Systems Dynamics

Course ID: 267MI – Fall 2022

Thomas Parisini
Gianfranco Fenu

University of Trieste
Department of Engineering and Architecture

267MI –Fall 2022

Lecture 3
Stability of Discrete-Time Dynamic Systems

Lecture 3: Table of Contents

3. Stability of Discrete-Time Dynamic Systems

Stability of Discrete-Time Dynamic Systems

When dealing with stability in the context of dynamic systems we consider three different cases (listed in order of decreasing generality):

1. Stability of state movements
2. Stability of equilibrium states
3. Stability of linear systems

Remark. Concerning case 1., we provide definitions and concepts in the context of general abstract dynamic systems so, for example, time-instants belong to any legitimate set of times T.

Stability of State Movements

Stability of State Movements

- Consider a general abstract dynamic system characterised by the state-transition function $\varphi(t, t_0, x_0, u(\cdot))$
- Then, consider a generic **nominal state movement** for a given initial state $\bar{x}_0$ and a given input function $u(\cdot)$:

$$\bar{x}(\cdot) = \varphi(t, t_0, \bar{x}_0, \bar{u}(\cdot))$$

- Now, consider the **perturbed state movement** generated by a **perturbation of the initial state** and a **perturbation of the input function**:

$$\begin{aligned} x(0) &= \bar{x}_0 + \delta\bar{x} \\ u(\cdot) &= \bar{u}(\cdot) + \delta u(\cdot) \end{aligned} \quad \Longrightarrow \quad \varphi(t, t_0, \bar{x}_0 + \delta\bar{x}, \bar{u}(\cdot) + \delta u(\cdot))$$

Perturbed State Movement

Stability with Respect to Perturbations of the Initial State

The **nominal state movement**

$$\bar{x}(\cdot) = \varphi(t, t_0, \bar{x}_0, \bar{u}(\cdot))$$

is **stable** with respect to perturbations of the initial state $\bar{x}_0$ if

$$\forall\, \varepsilon > 0\,,\, \forall t_0 > 0\; \exists\, \delta(\varepsilon, t_0) > 0 \;\text{ such that if }\; \|\delta\bar{x}\| < \delta(\varepsilon, t_0)$$

then, it follows that

$$\|\varphi(t, t_0, \bar{x}_0 + \delta\bar{x}, \bar{u}(\cdot)) - \varphi(t, t_0, \bar{x}_0, \bar{u}(\cdot))\| < \varepsilon\,, \forall\, t \geq t_0$$

Asymptotic Stability with Respect to Perturbations of the Initial State

The **nominal state movement** $\bar{x}(\cdot) = \varphi(t, t_0, \bar{x}_0, \bar{u}(\cdot))$ is **asymptotically stable** with respect to perturbations of the initial state $\bar{x}_0$ if:

- it is **stable**, that is, if

  $$\forall\, \varepsilon > 0\,,\, \forall t_0 > 0\; \exists\, \delta(\varepsilon, t_0) > 0 \;\text{ such that if }\; \|\delta\bar{x}\| < \delta(\varepsilon, t_0)$$

  then, it follows that

  $$\|\varphi(t, t_0, \bar{x}_0 + \delta\bar{x}, \bar{u}(\cdot)) - \varphi(t, t_0, \bar{x}_0, \bar{u}(\cdot))\| < \varepsilon\,, \forall\, t \geq t_0$$

- it is **attractive**, that is, $\forall t_0 > 0\; \exists\; \eta(t_0) > 0$ such that

  $$\lim_{t \to +\infty} \|\varphi(t, t_0, \bar{x}_0 + \delta\bar{x}, \bar{u}(\cdot)) - \varphi(t, t_0, \bar{x}_0, \bar{u}(\cdot))\| = 0\,,\; \forall\, \|\delta\bar{x}\| < \eta(t_0)$$

Unstability with Respect to Perturbations of the Initial State

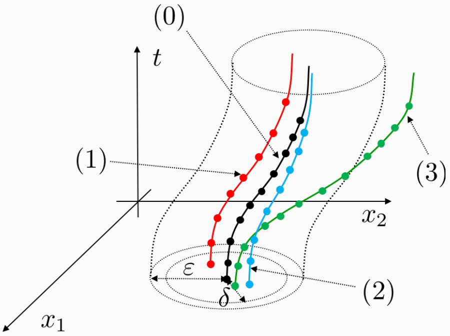
The **nominal state movement**

$$\bar{x}(\cdot) = \varphi(t, t_0, \bar{x}_0, \bar{u}(\cdot))$$

is **unstable** with respect to perturbations of the initial state $\bar{x}_0$ if it is not stable with respect to such a kind of perturbations.

