Martedì 18 ottobre pomeriggio

Def Sia $X \subseteq \mathbb{R}$. Se $\inf X \in X$ allora si dice che X ha minimo e si pone $\min X = \inf X$

se $\sup X \in X$ allora si dice che X ha massimo e si pone $\max X = \sup X$.

Es $(0,1)$ $\quad \sup(0,1) = 1 \notin (0,1)$

$\inf(0,1) = 0 \notin (0,1)$ non esistono $\max(0,1)$

e $\min(0,1)$.

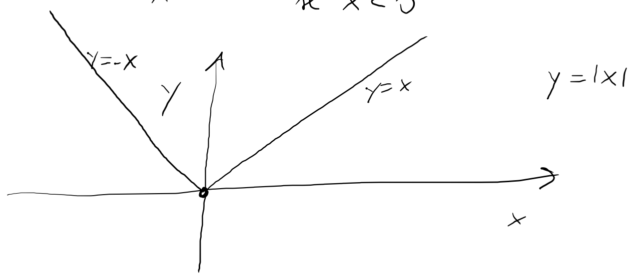
$\min [0,1) = 0$ $\quad \max[0,1)$ non esiste

Es $X \subseteq \mathbb{R}$ è finito se contiene un numero finito di elementi. Dimostrare che esistono $\min X$ e $\max X$.

Es Se $X \subseteq \mathbb{N}$ $\exists \min X$.

Funzione valore assoluto $|\cdot| : \mathbb{R} \longrightarrow [0,+\infty)$

$$|x| = \begin{cases} x & \text{se } x > 0 \\ 0 & x = 0 \\ -x & \text{se } x < 0 \end{cases}$$



Lemma Sia $a \geq 0$. Allora

$$|x| \leq a \iff -a \leq x \leq a$$

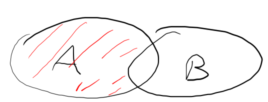
$$(\ |x| < a \iff -a < x < a \ )$$

1) $|x| = 0 \iff x = 0$

2) $|xy| = |x|\,|y| \qquad \forall x, y \in \mathbb{R}$

2') $\left|\frac{x}{y}\right| = \frac{|x|}{|y|} \qquad \forall x \in \mathbb{R} \text{ e } \forall y \in \mathbb{R} \setminus \{0\}$

3) $|x+y| \leq |x| + |y|$



$A \setminus B = \{a \in A : a \notin B\}$

$A - B$

Dim di 3 $\quad |x| \leq |x| \qquad\qquad |y| \leq |y|$

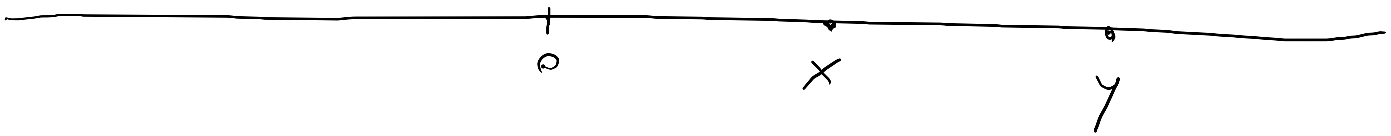
$$\Updownarrow$$

$$\left.\begin{array}{l} -|x| \leq x \leq |x| \\ -|y| \leq y \leq |y| \end{array}\right\} \text{ sommiamo}$$

$$-(|x|+|y|) = -|x| - |y| \leq x+y \leq |x| + |y| \iff |x+y| \leq |x| + |y|.$$

Esercizio Per ogni scelta $x_1, \dots, x_n$ di numeri reali

$$\left| \sum_{j=1}^{n} x_j \right| \leq \sum_{j=1}^{n} |x_j|$$



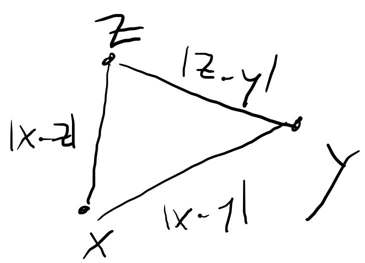
Dati due punti $x, y \in \mathbb{R}$, la

