Systems Dynamics

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Lecture 5
A (Very) Short Glimpse on Probability Theory, Random Variables and Discrete-Time Stochastic Processes

Lecture 5: Table of Contents

A Glimpse on Probability Theory and Random Variables

A Glimpse on Probability Theory and Random Variables

Basic Definitions

Random Variables

- **Random experiment**: analysis of characteristic elements of phenomena yielding unpredictable results.
- **Results space**: we denote by S the set of all possible results of the experiment. Result: $s \in S$.
- **Events**: sets of results of specific interest. Hence an event is a subset of S.

Random variable

Given a random experiment, a **random variable** (r.v.) is a variable $v(s)$ taking values depending on the result $s \in S$ of a random experiment via a function $\varphi(\cdot)$.

A Glimpse on Probability Theory and Random Variables

Probability Distribution & Density Functions

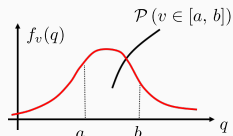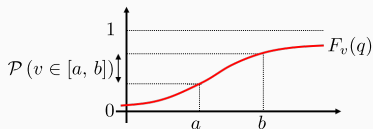# Probability Distribution & Density Functions

Probability distribution function

Provides information on the random variable v and it is defined as

$$F_v(q) = \mathcal{P}\left(v \leq q\right)$$

According to the definition $\qquad \mathcal{P}\left(v \in [a,\, b]\right) = F_v(b) - F_v(a)$

Probability density function

$$f_v(q) = \frac{d\,F_v}{d\,q}$$

Clearly $\mathcal{P}\left(v \in [a,\, b]\right)$ is the area "under" the diagram of $f(q)$ in the interval $[a,\, b]$.

A Glimpse on Probability Theory and Random Variables

Functions of Random Variables

Functions of Random Variables

- **Expected value (average value, average)**

$$\mathrm{E}\left(v\right) = \int_{-\infty}^{+\infty} q\, f_v(q) dq$$

- **Variance**

$$\mathrm{var}(v) = \int_{-\infty}^{+\infty} \left[q - \mathrm{E}(v)\right]^2 f_v(q)\, dq$$

- **Standard deviation**

$$\sigma(v) = \sqrt{\mathrm{var}(v)}$$

Tchebicev inequality

$$\mathcal{P}\left(|v - \mathrm{E}(v)| > \epsilon\right) \leq \frac{\mathrm{var}(v)}{\epsilon^2} \qquad \forall \epsilon > 0$$

Sum of random variables

Caution! Given two random variables $v_1(s)$, $v_2(s)$:

$$v(s) = v_1(s) + v_2(s) \implies \begin{array}{l} \mathrm{E}(v) = \mathrm{E}(v_1) + \mathrm{E}(v_2) \\ \mathrm{var}(v) \neq \mathrm{var}(v_1) + \mathrm{var}(v_2) \end{array}$$
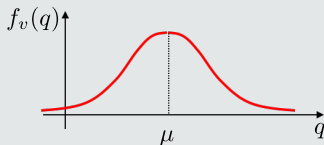
Important specific case: Gaussian random variable

A r.v. v is Gaussian if:

$$f_v(q) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(q-\mu)^2}{2\sigma^2}}$$

$$\mu = \mathrm{E}(v)$$

$$\sigma^2 = \mathrm{var}(v)$$

Vector Random Variable

- For example, given two random variables v_1, v_2 we can build a **random vector** in the obvious way:

$$v = \left[\begin{array}{c} v_1 \\ v_2 \end{array} \right]$$

- Consequently, expectation and variance of a random vector are

$$\mathrm{E}(v) \;=\; \left[\begin{array}{c} \mathrm{E}(v_1) \\ \mathrm{E}(v_2) \end{array} \right]$$

$$\mathrm{var}(v) \;=\; \mathrm{E}\left\{ [v - \mathrm{E}(v)] \, [v - \mathrm{E}(v)]^{\mathrm{T}} \right\}$$

Please note: $\mathrm{var}(v)$ is a matrix!

