October 21st.

X, Y normed spaces

$\mathcal{L}(X, Y)$

$T \in \mathcal{L}(X, Y) \iff$

$\|T\| := \sup\{\|Tx\|_Y : x \in D_X(0,1)\}$

$\|T\| < \infty$

Notice that this defines a norm in $\mathcal{L}(X, Y)$ $\quad K$

$\|T + S\| \leq \|T\| + \|S\| \quad \forall\, T, S \in \mathcal{L}(X, Y)$

$\|\lambda T\| = |\lambda| \, \|T\| \quad \forall\, \lambda \in K$

$T \in \mathcal{L}(X, Y)$

$\|T\| = 0 \iff T = 0$

It can be seen that Y Banach

$\implies \mathcal{L}(X, Y)$ is Banach

$Y = K = \mathbb{R}, \mathbb{C}$

$\implies X'$ is a Banach space

$f \in X'$

$\|f\|_{X'} = \sup\{ |f(x)| : x \in D_X(0, 1) \}$

$f(x) = \langle f, x \rangle_{X' \times X}$

$X' \times X \longrightarrow K$

$(f, x) \longrightarrow f(x) = \langle f, x \rangle_{X' \times X}$

$= \langle x, f \rangle_{X \times X'}$

$1 \leq p < +\infty$

$$\left(L^p(\Omega, d\mu)\right)' = L^{p'}(\Omega, d\mu)$$

$\mathcal{L}(X, Y)$ $\overline{\{T_n\}}$ we say

it converges uniformly to a $T \in \mathcal{L}(X, Y)$

if $\lim_{n \to +\infty} \| T - T_n \| = 0$

(it is a uniform convergence in

$C^0(D_X(0,1), Y)$ ⊗

$$\sup_{x \in D_X(0,1)} \| Tx \|_Y$$

We say $s\text{-}\lim_{n \to +\infty} T_n = T$

if $T_n x \xrightarrow{n \to +\infty} Tx \quad \forall x \in X.$

$\mathbb{R}^d$ $f \in L^p(\mathbb{R}^d, dx)$ $1 \le p < +\infty$

$$\chi_{D_{\mathbb{R}^d}(0,\lambda)} f \xrightarrow{\lambda \to +\infty} f \text{ in } L^p(\mathbb{R}^d, dx) \quad \forall f.$$

$$\int_{\mathbb{R}^d} \left(1 - \chi_{D_{\mathbb{R}^d}(0,\lambda)}\right)^p |f(x)|^p \, dx$$

$$= \int_{\mathbb{R}^d} \underbrace{\chi_{C D_{\mathbb{R}^d}(0,\lambda)}(x) \, |f(x)|^p}_{\substack{\downarrow \, \lambda \to +\infty \\ 0 \qquad \forall x \in \mathbb{R}^d}} dx$$

$$\Rightarrow \lim_{\lambda \to +\infty} \| \chi_{D_{\mathbb{R}^d}(0,\lambda)} f - f \|_{L^p} = 0$$

$$f \rightsquigarrow \chi_{D_{\mathbb{R}^d}(0,\lambda)} f$$

$$s\text{-}\lim_{\lambda \to +\infty} \chi_{D_{\mathbb{R}^d}(0,\lambda)} = 1 \quad \text{in } L^p, \; p < +\infty$$

$$\| 1 - \chi_{D_{\mathbb{R}^d}(0,\lambda)} \| = 1 \qquad \forall \lambda.$$

$$\| \chi_{C D_{\mathbb{R}^d}(0,\lambda)} f \|_{L^p} = \| f \|_{L^p}$$

if $\quad \operatorname{supp} f \subset \subset \mathbb{D}_{\mathbb{R}^d}(0,\lambda)$

$$\varphi \in C^0_c(\mathbb{R}^d, \mathbb{C})$$

$$\mathcal{S}(\mathbb{R}^d, \mathbb{C})$$

$$\varphi(0) = 1$$

$$\varphi_\lambda(x) \doteq \varphi\left(\frac{x}{\lambda}\right)$$

$$\varphi_\lambda f \xrightarrow{\lambda \to +\infty} f \quad \text{in } L^p(\mathbb{R}^d)$$

$$p < +\infty$$

but $\varphi_\lambda \not\to 1$ in uniform sense

$$\rho \in C^0_c(\mathbb{R}^d, \mathbb{C})$$

$$\rho(0) = 1 \qquad \int_{\mathbb{R}^d} \rho \, dx = 1$$

$$\rho_\varepsilon(x) = \varepsilon^{-d} \rho\left(\frac{x}{\varepsilon}\right)$$

$$\rho_\varepsilon * f = \varepsilon^{-d} \int_{\mathbb{R}^d} \rho\left(\frac{x-y}{\varepsilon}\right) f(y)\, dy$$

