19 october

Locally convex spaces

Def Given a vector space X a subset $\Omega \subseteq X$ is convex iff $\forall\ x_0, x_1 \in \Omega$ then we have

$$x_t := (1-t)\, x_0 + t\, x_1 \in \Omega$$
$$\forall\ t \in [0,1]$$

Def A t.v.s X is locally convex if there is a basis of neigh. of $0 \in X$ formed by convex set.

<u>Theorem</u> Let X a t.v.s

1) If $\Omega \subseteq X$ is convex then $\overline{\Omega}$ and $\mathring{\Omega}$ are convex set.

2) If $\{\Omega_j\}_{j\in J}$ is a family of convex subspaces, then

$$\bigcap_{j\in J} \Omega_j \quad \text{is convex}$$

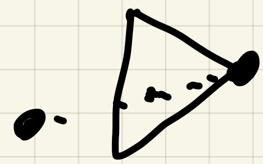
3) For any subset $Y \subseteq X$ (vector space) there is a smallest convex set Ω with $Y \subseteq \Omega$.

Ω is the convex hull.

4) Let X be a loc convex t.v.s. Then given any neigh^U of ~~0~~ 0 in X, there exists a neigh.

V of 0 s.t. $V \subseteq U$

and V is convex and balanced.



<u>Def</u> (Seminorms)

X vector space, a function

$p : X \to [0, +\infty)$ is a seminorm

if

$$p(x+y) \leq p(x) + p(y) \qquad \forall\, x, y \in X$$

$$p(\lambda x) = \lambda\, p(x) \qquad \forall\, x \in X \text{ and } \forall\, \lambda \geq 0.$$

$$p(0) = 0$$

<u>Remark</u> If $p : X \to [0, \infty)$

then $\boxed{C := \{ x \in X : \; p(x) < 1 \}}$ $\circledast$

is convex, $0 \in C$, and is

absorbing.

<u>Lemma</u> Let X be a t.v.s, C an open convex neigh of $0 \in X$. Then $\exists$ a seminorm $p : X \longrightarrow [0, +\infty)$ such that $\circledast$ is satisfied. Furthermore, $p \in C^0(X, [0, +\infty))$

$$C = \{ x \in X : p(x) < 1 \} \quad \circledast$$

<u>Pf</u> Set

$$p(x) := \inf \{ a > 0 : \frac{x}{a} \in C \}$$

We claim $p(x) \leq 1$ in C

Obviously $1 \in \{ a > 0 : \frac{x}{a} \in C \}$

if $x \in C \implies p(x) \leq 1$

Since C is open, for $x \in C$ and for $\varepsilon > 0$ small enough $\Rightarrow$ $(1+\varepsilon)x \in C$

$\Rightarrow$ $p((1+\varepsilon)x) \leq 1 \overset{?}{\Rightarrow} p(x) \leq \frac{1}{1+\varepsilon}$

Claim is $p(x) < 1$

$(1+\varepsilon)x \in C \Leftrightarrow \frac{x}{\frac{1}{1+\varepsilon}} \in C$

$\Rightarrow \frac{1}{1+\varepsilon} \in \{ a > 0 : \frac{x}{a} \in C \}$

$\Rightarrow p(x) \leq \frac{1}{1+\varepsilon} \Rightarrow p(x) < 1$

Next we show that if for $x \in X$

$\boxed{p(x) < 1 \Rightarrow x \in C}$

$p(x) = \inf\{ a > 0 : \frac{x}{a} \in C \}$

$\exists$ $0 < \alpha < 1$ s.t. $\frac{x}{\alpha} \in C$

$0 \in C$

$x = \alpha \frac{x}{\alpha} + (1-\alpha) 0 \in C$

$0 < \alpha < 1$

$\lambda > 0$ $x \in X \Rightarrow p(\lambda x) = \lambda p(x)$

$$p(\lambda x) = \inf\left\{ a > 0 : \frac{\lambda x}{a} \in C \right\}$$

$$a = \lambda\alpha$$

$$= \inf\left\{ \lambda\alpha > 0 : \frac{\cancel{\lambda} x}{\cancel{\lambda}\alpha} \in C \right\}$$

$$= \lambda \inf\left\{ \alpha > 0 : \frac{x}{\alpha} \in C \right\}$$

$$= \lambda \; p(x)$$

$x, y \in X \qquad \varepsilon > 0$

$$\frac{x}{p(x)+\varepsilon}, \quad \frac{y}{p(y)+\varepsilon} \in C$$

$$t \frac{x}{p(x)+\varepsilon} + (1-t)\frac{y}{p(y)+\varepsilon} \in C$$

$\forall \qquad t \in [0, 1]$

$$t = \frac{p(x)+\varepsilon}{p(x)+p(y)+2\varepsilon}$$

$$1-t = \frac{p(y)+\varepsilon}{p(x)+p(y)+2\varepsilon}$$

$$\cancel{p} \quad \frac{x+y}{p(x)+p(y)+2\varepsilon} \in C$$

$$p(x+y) < p(x) + p(y) + 2\varepsilon \quad \forall \varepsilon > 0$$

$$\Rightarrow \quad p(x+y) \leq p(x) + p(y)$$

<u>Lemma</u> Given X t.v.s. locally convex, then there exists a family $\{p_j\}_{j \in J}$ of continuous seminorms s.t. $\forall\, x_0 \neq 0$ $\exists\, j_0$ s.t. $p_{j_0}(x_0) \neq 0$ and such that the family

$$\{\, p_j^{-1}([0,r)) : r > 0 \text{ and } j \in J \,\}$$

is a sub-basis of neigh. of 0.