Geometrical Interpretation

- **(0)**: nominal state movement
- **(1)**: perturbed state movement remaining confined in the ”tube” of radius ε
- **(2)**: perturbed state movement remaining confined in the ”tube” of radius ε and asymptotically converging to the nominal movement
- **(3)**: perturbed state movement crossing the ”tube” of radius ε



Stability with Respect to Perturbations of the Input Function

The **nominal state movement**

$$\bar{x}(\cdot) = \varphi(t, t_0, \bar{x}_0, \bar{u}(\cdot))$$

is **stable** with respect to perturbations of the input function $\bar{u}(\cdot)$ if

$$\forall\, \varepsilon > 0\,,\, \forall t_0 > 0\; \exists\, \delta(\varepsilon, t_0) > 0 \;\; \text{such that if} \;\; \forall \|\delta\bar{u}(\cdot)\| < \delta(\varepsilon, t_0)$$

then, it follows that

$$\|\varphi(t, t_0, \bar{x}_0, \bar{u}(\cdot) + \delta\bar{u}(\cdot)) - \varphi(t, t_0, \bar{x}_0, \bar{u}(\cdot))\| < \varepsilon\,, \forall\, t \geq t_0$$

Unstability with Respect to Perturbations of the Input Function

The **nominal state movement**

$$\bar{x}(\cdot) = \varphi(t, t_0, \bar{x}_0, \bar{u}(\cdot))$$

is **unstable** with respect to perturbations of the input function $\bar{u}(\cdot)$ if it is not stable with respect to such a kind of perturbations.

Stability of Equilibrium States

Stability of Equilibrium States

- Consider the discrete-time dynamic system

$$\begin{cases} x(k+1) = f(x(k), u(k)) \\ y(k) = g(x(k), u(k)) \end{cases}$$

and the equilibrium state $\bar{x}$ corresponding to a constant input sequence $u(k) = \bar{u}, \forall\, k \geq 0$, that is:

$$\bar{x} = f(\bar{x}, \bar{u})$$

- Now, consider a perturbation of the initial state with respect to the equilibrium state $\bar{x}$:

$$\begin{matrix} x(0) = \bar{x} + \delta\bar{x} \\ u(k) = \bar{u},\ k \geq 0 \end{matrix} \quad \Longrightarrow \quad \begin{matrix} x(k) \neq \bar{x},\ k \geq 0 \\ \text{perturbed state movement} \end{matrix}$$

Stability of Equilibrium States (cont.)

The **equilibrium state** is **asymptotically stable** if:

- It is **stable**, that is:

$$\forall\, \varepsilon > 0 \;\; \exists\, \delta(\varepsilon) > 0 \;\; \text{such that :} \\ \forall\, x(0) : \; \|\delta\bar{x}\| < \delta(\varepsilon) \quad \Longrightarrow \quad \|x(k) - \bar{x}\| < \varepsilon, \; \forall\, k \geq 0$$

- It is **attractive**, that is:

$$\lim_{k \to \infty} \|x(k) - \bar{x}\| = 0$$

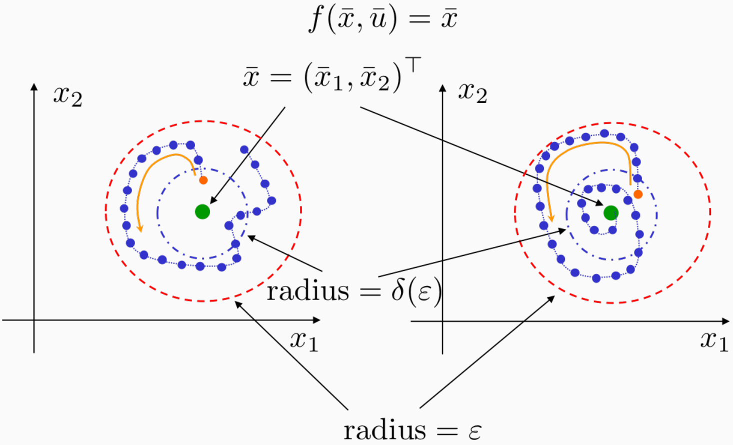
In **qualitative** terms:

when the initial state is perturbed, the state remains ”close” to the nominal equilibrium state and tends to return asymptotically to this equilibrium state.

Stability of Equilibrium States (cont.)

The **equilibrium state** is **unstable** if it is not stable.

Stability of Equilibrium States: Geometric Interpretation



Stability **Asymptotic Stability**

Stability of Equilibrium States

Stability of State Movements and of Equilibrium States

Stability of State Movements and of Equilibrium States

- Consider the general discrete-time dynamic system

$$x(k+1) = f(x(k), u(k), k)$$

and consider a **nominal** state movement

$$\bar{x}(k) = \varphi(k, k_0, \bar{x}_0, \{u(k_0, \ldots, u(k-1)\})$$

starting from the initial state $\bar{x}(k_0) = \bar{x}_0$.

- We analyse the stability of the nominal movement $\bar{x}(k)$ with respect to perturbations of the initial state $\bar{x}_0$, that is, we consider the perturbed state movement

$$x(k) = \varphi(k, k_0, x_0, \{u(k_0, \ldots, u(k-1)\})$$

starting from the perturbed initial state $x_0 \neq \bar{x}_0$.

