 \cos(6\vartheta) - r^4 + r^2 = 1$$

$$\cos(6\vartheta) = 1$$

$$r^6 - r^4 + r^2 - 1 = 0$$

$$r^4(r^2-1) + (r^2-1) = (r^2-1)(r^4+1) = 0$$

$$\iff \quad r^2 - 1 = 0 \iff \boxed{r = 1}$$

$$\begin{cases} \cos(6\vartheta) = 1 \\ \sin(6\vartheta) = 0 \end{cases} \qquad r = 1$$

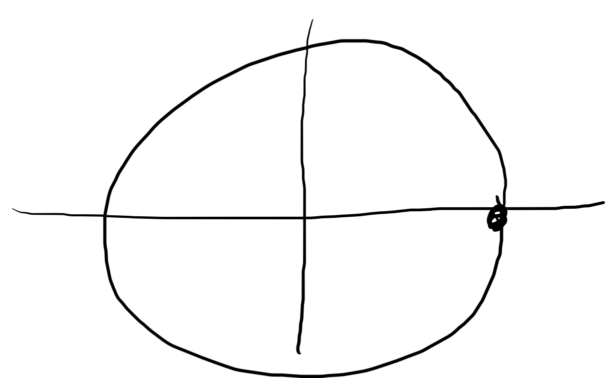
$$\Updownarrow$$

$$6\vartheta = 2\pi k \qquad k \in \mathbb{Z}$$

$$\vartheta = \frac{2\pi k}{6} = \frac{\pi k}{3} \qquad k = 0, \dots, 5$$

$$r = 1$$

$$z^6 - |z|^4 + |z|^2 = 1$$



$z^4 + 2|z|^2 = 1$ $\qquad z = r(\cos\vartheta + i\sin\vartheta)$

$r^4\cos(4\vartheta) + i\, r^4 \sin(4\vartheta) + 2r^2 = 1$

$$\begin{cases} r^4\cos(4\vartheta) + 2r^2 = 1 \\ r^4\sin(4\vartheta) = 0 \end{cases}$$

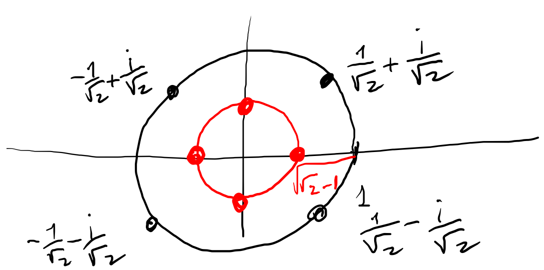
$\sin(4\vartheta) = 0 \Rightarrow \cos(4\vartheta) \begin{cases} -1 \\ 1 \end{cases}$

$\cos(4\vartheta) = -1$

$-r^4 + 2r^2 - 1 = 0 \iff r^4 - 2r^2 + 1 = 0 = (r^2-1)^2$

$(r^2-1)^2 = 0 \iff r^2 - 1 = 0 \iff r^2 = 1 \iff r = 1$

$\begin{cases} \sin(4\vartheta) = 0 \\ \cos(4\vartheta) = -1 \end{cases}$ $\qquad 4\vartheta = \pi + 2\pi k$ $\qquad \vartheta = \frac{\pi}{4} + \frac{2\pi k}{4}, \quad k = 0,1,2,3$



$z = \pm\sqrt{\sqrt2 - 1}\ ; \ i\sqrt{\sqrt2 - 1}$

$\cos(4\vartheta) = 1$ $\qquad r^4\cos(4\vartheta) + 2r^2 = 1$

$r^4 + 2r^2 - 1 = 0$ $\qquad u = r^2$

$u^2 + 2u - 1 = 0$ $\qquad u_\pm = -1 \pm \sqrt{2} \begin{cases} \sqrt2 - 1 \\ \cancel{-\sqrt2 - 1} \end{cases}$

$u = r^2$

$r^2 = \sqrt2 - 1$ $\qquad r = \sqrt{\sqrt2 - 1}$

$\begin{cases} \cos(4\vartheta) = 1 \\ \sin(4\vartheta) = 0 \end{cases}$ $\qquad 4\vartheta = 2\pi k$ $\qquad \vartheta = \frac{2\pi}{4} k$ $\qquad k = 0,1,2,3$

$\mathbb{N} = \{1, 2, 3, 4, \dots\}$

$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$

$\mathbb{Q} = \{ \frac{a}{b} : a \in \mathbb{Z}, \ b \in \mathbb{Z} \setminus \{0\} \}$

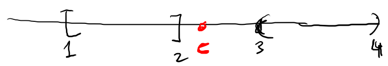
$\mathbb{R}$

<u>Def</u> Una coppia ordinata A, B di sottoinsiemi di $\mathbb{R}$ è separata se $a \leq b$ $\forall a \in A$ e $\forall b \in B$.

Es $[1, 2]$ $(3, 4)$

$[1,2] = \{x \in \mathbb{R} : 1 \leq x \leq 2\}$

$(3,4) = \{x \in \mathbb{R} : 3 < x < 4\}$



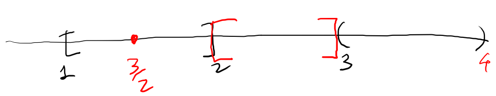
$[1,2]$ e $(3,4)$ sono una coppia separata.

Assioma di Separazione (di Dedekind) Data una coppia separata A e B in $\mathbb{R}$ $\exists\, c \in \mathbb{R}$ t.c. $a \leq c \leq b$ $\forall a \in A$ e $\forall b \in B$.

c è un elemento di separazione della coppia

Ad esempio, un qualsiasi $c \in [2, 3]$ separa $A = [1,2]$ e $B = (3,4)$

$a \leq c \leq b$ $\forall a \in [1,2]$ e $b \in B = (3,4)$



$2 \geq a$ $\forall a \in [1,2]$

$2 < b$ $\forall b \in (3,4)$

$\frac{3}{2}$ non è un elemento di separazione

<u>Definizione</u> Due classi separate A e B sono contigue quando ammettono in $\mathbb{R}$ un unico elemento di separazione

<u>Esercizio</u> $A = \{x : 0 < x, \ x^2 < 2\}$

$B = \{x : 0 < x, \ x^2 > 2\}$

Dimostrare che sono separati e contigui

$\mathbb{R}_+ = \{ x : x > 0 \} = (0, +\infty)$

$\mathbb{R}_- = \{ x : x < 0 \} = (-\infty, 0)$

Per retta reale estesa $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$

$+\infty \geq x \qquad \forall x \in \overline{\mathbb{R}}$

$-\infty \leq x \qquad \forall x \in \overline{\mathbb{R}}$

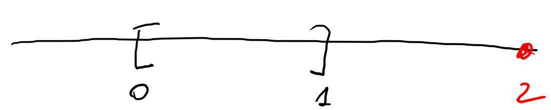
$[0, +\infty) = \overline{\mathbb{R}_+}$

Teorema: Per ogni sottoinsieme non vuoto X di $\overline{\mathbb{R}}$ esiste, ed è unico, un elemento di $\overline{\mathbb{R}}$, che denotiamo con $\sup X$ t.c.

1) $\sup X \geq x \quad \forall x \in X$

2) $M \geq x \quad \forall x \in X \implies M \geq \sup X$

$\sup [0,1]$



$= \sup \{ x : 0 \leq x \leq 1 \} = 1$

Infatti $1 \geq x \quad \forall \; 0 \leq x \leq 1$ (1)

e se $M \geq x \quad \forall \; 0 \leq x \leq 1 \implies M \geq 1$ (2)

$\sup (0,1) = 1$