- In two dimensions

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad \mu_1 = \mathrm{E}(v_1) \,, \quad \mu_2 = \mathrm{E}(v_2)$$

- Therefore

$$\begin{aligned}
\mathrm{var}(v) &= \mathrm{E}\left\{ [v - \mathrm{E}(v)]\,[v - \mathrm{E}(v)]^{\mathrm{T}} \right\} \\
&= \mathrm{E}\left\{ \begin{bmatrix} v_1 - \mu_1 \\ v_2 - \mu_2 \end{bmatrix} \begin{bmatrix} v_1 - \mu_1 & v_2 - \mu_2 \end{bmatrix} \right\} \\
&= \mathrm{E}\begin{bmatrix} (v_1 - \mu_1)^2 & (v_1 - \mu_1)(v_2 - \mu_2) \\ (v_2 - \mu_2)(v_1 - \mu_1) & (v_2 - \mu_2)^2 \end{bmatrix} \\
&= \begin{bmatrix} \sigma_1^2 & \sigma_{1\,2} \\ \sigma_{2\,1} & \sigma_2^2 \end{bmatrix} = \Sigma \qquad \text{\textcolor{red}{variance matrix}}
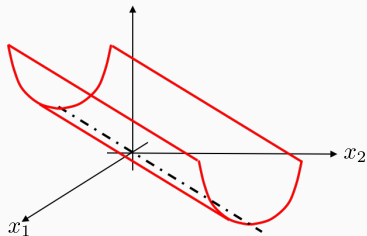\end{aligned}$$

covariance

- The matrix $\Sigma = \mathrm{var}(v)$ in general is symmetric and positive semidefinite
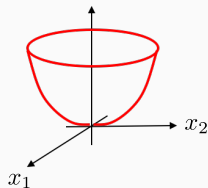
$$x^T \Sigma x \,, \quad \Sigma > 0$$

$$x^T \Sigma x \,, \quad \Sigma \geq 0$$

A Glimpse on Probability Theory and Random Variables

Random Variables: Correlation and Independence

Correlation and Independence

- Two random variables v_1, v_2 are uncorrelated if

$$\mathrm{E}\left\{[v_1 - \mathrm{E}(v_1)]\,[v_2 - \mathrm{E}(v_2)]\right\} = 0$$

that is $\quad \mathrm{E}\left(v_1\,v_2\right) = \mathrm{E}(v_1) \cdot \mathrm{E}(v_2)$

- Two random variables v_1, v_2 are independent if

$$f_{v_1,\,v_2}\left(a,\,b\right) = f_{v_1}\left(a\right) \cdot f_{v_2}\left(b\right)$$

Independence vs correlation

r.v. independent ➡ r.v. uncorrelated

Discrete-Time Stochastic Processes

Discrete-Time Stochastic Processes

Definition

Stochastic Processes

A **stochastic process** is a random phenomenon evolving over time according to a probabilistic law.

In practice: a two-variable function $v(t, s)$, where t is the time and s is the instance of the random experiment associated with the stochastic process.

Hence

- given $t = \bar{t}$, $v(\bar{t}, s)$ is a r.v. with a certain probability distribution
- given $\bar{s}$, $v(t, \bar{s})$ is a function of time that takes on the name of **realization** of the stochastic process
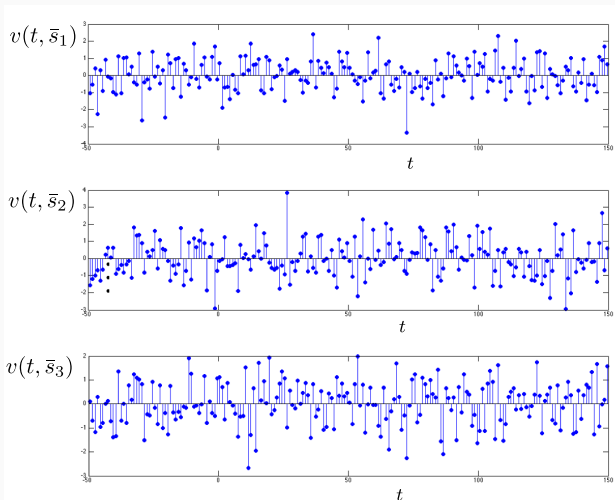
In practice a stochastic process is a set of infinite r.v. ordered with respect to time.
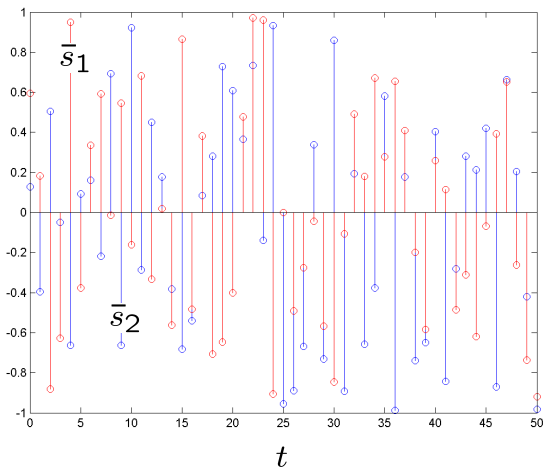
In practice a stochastic process is a set of infinite r.v. ordered with respect to time.