$f : X \to \mathbb{R}$ is homogeneous of order $\alpha \geq 0$ if

$$f(\lambda x) = \lambda^\alpha f(x) \quad \forall \lambda \geq 0$$
$$\forall x \in X$$

<u>Exercise</u> Show that if have ~~for~~ X with a sub-basis $\{P_j\}_{j \in J}$ and if $f : X \to K$ is homogeneous degree 1 then f is continuous in 0 iff $\exists\ C > 0$ and finitely many indexes $j_1, \dots, j_n$ s.t.

$$|f(x)| \leq C \left(P_{j_1}(x) + \dots + P_{j_n}(x)\right)$$

$$\forall x \in X.$$

Example $L^p(0,1)$ $0<p<1$

$$d(f,g) := \int_0^1 |f(t)-g(t)|^p \, dt$$

The only convex open sets in $L^p(0,1)$ are ϕ and $L^p(0,1)$

As a consequence if

$T : L^p(0,1) \to X$ is cont. and linear, X loc. convex t.v.s

$\Longrightarrow$ $T \equiv 0$

Indeed for any open convex neigh V of 0 in X, $T^{-1}V$ is an open convex subset of $L^p(0,1)$

$$L^p(0,1) = T^{-1} V$$

$\Longrightarrow$ $T\, L^p(0,1) \subseteq V$

$\forall$ V convex open neig. of $0 \in X$

$\Rightarrow$ $\{0\} = R(T) = T(L^p(0,1))$

$\Rightarrow$ $T \equiv 0$

$X = \mathbb{K}$ $\Rightarrow$ $(L^p(0,1))' = \{0\}$

<u>Ex</u> If ~~X~~ $\{p_j\}_{j \in J}$

then $J \subseteq \mathbb{N}$ $\quad p_j(y-x) = p_j(x-y)$

$$d(x,y) \doteq \sum_{j \in J} 2^{-j} \frac{p_j(x-y)}{1 + p_j(x-y)}$$

is a metric and the topology induced on X by d is the same of that induced by the seminorms

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Schwartz

$$\mathcal{S}(\mathbb{R}^d, \mathbb{C}) = \Big\{ \phi \in C^\infty(\mathbb{R}^d) : \text{and}$$

$$\mathcal{S}(\mathbb{R}^d) \qquad p_{\alpha,\beta}(\phi) := \sup_{x \in \mathbb{R}^d} |x^\beta \partial^\alpha \phi(x)| < +\infty \Big\}$$

$$\alpha = (\alpha_1, \dots, \alpha_d) \in (\mathbb{N} \cup \{0\})^d$$

$$p_{\alpha,\beta}(\phi) \qquad \partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$$

$$x^\beta = x_1^{\beta_1} \cdots x_d^{\beta_d}$$

$$\mathcal{S}'(\mathbb{R}^d, \mathbb{C}) = \cancel{\{T \in}\ \mathcal{L}\big(\mathcal{S}(\mathbb{R}^d, \mathbb{C}), \mathbb{C}\big)$$

space of Tempered distributions.

Lemma $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$

$T : X \to Y$ linear

The following are equivalent

1) T is continuous

2) The following happens

$$\|T\|_{\mathcal{L}(X,Y)} := \sup_{x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X}$$

$$= \sup_{x \in D_X(0,1) \setminus \{0\}} \frac{\|Tx\|_Y}{\|x\|_X}$$

$$= \sup_{x \in D_X(0,1)} \|Tx\|_Y < +\infty$$

Pf If T is ~~is~~ continuous

$\forall \varepsilon > 0 \ \exists \ \delta > 0$ s.t.

$$\|x\|_X \leq \delta \Rightarrow \|Tx\|_Y < \varepsilon$$

Let $x \neq 0$

$$\tilde{x} = \frac{\delta}{\|x\|_X} x \qquad \|\tilde{x}\|_X = \delta$$

$$\|T\tilde{x}\|_Y < \varepsilon$$

$$\frac{\|Tx\|_Y}{\|x\|_X} = \frac{\|T\tilde{x}\|_Y}{\|\tilde{x}\|_X} \leq \frac{\varepsilon}{\delta}$$

$\Rightarrow \ \forall \ x \neq 0$

$$\|T\|_{\mathcal{L}(X,Y)} \leq \frac{\varepsilon}{\delta} < +\infty$$