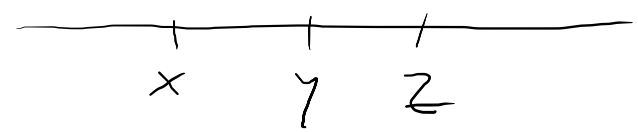
$$\text{distanza}(x,y) = |x-y|$$

1) $|x-y| = 0 \iff x = y$

2) $|x-y| = |y-x| \quad \forall\, x, y \in \mathbb{R}$

3) $|x-y| \leq |x-z| + |z-y| \quad \forall\, x, y, z \in \mathbb{R}$





Es. Dati $L_1, L_2 \in \mathbb{R}$ allora quanto segue è vero

$$|L_1 - L_2| < \varepsilon \ \forall \varepsilon > 0 \implies L_1 = L_2$$

Def (Successioni) Dato un insieme non vuoto X una successione $\{x_n\}_{n\in\mathbb{N}}$ di elementi di X è una funzione $f:\mathbb{N}\to X$, dove

$$f(n)=x_n$$

Esempi $\{n\} = \{1,2,3,4,\dots\}$

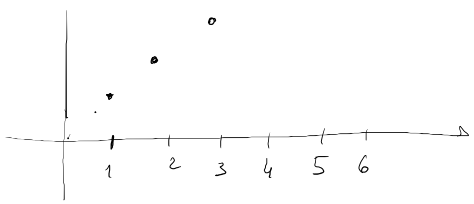
$\{1\} = \{1,1,1,\dots\dots 1\}$

$\{(-1)^n\} = \{-1,1,-1,1,-1,\dots\}$

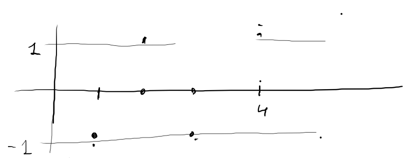
$\{\frac{1}{n}\} = \{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\dots\}$

$\lim_{n\to+\infty} n = +\infty$

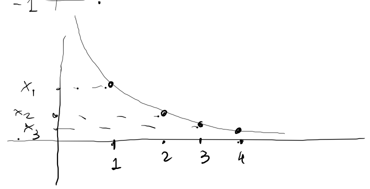
$x_n = n$



$\lim_{n\to+\infty} (-1)^n$ non esiste



$\lim_{n\to+\infty} \frac{1}{n} = 0$



Def Data una successione $\{x_n\}$ in $\mathbb{R}$ ed un $L\in\mathbb{R}$ diremo che L è il limite della successione, e scriveremo $L = \lim_{n\to+\infty} x_n$

$$\forall \varepsilon>0 \ \exists N_\varepsilon \ \text{t.c.} \ n>N_\varepsilon \Rightarrow |x_n-L|<\varepsilon$$

$$\forall \varepsilon\in\mathbb{R}_+ \ \exists N_\varepsilon \ \text{t.c.} \ n>N_\varepsilon \Rightarrow |x_n-L|<\varepsilon$$

$$\Updownarrow$$

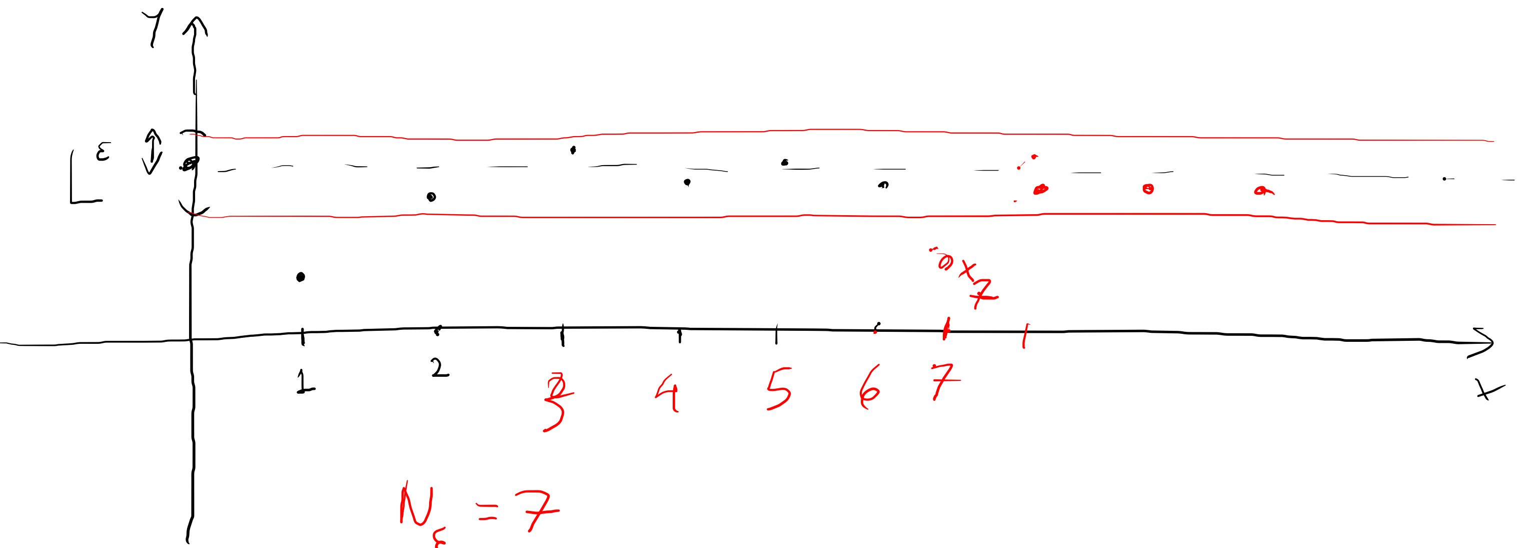
$$\forall \varepsilon\in(0,1/2) \ \exists N_\varepsilon \ \text{t.c.} \ n>N_\varepsilon \Rightarrow |x_n-L|<\varepsilon$$

