October 26

$L^p((0,1),\mathbb{C})$ $\quad 1 \le p \le \infty$

$m(x) \in C^0([0,1],\mathbb{C})$

$$f \longrightarrow m(x) f = T_m f$$

$$\sigma(T) = m([0,1]).$$

Hahn Banach

<u>Thm</u> X vector space on $\mathbb{R}$, $p : X \to [0,+\infty)$ a seminorm, $Y \subseteq X$ a vector space and $g : Y \longrightarrow \mathbb{R}$ linear s.t.

$$g(y) \le p(y) \quad \forall \; y \in Y.$$

Then $\exists\ f: X \to \mathbb{R}$ linear s.t.

$f|_Y = g$ and

$$f(x) \leq p(x) \quad \forall x \in X.$$

<u>Pf</u> Let $x_0 \notin Y$ $\quad \mathbb{R}x_0 \oplus Y$

in particular any $x \in \mathbb{R}x_0 + Y$ in a unique way $x = t x_0 + y \quad y \in Y$.

$$f(tx_0 + y) = t f(x_0) + f(y) =$$
$$= t\alpha + g(y)$$

We choose α so that

$$f(tx_0+y) \leq p(tx_0+y) \quad \forall t \in \mathbb{R} \quad \forall y \in Y$$

$$\boxed{(1) \quad t\alpha + g(y) \leq p(tx_0+y) \quad \forall t \in \mathbb{R} \quad \forall y \in Y}$$

To get for $t > 0$ it is enough

$$(2) \quad \alpha + g(y) \leq p(x_0+y) \quad \forall y \in Y$$

$$t\alpha + g(y) \leq p(tx_0+y) \qquad \frac{1}{t} \quad t > 0$$

$$\alpha + g\left(\frac{y}{t}\right) \le \frac{1}{t} p(tx_0 + y) \cancel{\le} p\left(x_0 + \frac{y}{t}\right) \ \&$$

this is equivalent to (2)

To get (1) for $t < 0$ is equivalent to

(3) $$-\alpha + g(y) \le p(-x_0 + y) \quad \forall y \in Y$$

$$t\alpha + g(y) \le p(tx_0 + y) \quad \forall t < 0, \ \forall y \in Y$$

$$\Updownarrow$$

$$-t\alpha + g(y) \le p(-tx_0 + y) \quad \forall t > 0, \ \forall y \in Y$$

$$-\alpha + g(y) \le p\left(-x_0 + \frac{y}{t}\right) \quad \forall t \ge 0, \ \forall y \in Y$$

$$-p(-x_0 + y) + g(y) \le \alpha \le p(x_0 + y) - g(y) \quad \forall y \in Y$$

$\forall \ y_1, y_2 \in Y$

$$-p(-x_0 + y_1) + g(y_1) \le p(x_0 + y_2) - g(y_2)$$

$$* \quad g(y_1) + g(y_2) \leq P(x_0 + y_2) + P(-x_0 + y_1)$$

$$g(y_1 + y_2) \leq P(y_1 + y_2) = P(-x_0 + y_1 + x_0 + y_2)$$

$$\leq P(-x_0 + y_1) + P(x_0 + y_2)$$

So $\alpha \in \mathbb{R}$ exists and we get the desired extension of g into

$$f : \mathbb{R} x_0 + Y \longrightarrow \mathbb{R}$$

We consider the set of pairs $(\mathcal{P}, \leq)$

(h, D)

$Y \subseteq D \subseteq X$ is a vector space

$h : D \longrightarrow \mathbb{R}$ linear

$h|_Y = g$

$h(x) \leq p(x) \quad \forall x \in D$

$(h, D) \leq (h_1, D_1)$ if

$D_1 \supseteq D$ and $h_1|_D = h$

$(\mathcal{P}, \leq)$ is "inductive

If $\mathcal{Q}$ is a totally ordered subset of $\mathcal{P}$ then in $\mathcal{P}$ there is an element larger than all the elements

$\mathcal{Q} \ni (h_q, D_q)_{q \in \mathcal{Q}}$

$$D = \bigcup_{q \in \mathcal{Q}} D_q \xrightarrow{h} \mathbb{R}$$

By Zorn's Lemma in $\mathcal{P}$ there are maximal elements

(D, h) $\qquad h|_Y = g \qquad D \supseteq Y$

$$h(x) \leq p(x) \quad \forall x \in D$$

without strict extension

If $D = X$ the proof is finished

If $D \neq X$ and $x_0 \notin D$

then we can define a strict

extension on $\mathbb{R} x_0 \oplus D$

But this ~~will~~ contradicts the fact that (h, D) is maximal

$$\boxed{D = X}.$$

Corollary $(X, \|\cdot\|_X)$ $\quad$ $K = \mathbb{R}, \mathbb{C}$ $\quad$ $Y \subseteq X$

If $g : Y \to K$ is a linear functional, then $\exists$ $f \in X'$ s.t.

$$f|_Y = g$$

$$\|f\|_{X'} = \sup\{|f(x)| : \|x\|_X \le 1\}$$
$$\geqq \underbrace{\sup\{|g(y)| : \|y\|_X \le 1\}}_{c_0}$$