$$\rho_\varepsilon * f \xrightarrow{\varepsilon \to 0^+} f \qquad \forall \; f \in L^p(\mathbb{R}^d)$$

$$p < +\infty$$

$\rho_\varepsilon * \not\to 1$, uniformly

$$\widehat{\rho_\varepsilon * f} = \hat{\rho}(\varepsilon x)\,\hat{f} \qquad \varepsilon \to 0^+$$

$$e^{t\Delta}$$

$$\Delta = \partial_1^2 + \cdots + \partial_d^2$$

$$u(t) = e^{t\Delta} f(x) = (4\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} f(y)\,dy$$

$$\begin{cases} (\partial_t - \Delta) u = 0 \\ u(0) = f \end{cases}$$

$$t \mapsto e^{t\Delta} \in \mathcal{L}(L^p(\mathbb{R}^d))$$

$$\| e^{t\Delta} \| = 1 .$$

$t \to e^{t\Delta}$ is strongly continuous in t $\quad p < +\infty$

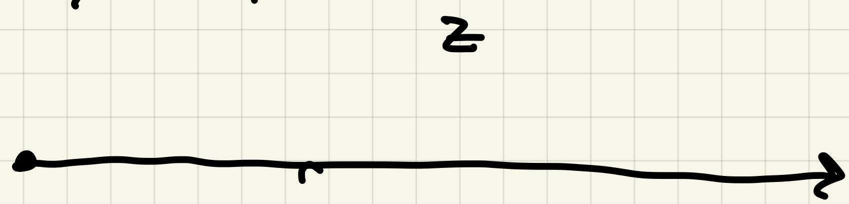
Y ~~Banach~~ normed space $\overline{X}$ X is normed

$$T \in \mathcal{L}(X, Y)$$

$$\overline{T} \in \mathcal{L}(\overline{X}, \overline{Y})$$

$$\left(-\frac{d^2}{dx^2} - z\right) u = f \in L^2(\mathbb{R}, \mathbb{C})$$

$$z \notin [0, +\infty)$$



$$f \xrightarrow{R_0(z)} u \in L^2(\mathbb{R}, \mathbb{C})$$

$$R_0(z)u(x) = \frac{i}{2\sqrt{z}} \int_{\mathbb{R}} e^{i\sqrt{z}|x-y|} f(y)\,dy$$

$$\operatorname{Im}\sqrt{z} > 0 \qquad \sqrt{z} = \sqrt{|z|} + \frac{i}{2}\arg z$$

$$0 < \arg z < 2\pi$$

$$\left(-\frac{d^2}{dx^2} - z\right) R_0(z) = 1$$

$$\left(-\frac{d^2}{dx^2} - z\right) u = 0$$

$$\psi_\pm(x,\sqrt{z}) = e^{\pm i\sqrt{z}\,x}$$

$$\psi_\pm(x,\sqrt{z}) \xrightarrow{x\to\pm\infty} 0$$

$$e^{i\sqrt{z}\,x} = e^{i\sqrt{|z|}\,x} \quad e^{-\frac{1}{2}\arg z\, x} \xrightarrow{x\to+\infty} 0$$

$$W(f,g) = f'g - fg'$$

$$W\big(\psi_+(x,z), \psi_-(x,z)\big) = 2i\sqrt{z}$$

$$(-\partial_x^2 - z)u = f$$

$$R_0(x,y,z) = \begin{cases} -\dfrac{\psi_+(x,\sqrt{z})\,\psi_-(y,\sqrt{z})}{W(\psi_+(y,\sqrt{z}),\psi_-(y,\sqrt{z}))} & x>y \\ -\dfrac{\psi_+(y,\sqrt{z})\,\psi_-(x,\sqrt{z})}{W(\psi_+(y,\sqrt{z}),\psi_-(y,\sqrt{z}))} & x<y \end{cases}$$

$$\| \quad \frac{i}{2\sqrt{z}} e^{i\sqrt{z}|x-y|}$$

$$(-\partial_x^2 - z)$$

$$\int_{\mathbb{R}} R_0(x,y,z) f(y)\,dy =$$

$$(-\partial_x^2 - z)$$

$$= \Big(-\int_{-\infty}^{x} \frac{\psi_+(x,\sqrt{z})\,\psi_-(y,\sqrt{z})}{W(\psi_+(y,\sqrt{z}),\psi_-(y,\sqrt{z}))} f(y)\,dy$$

$$- \int_{x}^{+\infty} \frac{\psi_+(y,\sqrt{z})\, \psi_-(x,\sqrt{z})}{W(\psi_+(y,\sqrt{z}), \psi_-(y,\sqrt{z}))} f(y)\, dy$$