- Hence, introducing the difference between the perturbed and the nominal state movement $z(k) := x(k) - \bar{x}(k)$, one gets:

$$z(k+1) = x(k+1) - \bar{x}(k+1) = f(z(k) + \bar{x}(k), \bar{u}(k), k) - f(\bar{x}(k), \bar{u}(k), k)$$

Stability of State Movements and of Equilibrium States (cont.)

- Letting:

$$w_{\bar{x},\bar{u}}(z(k),k) := f(z(k)+\bar{x}(k),\bar{u}(k),k) - f(\bar{x}(k),\bar{u}(k),k)$$

  it follows that the dynamics of $z(k)$ can be described by the autonomous (in general time-varying) system

$$z(k+1) = w_{\bar{x},\bar{u}}(z(k),k) \quad (\star)$$

  where the function $w_{\bar{x},\bar{u}}$ is parametrised by the nominal state movement $\{\bar{x}(k)\}$ and the nominal input $\{\bar{u}(k)\}$.
- The function $w_{\bar{x},\bar{u}}$ satisfies:

$$w_{\bar{x},\bar{u}}(0,k) = 0\,, \quad \forall k \geq k_0$$

  Hence, the **constant movement**

$$\tilde{z}(k) = 0, \quad \forall\, k \geq k_0$$

  is an **equilibrium state** of the system $(\star)$.

Stability of State Movements and of Equilibrium States (cont.)

State Movement Stability Analysis

The stability analysis of a generic nominal state movement can always be carried out by analysing the stability of the zero-state as an equilibrium state of a suitable autonomous system.

Therefore:

There is no loss of generality in dealing only with the stability analysis of equilibrium states

Stability of Linear Discrete-Time Systems

Stability of Linear Discrete-Time Systems

- Consider the general discrete-time linear dynamic system

$$x(k+1) = A(k)x(k) + B(k)u(k)$$

  and consider a **nominal** state movement

$$\bar{x}(k) = \varphi(k, k_0, \bar{x}_0, \{u(k_0, \ldots, u(k-1)\})$$

  starting from the initial state $\bar{x}(k_0) = \bar{x}_0$.
- We analyse the stability of the nominal movement $\bar{x}(k)$ with respect to perturbations of the initial state $\bar{x}_0$, that is, we consider the perturbed state movement

$$x(k) = \varphi(k, k_0, x_0, \{u(k_0, \ldots, u(k-1)\})$$

  starting from the perturbed initial state $x_0 \neq \bar{x}_0$.

Stability of Linear Discrete-Time Systems (cont.)

- Hence, introducing the difference between the perturbed and the nominal state movement $z(k) := x(k) - \bar{x}(k)$, one gets:

$$
\begin{aligned}
z(k+1) &= x(k+1) - \bar{x}(k+1) \\
&= A(k)[z(k) + \bar{x}(k)] + B(k)\bar{u}(k) - A(k)\bar{x}(k) - B(k)\bar{u}(k) \\
&= A(k)z(k)
\end{aligned}
$$

- It follows that the dynamics of $z(k)$ can be described by the autonomous (in general time-varying) system

$$z(k+1) = A(k)z(k) \quad (\star)$$

- Hence, the **constant movement**

$$\tilde{z}(k) = 0, \quad \forall\, k \geq k_0$$

is an **equilibrium state** of the system $(\star)$.

Stability of Linear Discrete-Time Systems (cont.)

Summing up:

For **linear systems** the dynamics of the difference between the perturbed and the nominal state movement $z(k) = x(k) - \bar{x}(k)$ satisfies:

$$z(k+1) = A(k)z(k)$$

and:

- The dynamics of $z(k)$ does not depend on the specific initial state $\bar{x}_0$ but on the magnitude of the initial state perturbation $z(k_0) = x(k) - \bar{x}_0$
- All state movements have the same stability properties or, in other terms, stability is not a property of a specific nominal state movement but, instead, is a **global property of the linear dynamic system**

Stability of Linear Discrete-Time Systems

Analysis of the Free State Movement

Stability of Linear Systems via Analysis of the Free State Movement

- Given the linear time-invariant discrete-time dynamic system

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases}$$

- In **equilibrium** conditions:

$$\begin{aligned} & x(0) = \bar{x} \\ & u(k) = \bar{u},\ k \geq 0 \\ & \quad \Longrightarrow x(k) = A^k\,\bar{x} + \sum_{i=0}^{k-1} A^{k-i-1}\,B\bar{u} = \bar{x},\ \forall k \geq 0 \end{aligned}$$

Stability of Linear Systems via Analysis of the Free State Movement (cont.)