Discrete-Time Stochastic Processes

How To Describe a Stochastic Process? Stationary Stochastic Processes

Description of a Stochastic Process

- From a formal point of view, the full description of a stochastic process entails the knowledge of the probability distribution function:

$$\mathcal{P}\left[x(t_1) \leq x_1,\ x(t_2) \leq x_2,\ \cdots,\ x(t_k) \leq x_k\right]$$

for every arbitrary value of

$$k,\ x_1,\ x_2,\ \cdots,\ x_k,\ t_1,\ t_2,\ \cdots,\ t_k$$

- Such description is clearly not practical. Therefore, we assume that the stochastic process is fully described by the first- and second-order moments.
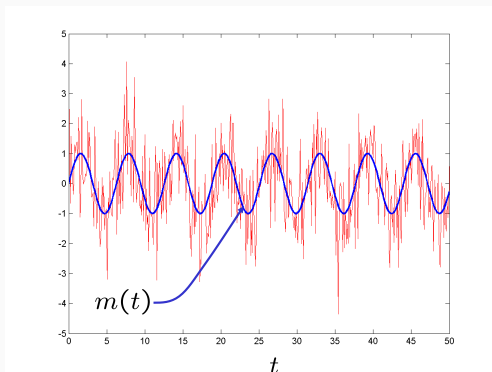
- First-order moment (**expected value** or **average**):

function of t

$$m(t) = \mathrm{E}\left[v_t(s)\right]$$

function
of s for
given t

- Second-order moment (**covariance function**):

function of t_1

function of t_2

$$\gamma(t_1, t_2) = \mathrm{E}\left\{\left[\; v_{t_1}(s) \;-\; m(t_1)\;\right]\left[\; v_{t_2}(s) \;-\; m(t_2)\;\right]\right\}$$

function of s
for given t_1

function of s
for given t_2

- **Correlation function**:

$$\mathrm{E}\left[v_{t_1}(s) \cdot v_{t_2}(s)\right]$$

Coincides with covariance function when $m(t) \equiv 0\ \forall t$.

Therefore:

> For our purposes, we assume that a stochastic process is fully described by first- and second-order moments: $m(t)$, $\gamma(t_1, t_2)$.

Two stochastic processes with the same first- and second-order moments are **undistinguishable by hypothesis**.

Stationary stochastic process

A stochastic process is stationary (in weak sense) if:

- $m(t) \equiv m = \text{const}$
- $\gamma(t_1, t_2) = \gamma(\tau), \quad \tau = t_2 - t_1$

This assumption greatly simplifies several derivations and, especially, implies the possibility of analyzing the probability distribution without caring about the specific time-instant.

- Consider a stationary stochastic process for which:
  - $m(t) \equiv m = \text{const}$
  - $\gamma(t_1, t_2) = \gamma(\tau), \quad \tau = t_2 - t_1$

Clearly, the variance of the process is $\gamma(0)$ and we define the **normalized covariance**:

$$\rho(\tau) = \frac{\gamma(\tau)}{\gamma(0)}$$

Discrete-Time Stochastic Processes

Gaussian Stochastic Processes

Gaussian Stochastic Processes

Gaussian processes

irrespective of the choice of the time-instants t_1, t_2, ..., t_N the random variables $v_{t_1}(s)$, $v_{t_2}(s)$, ..., $v_{t_N}(s)$ are jointly Gaussian, that is:

$$f\left(v_1,\, v_2,\, \ldots,\, v_N\right) = \alpha \exp\left\{-\frac{1}{2}\left(v-\mu\right)^T \Sigma^{-1}\left(v-\mu\right)\right\}$$

where

$$v = \left[v_1,\, v_2,\, \ldots,\, v_N\right]^T \qquad \mu = \mathrm{E}(v) \qquad \Sigma = \mathrm{var}(v)$$

Discrete-Time Stochastic Processes

White Stochastic Processes

White process

A stochastic process $\varepsilon(t)$ is defined **white** if

- $\mathrm{E}\left[\varepsilon(t)\right] = 0$
- $\gamma(\tau) = \begin{cases} \lambda^2, & \tau = 0 \\ 0, & \tau \neq 0 \end{cases}$

and we denote: $\quad \varepsilon \sim \mathsf{WN}\left(0, \, \lambda^2\right)$

In a white process what happens at different time-instants is unrelated, thus the knowledge of $\varepsilon(t)$ does not help in gaining knowledge about $\varepsilon(t+1)$.

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