Pf $\quad g(y) \le c_0 \|y\| \quad \forall\, y \in Y$

$\exists\, f : X \to \mathbb{R}$, $f|_Y = g$

$$* \quad f(x) \leq c_0 \|x\| \qquad \forall x \in X$$

$$\Longrightarrow \quad |f(x)| \leq c_0 \|x\| \qquad \forall x \in X$$

If this was false

$$f(x) < -c_0 \|x\| \qquad \cdot \sim 1$$

$$f(-x) > c_0 \|-x\| \quad \text{contradicting } *$$

$$\Longrightarrow \quad \cancel{\ker_X} \; f \in X'$$

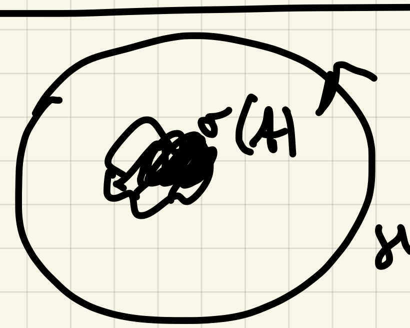
$$\|f\|_{X'} \leq c_0 \Longrightarrow \|f\|_{X'} = c_0$$

<u>Corollary</u> $(X, \|.\|)$ then $\forall x_0 \in X$

$\exists f \in X'$ st

$$\|f\|_{X'} = \|x_0\|_X \quad , \quad f(x_0) = \|x_0\|_X^2$$

$$\left(\|f\|_{X'} = 1 \qquad f(x_0) = \|x_0\|_X \right)$$



$$-\frac{1}{2\pi i} \int_\gamma R_A(z)\, dz = 1$$

$f(A)$.

$$f(z) = \sum a_n z^n$$

$$f(A) = \sum a_n A^n$$

$$-\frac{1}{2\pi i}\int_\gamma f(z)\, R_A(z)\, dz = f(A)$$

$L^p(0,1) \qquad 0 < p < 1$

$$\left(L^p(0,1)\right)' = 0$$

$Y = \mathbb{R}\, x_0$

$$g(\lambda x_0) = \lambda \, \|x_0\|_X^2$$

$$\|g\|_{Y'} = \sup\{ |\lambda| \, \|x_0\|_X^2 : \|\lambda x_0\|_X \le 1\}$$

$$\left(= \sup\{ |g(y)| : \|y\|_X \le 1\}\right)$$

$$|\lambda| \, \|x_0\|_X = 1 \qquad |\lambda| = \frac{1}{\|x_0\|_X}$$

$$\|g\|_{Y'} = |\lambda| \, \|x_0\|_X^2 = \|x_0\|_X$$

$f \in X'$ extends g

$$\|f\|_{X'} = \|x_0\|_X$$

$$f(x_0) = \|x_0\|_X^2 .$$

$$T : X \longrightarrow Y \xrightarrow{\;y'\;} \mathbb{R}$$

$$T^* : Y' \longrightarrow X'$$

$$y' \in Y' \longrightarrow \underset{T^* y'}{\overset{}{y' \circ T}} \in X'$$

$$T^* \in \mathcal{L}(Y', X')$$

$$\|T^*\|_{\mathcal{L}(Y',X')} = \|T\|_{\mathcal{L}(X,Y)}$$

$$\langle T^* y', x \rangle_{X' \times X} = \langle y' \circ T, x \rangle_{X' \times X} =$$
$$= \langle y', Tx \rangle_{Y' \times Y}$$

$$\|T^*\| = \sup\{\|T^*y'\|_{X'} : \|y'\|_{Y'} = 1\}$$

$$\|T^*y'\|_{X'} \overset{\|y'\|_{Y'}=1}{=}$$

$T^*y'(x) = y' \circ T(x) = y'(Tx)$

$$= \sup\{|\langle T^*y', x\rangle_{X' \times X}| : \|x\|_X = 1\}$$

$$= \sup\{|\langle y', Tx\rangle_{Y' \times Y}| : \|x\|_X = 1\}$$

$$\leq \sup\{\cancel{\|y'\|_{Y'}} \|Tx\|_Y : \|x\|_X = 1\}$$

$$= \|T\| \implies \|T^*\|_{\mathcal{L}(Y',X')} \leq \|T\|_{\mathcal{L}(X,Y)}$$

$\geq$

$$\|T\| = \sup\{\|Tx\|_Y : \|x\|_X = 1\}$$

$$\|Tx\|_Y = \sup\{|\langle Tx, y'\rangle_{Y' \times Y}| : \|y'\|_{Y'} = 1\}$$

$$= \sup\{|\langle x, T^*y'\rangle_{X \times X'}| : \|y'\|_{Y'} = 1\}$$

$$\leq \sup\{\cancel{\|x\|_{X'}} \|T^*y'\|_{X'} : \|y'\|_{Y'} = 1\}$$

$$= \|T^*\|$$

$$\implies \|T\| \leq \|T^*\|$$

Corollary $U = D_{\mathbb{C}}(0,1)$ $\mathbb{T} = \partial U$

Let $A \subseteq C^0(\bar{U}, \mathbb{C})$ a vector space

$A \supseteq \mathbb{C}[z]$ and assume

$$\| f \|_{L^\infty(\bar{U})} = \| f \|_{L^\infty(\mathbb{T})}$$

$$\forall f \in A.$$

Then $\forall$ $z \in U$

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1-|z|^2}{|z-e^{it}|^2} f(e^{it})\, dt$$

$$\Delta \quad \frac{1-|z|^2}{|z-z_0|^2} = 0 \qquad z_0 \in \mathbb{T}$$

$$\Delta = \partial_x^2 + \partial_y^2$$