- **Perturbing the equilibrium** conditions:

$$\begin{aligned} x(0) &= \bar{x} + \delta\bar{x} \\ u(k) &= \bar{u},\ k \geq 0 \end{aligned} \quad \Longrightarrow \quad \begin{array}{c} x(k) \neq \bar{x}, k \geq 0 \\ \text{perturbed state movement} \end{array}$$

$$\begin{aligned} \Longrightarrow x(k) &= A^k\ (\bar{x} + \delta\bar{x}) + \sum_{i=0}^{k-1} A^{k-i-1}\, B\bar{u} \\ &= \bar{x} + A^k\, \delta\bar{x} \end{aligned}$$

Hence:

$$\delta x(k) = A^k \delta\bar{x}$$

- Also, recall that:

$$x_l(k) = A^k\, x(0)$$

Stability of Linear Systems via Analysis of the Free State Movement (cont.)

Stability and A^k

- The stability properties do not depend on the specific value taken on by the equilibrium state $\bar{x}$
- Hence, the stability properties are a structural property of the linear dynamic system as a whole
- The stability properties depend on the time-behaviour of the $n \times n$ elements of the matrix A^k:
  - Stability $\iff$ all elements of A^k are bounded $\forall\, k \geq 0$
  - Asymptotic stability $\iff$ $\lim_{k \to \infty} A^k = 0$
  - Instability $\iff$ at least one element of A^k diverges

Stability of Linear Systems via Analysis of the Free State Movement (cont.)

Recall that the matrix A^k can be expressed as a sum of the so-called **response modes** (**Part 2**):

- Let $\lambda_1, \ldots, \lambda_\sigma$ the **distinct** eigenvalues of A and n_i the **algebraic multiplicity** of such eigenvalues (with $\sum_{i=1}^{\sigma} n_i = n$).
- If $\lambda_i \neq 0\,,\ i = 1\,, \ldots \sigma$ then

$$A^k = \sum_{i=1}^{\sigma} \left[A_{i0} \lambda_i^k 1(k) + \sum_{l=1}^{n_i - 1} A_{il} l! \begin{pmatrix} k \\ l \end{pmatrix} \lambda_i^{k-l} 1(k-l) \right]$$

- if $\lambda_j = 0\,,\ \lambda_j \in \{\lambda_1, \ldots, \lambda_\sigma\}$ then the corresponding response modes are

$$A_{j0} \cdot \delta(k) + \sum_{l=1}^{n_j - 1} A_{jl}\, l!\, \delta(k-l)$$

Stability of Linear Systems via Analysis of the Free State Movement (cont.)

- The matrices A_{il} can be determined as

$$A_{il} = \frac{1}{l!} \frac{1}{(n_i - 1 - l)!} \lim_{z \to \lambda_i} \left\{ \frac{d^{n_i - 1 - l}}{dz^{n_i - 1 - l}} \left[(z - \lambda_i)^{n_i} (zI - A)^{-1} \right] \right\}$$

where $l = 0,\ 1,\ 2,\ \ldots,\ n_i - 1$.

Stability of Linear Systems via Analysis of the Free State Movement (cont.)

Stability and A^k

Using the response modes

$$A^k = \sum_{i=1}^{\sigma} \sum_{l=0}^{n_i - 1} \left[A_{il} l! \begin{pmatrix} k \\ l \end{pmatrix} \lambda_i^{k-l} 1(k-l) \right]$$

For the stability analysis, the **boundedness of the free-state movement** has to be ascertained. Since the matrices A_{jl} does not depend on k, it suffices to **analyse the boundedness of the terms**

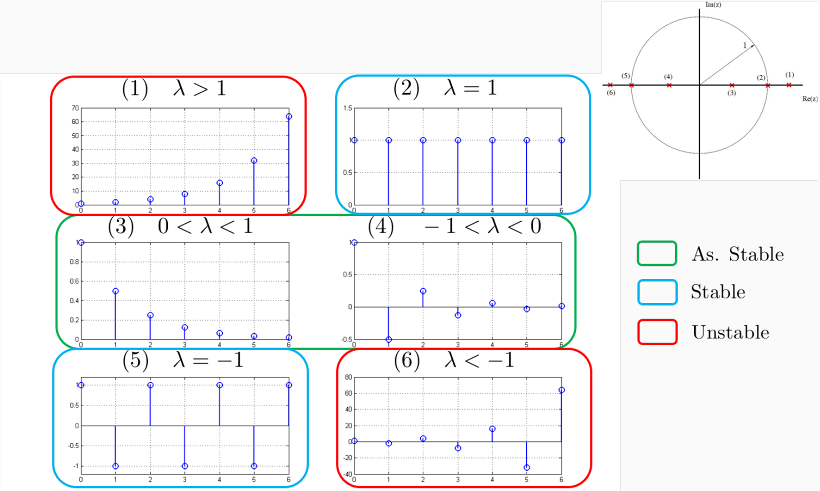
$$\begin{pmatrix} k \\ l \end{pmatrix} \lambda_i^{k-l} 1(k-l) \qquad l = 0,\, 1,\, 2,\, \ldots,\, n_i - 1$$

where n_i is the **algebraic multiplicity** of the eigenvalue λ_i.