$$= \left(\frac{\psi_+'(x,\sqrt{z})\, \psi_-(x,\sqrt{z})}{W} - \frac{\psi_+(x,\sqrt{z})\, \psi_-'(x,\sqrt{z})}{W} \right)$$

$$f(x)$$

$$= f(x)$$

~~$R(x)$~~ R_0

$$\left(\frac{d^2}{dx^2} + V - z\right) u = f$$

$$V(x) \in C_c^{\infty}(\mathbb{R}^d, \mathbb{R})$$

$$\psi_\pm(x, \sqrt{z}) \qquad z \notin [0, +\infty)$$

$$\psi_\pm(x,\sqrt{z}) \xrightarrow[]{x\to\pm\infty} 0$$

$$\left(-\frac{d^2}{dx^2}+V-z\right)u = 0$$

$$R_V(x,y,z) = \begin{cases} -\dfrac{\psi_+(x,\sqrt{z})\,\psi_-(y,\sqrt{z})}{W(\psi_+(y,\sqrt{z}),\psi_-(y,\sqrt{z}))} & x>y \\ -\dfrac{\psi_+(y,\sqrt{z})\,\psi_-(x,\sqrt{z})}{W(\psi_+(y,\sqrt{z}),\psi_-(y,\sqrt{z}))} & x<y \end{cases}$$

$$f \to \overbrace{\int_{\mathbb{R}^d} R_V(x,y,z)\,f(y)\,dy}^{R_V(z)f}$$

$L^2(\mathbb{R})$ ↻ which inverts

$$(-\partial_x^2+V-z)\,R_V(z) = 1$$

<u>Lemma</u> Let X be Banach

on $T\in\mathcal{L}(X)$. Then

if $\|T\| < 1$ then exists $(1+T)^{-1} \in \mathcal{L}(X)$

<u>Remark</u> $\|T^n\| \leq \|T\|^n \quad \forall\, n \in \mathbb{N}$

Proof We will show that

$$(1+T)^{-1} = \sum_{n=0}^{\infty} (-1)^n T^n \quad \mathcal{L}(X)$$

Indeed

$$(1+T) \sum_{n=0}^{N} (-1)^n T^n =$$

$$= 1 + (-1)^N T^{N+1}$$

$$(1+T) \sum_{n=0}^{N} (-1)^n T^n =$$

$$= \sum_{n=0}^{N} (-1)^n T^n + \sum_{n=0}^{N} (-1)^n T^{n+1}$$

$$= 1 + (-1)^N T^{N+1}$$

$$(1+T) \sum_{n=0}^{N} (-1)^n T^n =$$

$$= 1 + (-1)^N T^{N+1}$$

$$\left\| (1+T) \sum_{n=0}^{N} (-1)^n T^n - 1 \right\| =$$

$$= \| T^{N+1} \| \leq \| T \|^{N+1} \xrightarrow{N \to +\infty} 0$$

$\|T\| < 1$

Also notice that

$$\lim_{N \to +\infty} \sum_{n=0}^{N} (-1)^n T^n \quad \text{is convergent}$$

uniform tone

$N_1 < N_2$

$$\left\| \sum_{n=0}^{N_2} - \sum_{n=0}^{N_1} \right\| = \left\| \sum_{n=N_1+1}^{N_2} (-1)^n T^n \right\|$$

$$\leq \sum_{n=N_1+1}^{N_2} \|T\|^n \xrightarrow{N_2 > N_1 \to +\infty} 0$$

$$(1+T) \sum_{n=0}^{\infty} (-1)^n T^n - 1 = 0$$

$$\sum_{n=0}^{\infty} (-1)^n T^n \quad (1+T) - 1 = 0$$

Spectrum

X Banch on $\mathbb{C}$ $\quad T \in \mathcal{L}(X)$

Resolvent set

$$\rho(T) = \{ z \in \mathbb{C} : T - z \text{ is invertible } (T-z)^{-1} \in \mathcal{L}(X) \}$$

Notice, it is open

$$\sigma(T) = \mathbb{C} \setminus \rho(T)$$

spectrum

Remarks 1) If λ is an eigenvalue of T, then $\lambda \in \sigma(T)$

$\sigma_p(T)$

2) $\sigma(T) \neq \emptyset$

3) $\rho(T)$ open, $\sigma(T)$ closed

4) $\sigma(T) \subseteq \overline{D_{\mathbb{C}}(0, \|T\|)}$

$z \in \mathbb{C}$ $\quad |z| > \|T\| \geq 0$

$\Rightarrow \quad z \in \rho(T)$

$z \neq 0$

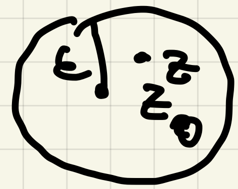
$$T - z = z\left(\frac{T}{z} - 1\right) =$$