$$\Updownarrow$$

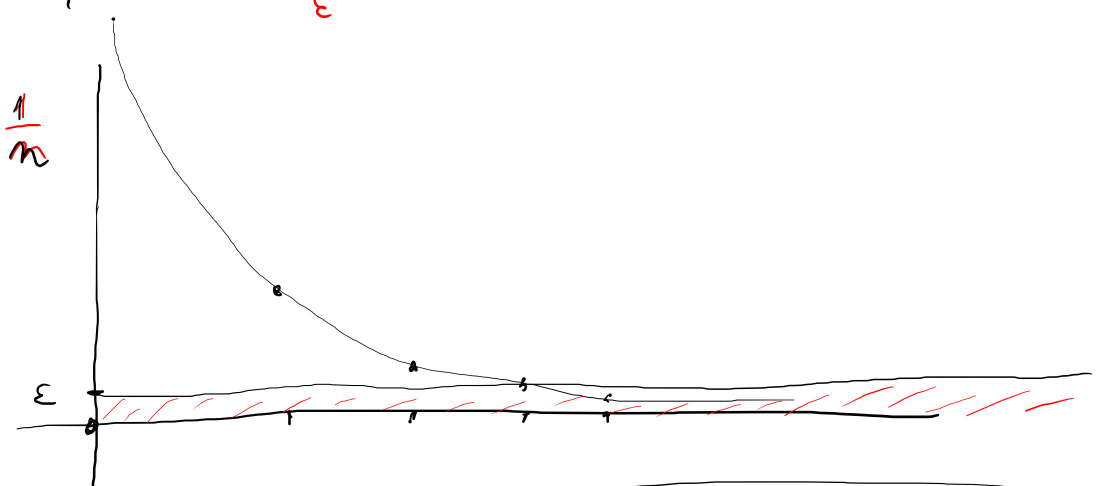
sia $X\subseteq\mathbb{R}_+$ t.c. $\inf X = 0$

$$\forall \varepsilon\in X \ \exists N_\varepsilon \ \text{t.c.} \ n>N_\varepsilon \Rightarrow |x_n-L|<\varepsilon$$

$\forall \varepsilon > 0 \ \exists \ N_\varepsilon$ t.c. $n > N_\varepsilon \Rightarrow |x_n - L| < \varepsilon$



$N_\varepsilon = 7$



$\forall \varepsilon > 0 \ \exists \ N_\varepsilon$ t.c. $\boxed{n > N_\varepsilon \Rightarrow |x_n - L| < \varepsilon}$

$\lim_{n \to +\infty} \frac{1}{n} = 0$ scelgo $\varepsilon > 0$ arbitrario

$$\frac{1}{n} < \varepsilon \iff n > \frac{1}{\varepsilon}$$

Se scelgo $N_\varepsilon = \frac{1}{\varepsilon}$ ottengo che $n > \frac{1}{\varepsilon} = N_\varepsilon \Rightarrow$

$$|x_n - 0| = \frac{1}{n} < \varepsilon$$

$\lim_{n\to+\infty} (-1)^n$ non esiste. Se esistesse un limite

$L \in \mathbb{R}$ allora

$\forall \varepsilon > 0 \quad \exists\, N_\varepsilon$ t.c. $n > N_\varepsilon \Rightarrow |(-1)^n - L| < \varepsilon$.

Ma per ogni $N_\varepsilon \in \mathbb{R}$ esistono numeri naturali

$n > N_\varepsilon$ che sono pari $\Rightarrow (-1)^n = 1$

$\Rightarrow \forall \varepsilon > 0 \Rightarrow |1 - L| < \varepsilon \Rightarrow L = 1$

Per ogni $N_\varepsilon \in \mathbb{R}$ esistono numeri $n > N_\varepsilon$ che sono

dispari $\Rightarrow (-1)^n = -1$

$\Rightarrow \forall \varepsilon > 0 \Rightarrow |(-1) - L| < \varepsilon \Rightarrow L = -1$

$\Rightarrow 1 = -1$ falso