Stability of Linear Discrete-Time Systems

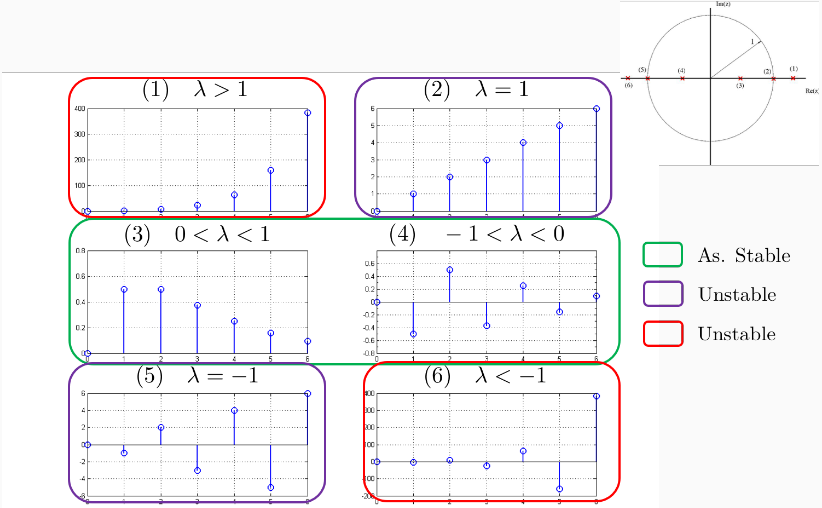
Stability Criterion Based on Eigenvalues

- $\binom{k}{l} \lambda_i^{k-l}$ with $\lambda_i \in \mathbb{R}$, multiplicity $n_i = 1$ (so $l = 0$).



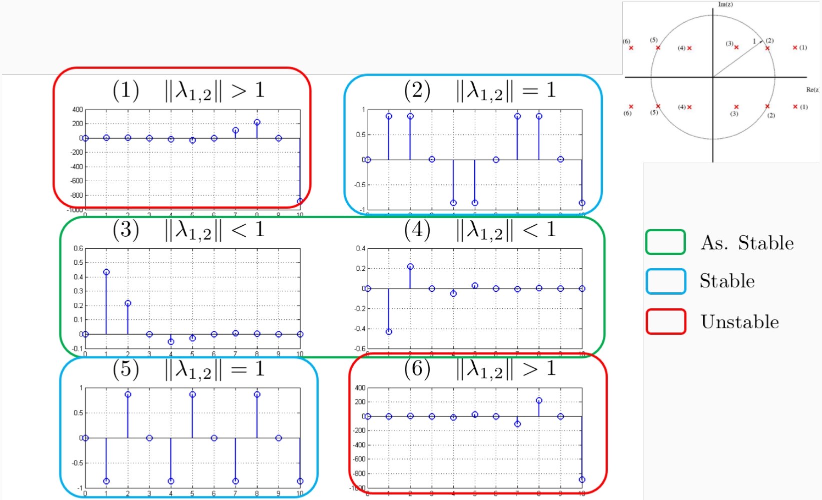
Stability & Qualitative Behaviour of Response Modes

- $\begin{pmatrix} k \\ l \end{pmatrix} \lambda_i^{k-l}$ with $\lambda_i \in \mathbb{R}$, mult. $n_i > 1$ ($l = 0,\ 1,\ \ldots n_i - 1$).



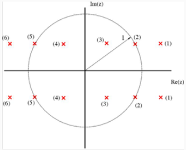
Stability & Qualitative Behaviour of Response Modes

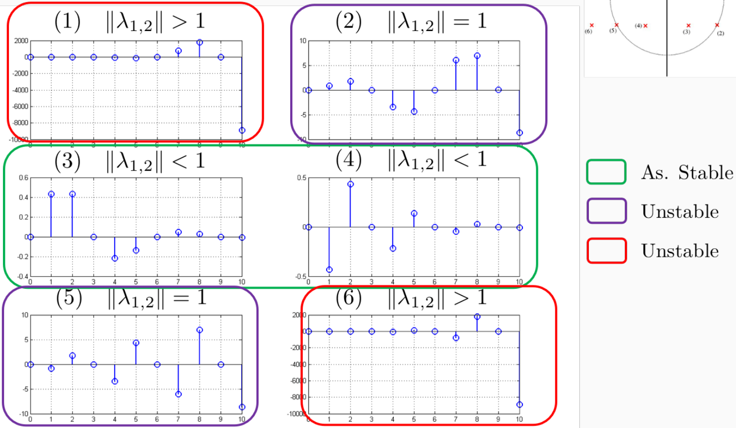
- $\begin{pmatrix} k \\ l \end{pmatrix} \lambda_i^{k-l}$ with $\lambda_i \in \mathbb{C}$, multiplicity $n_i = 1$



Stability & Qualitative Behaviour of Response Modes

- $\begin{pmatrix} k \\ l \end{pmatrix} \lambda_i^{k-l}$ with $\lambda_i \in \mathbb{C}$, multiplicity $n_i > 1$