$$= -z\left(1 - \frac{T}{z}\right)$$

$$\left\| - \frac{T}{z} \right\| = \frac{1}{|z|} \|T\| < 1$$

5) $\rho(T)$ is open

Let $z_0 \in \rho(T)$



$|z - z_0| \leq \varepsilon$

$\varepsilon > 0$

$T - z =$

$= (T - z_0) - (z - z_0) =$

$= (T - z_0)\left(1 - (z - z_0)(T - z_0)^{-1}\right)$

$\| (z - z_0)(T - z_0)^{-1} \| =$

$= |z - z_0| \, \underbrace{\| (T - z_0)^{-1} \|}_{\in [0, +\infty)}$

$\leq \varepsilon \, \| (T - z_0)^{-1} \| < 1$

for $\varepsilon < \dfrac{1}{\| (T - z_0)^{-1} \|}$

$\rho(T) \ni z \longrightarrow R_T(z) \doteq (T-z)^{-1} \in \mathcal{L}(X)$

is holomorphic.

If $\lambda \in \sigma_p(T)$ and

$n = \dim (T - \lambda) < +\infty$

n is the geometric dimension of λ.

Notice that for any $n \in \mathbb{N}$

$N_g(T-\lambda) \bigcup_{n \geq 1}^{\infty} \ker (T-\lambda)^n \supseteq \ker (T - \lambda)$

$(T-\lambda)x = 0 \Rightarrow (T-\lambda)^{n-1} \underbrace{(T-\lambda)x}_{0} = 0$

If $n = \dim N_g (T-\lambda) < +\infty$

n is the algebraic dimension of λ.

$f(A) \qquad A \in \mathcal{L}(X)$

$$e^{A} = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

$$e^{tA} = \sum_{n=0}^{\infty} t^n \frac{A^n}{n!}$$

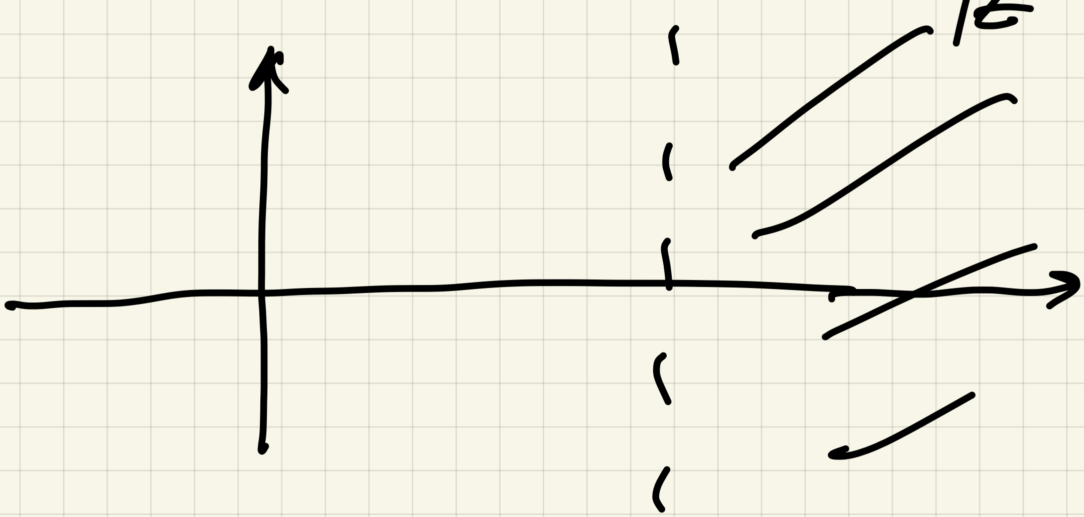
$$\begin{cases} \dot{x} = Ax + f \\ x(0) = x_0 \end{cases} \qquad x_0 \in X, \quad f \in C^0(\mathbb{R}, X)$$

$$x(t) = e^{tA} x_0 + \int_0^t e^{(t-s)A} f(s)\, ds$$

Duhamel

$$R_A(z) = -\int_0^{+\infty} e^{tA} e^{-tz}\, dt$$

$$\operatorname{Re} z > \|A\|$$



$$\int_0^{+\infty} e^{t(A-z)}\,dt =$$

$$= \lim_{R\to+\infty} \int_0^R e^{t(A-z)}\,dt$$

$$= \lim_{R\to+\infty} \left(\frac{-1}{A-z} + \frac{e^{R(A-z)}}{A-z} \right)$$

$\xrightarrow[R\to+\infty]{} 0$

$$e^{A} = -\frac{1}{2\pi i} \int_\gamma e^{z} R_A(z)\,dz$$