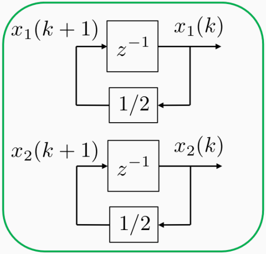
Stability & Behaviour of Response Modes: Example 1

Asymptotically Stable

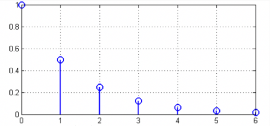
$$A = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$\lambda_1 = \lambda_2 = \frac{1}{2}$$

$$A^k = \begin{bmatrix} (1/2)^k & 0 \\ 0 & (1/2)^k \end{bmatrix}$$



Response modes for $x_1(k)$ and $x_2(k)$



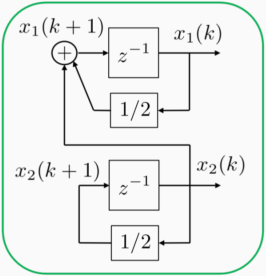
Stability & Behaviour of Response Modes: Example 2

Asymptotically Stable

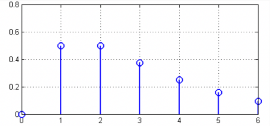
$$A = \begin{bmatrix} 1/2 & 1 \\ 0 & 1/2 \end{bmatrix}$$

$$\lambda_1 = \lambda_2 = \frac{1}{2}$$

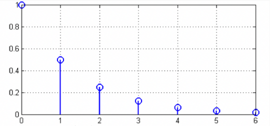
$$A^k = \begin{bmatrix} (1/2)^k & k(1/2)^{k-1} \\ 0 & (1/2)^k \end{bmatrix}$$



Response mode for $x_1(k)$



Response mode for $x_2(k)$

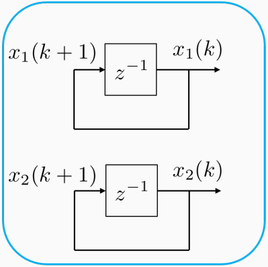


Stable (not asymptotically)

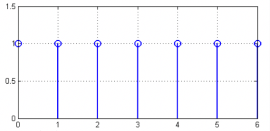
$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\lambda_1 = \lambda_2 = 1$$

$$A^k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



Response modes for $x_1(k)$ and $x_2(k)$



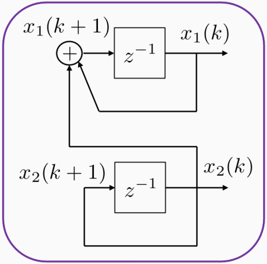
Stability & Behaviour of Response Modes: Example 4

Unstable

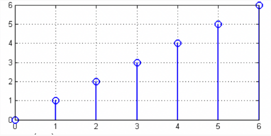
$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\lambda_1 = \lambda_2 = 1$$

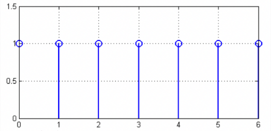
$$A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$



Response mode for $x_1(k)$



Response mode for $x_2(k)$



Algebraic and Geometrical Multiplicity of an Eigenvalue

Algebraic vs Geometrical Multiplicity of an Eigenvalue

- Let $\bar{\lambda}$ be an **eigenvalue** of A.
- The **eigenvectors** of A associated with $\bar{\lambda}$ are the nonzero vectors in the **nullspace** of $A - \bar{\lambda} I$, called **the eigenspace** of A for $\bar{\lambda}$ and denoted by

$$\mathrm{null}\left(A - \bar{\lambda} I\right) = \mathcal{E}_A\left(\bar{\lambda}\right)$$

- The **geometric multiplicity** of the eigenvalue $\bar{\lambda}$ of A is the dimension of $\mathcal{E}_A\left(\bar{\lambda}\right)$.
- The **algebraic multiplicity** of the eigenvalue $\bar{\lambda}$ of A is the multiplicity of $\bar{\lambda}$ as a root of the characteristic polynomial of A $p_A(z) = \det\left(zI - A\right)$.

Algebraic and Geometrical Multiplicity of an Eigenvalue (cont.)

Diagonalisable Matrices – Algebraic vs Geometrical Multiplicity of an Eigenvalue

- In general, an eigenvalue's algebraic and geometric multiplicity can differ. However, the geometric multiplicity can never exceed the algebraic one.
- Let $\lambda_1, \ldots, \lambda_\sigma$ the distinct eigenvalues of A and n_i the algebraic multiplicity of such eigenvalues. Of course $\sum_{i=1}^{\sigma} n_i = n$
- If for every eigenvalue of A, the geometric multiplicity equals the algebraic multiplicity, then A is said to be **diagonalisable**.

Complete Stability Criterion Based on Eigenvalues of A

Stability Criterion

Given the system $x(k+1) = Ax(k)$ and denoting by $\lambda_i,\ i = 1, \ldots n$ the eigenvalues of matrix A.

- $|\lambda_i| < 1,\ \forall\, i = 1, \ldots n \quad \Longleftrightarrow$ The system is as. stable
- $\exists\, i,\ 1 \leq i \leq n : \ |\lambda_i| > 1 \quad \Longrightarrow$ The system is unstable
- $\left.\begin{array}{l} |\lambda_i| \leq 1,\ \forall\, i = 1, \ldots n \\ \exists\, j,\ 1 \leq j \leq n : \ |\lambda_j| = 1 \end{array}\right\} \Longrightarrow$ The system is not as. stable
  - $\lambda_j : |\lambda_j| = 1$ have algebraic multiplicity $= 1$, then the system is stable (not as.)
  - $\lambda_j : |\lambda_j| = 1$ have algebraic multiplicity > 1 and the same value as geometrical multiplicity, then the system is stable (not as.)
  - $\lambda_j : |\lambda_j| = 1$ have algebraic multiplicity > 1, but the geometrical multiplicity is different, then the system is unstable

Stability of Linear Discrete-Time Systems

Analysis of the Characteristic Polynomial

Stability by Analysing the Characteristic Polynomial

- The previous complete stability criterion requires checking whether the eigenvalues of matrix A belong to the unit circle in the complex plane
- A number of techniques exist to perform the check above **without explicitly calculating** the eigenvalues of matrix A
- Considering the characteristic polynomial

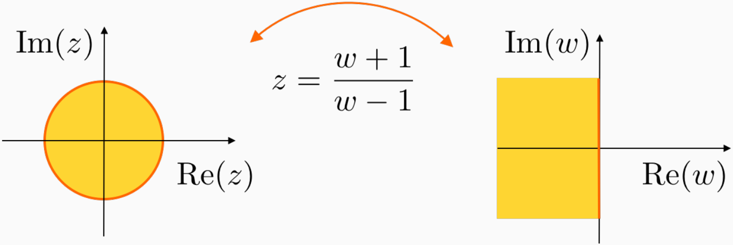
$$p_A(z) = \det(zI - A) = \varphi_0 z^n + \varphi_1 z^{n-1} + \cdots + \varphi_{n-1} z + \varphi_n$$

  a suitable **bi-linear transformation** allows to reduce the problem of checking whether the roots of polynomial $p_A(z)$ belong to the unit circle in the complex plane to an **equivalent problem** of checking whether the roots of a suitable polynomial $q_a(w)$ belong to the complex left half-plane
- This equivalent problem can then be solved by using the **Routh-Hurwitz** technique (see the course *Fundamentals of Automatic Control*)

Use of the Bi-linear Transformation

$$z = \frac{w+1}{w-1}, \;\; z,\, w \in \mathbb{C}$$

$$\begin{aligned} |z| < 1 &\iff \mathrm{Re}(w) < 0 \\ |z| = 1 &\iff \mathrm{Re}(w) = 0 \\ |z| > 1 &\iff \mathrm{Re}(w) > 0 \end{aligned}$$



Use of the Bi-linear Transformation (cont.)

Substitute

$$z = \frac{w+1}{w-1},\ z,\, w \in \mathbb{C}$$

into

$$p_A(z) = \varphi_0 z^n + \varphi_1 z^{n-1} + \cdots + \varphi_{n-1} z + \varphi_n$$

thus obtaining

$$q_A(w) = \ (w-1)^n \ \left[\varphi_0 \frac{(w+1)^n}{(w-1)^n} + \varphi_1 \frac{(w+1)^{n-1}}{(w-1)^{n-1}} + \cdots \right. \\ \left. + \varphi_{n-1} \frac{(w+1)}{(w-1)} + \varphi_n \right]$$

and hence one gets

$$q_A(w) = q_0\, w^n + q_1\, w^{n-1} + \cdots + q_{n-1}\, w + q_n$$

with suitable coefficients $q_0,\, q_1, \ldots, q_n$.

Use of the Bi-linear Transformation. Example 1

Given

$$p_A(z) \;=\; z^3 + 2\,z^2 + z + 1$$

one gets

$$q_A(w) \;=\; (w-1)^3 \;\left[\frac{(w+1)^3}{(w-1)^3} + 2\,\frac{(w+1)^2}{(w-1)^2} + \frac{w+1}{w-1} + 1\right]$$

and after some algebra

$$q_A(w) = 5\,w^3 + w^2 + 3\,w - 1$$

$$\begin{array}{c|cc} 3 & 5 & 3 \\ 2 & 1 & -1 \\ 1 & 8 & \\ 0 & -1 & \end{array}$$

Hence, there is one root of $q_A(w)$ on the complex right-half plane which in turn implies that one of the roots of $p_A(z)$ lies outside the unit circle.

Use of the Bi-linear Transformation. Example 2

Given

$$p_A(z) = z^2 + az + b$$

with $a, b \in R$. Thus, one gets:

$$q_A(w) \ = \ (w-1)^2 \left[\frac{(w+1)^2}{(w-1)^2} + a\,\frac{(w+1)}{(w-1)} + b \right]$$

and after some easy algebra

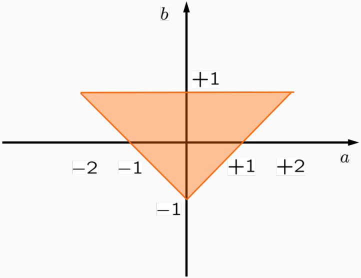
$$q_A(w) = (1+b+a)w^2 + 2(1-b)w - a + 1 + b$$

$$\begin{array}{c|cc} 2 & (1+b+a) & (1+b-a) \\ 1 & 2(1-b) & \\ 0 & (1+b-a) & \end{array} \qquad \left\{ \begin{array}{l} 1+b+a > 0 \\ 2(1-b) > 0 \\ 1+b-a > 0 \end{array} \right. \implies \left\{ \begin{array}{l} b > -a-1 \\ b < 1 \\ b > a-1 \end{array} \right.$$

Use of the Bi-linear Transformation. Example 2 (cont.)

The stability condition has a nice geometric interpretation:

$$\begin{cases} b > -a - 1 \\ b < 1 \\ b > a - 1 \end{cases}$$



Stability of Linear Discrete-Time Systems

Stability of Equilibrium States Through the Linearised System

Stability of Equilibrium States Through the Linearised System - Time-Invariant Systems

Recall from Part 1

- Consider the nonlinear time-invariant system:

$$x(k+1) = f(x(k), u(k))$$

- Moreover, consider an **equilibrium state** $\bar{x}$ obtained by the constant input sequence $u(k) = \bar{u},\ k \geq k_0$.
- Let us **perturb** the initial state and the nominal input sequence, thus getting a **perturbed state movement**:

$$x(k_0) = \bar{x}_0 + \delta x_0\,;\ u(k) = \bar{u} + \delta u(k) \implies x(k) = \bar{x} + \delta x(k)$$

- Hence:

$$\begin{aligned} x(k+1) = \bar{x} + \delta x(k+1) &= f(\bar{x} + \delta x(k), \bar{u} + \delta u(k)) \\ &\simeq f(\bar{x}, \bar{u}) + f_x(\bar{x}, \bar{u})\delta x(k) + f_u(\bar{x}, \bar{u})\delta u(k) \end{aligned}$$

Stability of Equilibrium States Through the Linearised System - Time-Invariant Systems (cont.)

- Since the equilibrium state $\bar{x}$ is the constant solution of the algebraic equation $\bar{x} = f(\bar{x}, \bar{u})$, it follows that

$$\delta x(k+1) \simeq A\delta x(k) + B\delta u(k)$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ are **constant matrices** defined as:

$$A = f_x(\bar{x}, \bar{u}) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial f_n}{\partial x_1} & \cdots & \dfrac{\partial f_n}{\partial x_n} \end{bmatrix}_{x(k)=\bar{x}, u(k)=\bar{u}}$$

$$B = f_u(\bar{x}, \bar{u}) = \begin{bmatrix} \dfrac{\partial f_1}{\partial u_1} & \cdots & \dfrac{\partial f_1}{\partial u_m} \\ \vdots & & \vdots \\ \dfrac{\partial f_n}{\partial u_1} & \cdots & \dfrac{\partial f_n}{\partial u_m} \end{bmatrix}_{x(k)=\bar{x}, u(k)=\bar{u}}$$

Stability of Equilibrium States Through the Linearised System - Time-Invariant Systems (cont.)

Summing up:

The linear time-invariant system obtained by linearization around a given equilibrium state $\bar{x}$ obtained by the constant input sequence $u(k) = \bar{u},\ k \geq k_0$ is

$$\delta x(k+1) = A\delta x(k) + B\delta u(k)$$

The Reduced Lyapunov Method for Discrete-Time Systems

- Consider the nonlinear time-invariant system:
$$x(k+1) = f(x(k), u(k))$$
- Moreover, consider an **equilibrium state** $\bar{x}$ obtained by the constant input sequence $u(k) = \bar{u},\ k \geq k_0$.
- Consider the free linear time-invariant system obtained by linearization around the equilibrium state $\bar{x}$ (the effect of the input is not considered in the stability of the equilibrium) and denote by $\lambda_i,\ i = 1, \ldots n$ the eigenvalues of matrix A:
$$\delta x(k+1) = A\delta x(k)$$
- $|\lambda_i| < 1, \forall\, i = 1, \ldots n \quad \Longrightarrow \quad \bar{x}$ is an **asymptotically stable equilibrium state**
- $\exists\, i,\ 1 \leq i \leq n :\ |\lambda_i| > 1 \quad \Longrightarrow \quad \bar{x}$ is an **unstable equilibrium state**
- In all other situations, **no conclusions** on the stability of the equilibrium state can be drawn from the analysis of the linearised system.

267MI –Fall 2022

Lecture 3
Stability of Discrete-Time Dynamic Systems

END