Martedì 25 pomeriggio

$$\lim_{x \to +\infty} \left(x - \sqrt{x^2+1} \right) = 0 \qquad (+\infty) - (+\infty) = ?$$

$$\lim_{x \to +\infty} x = +\infty$$

$$\boxed{\forall K \ \exists N(K) \ \text{t.c.} \quad x > N(K) \Rightarrow x > K}$$

E' vero, basta usare $N(K) = K$.

$$\lim_{x \to +\infty} \sqrt{x^2+1} = \lim_{x \to +\infty} \sqrt{x^2\left(1+\frac{1}{x^2}\right)}$$

$$= \lim_{x \to +\infty} x \sqrt{1+\frac{1}{x^2}} = +\infty$$

$$a^2 - b^2 = (a-b)(a+b)$$

$$x - \sqrt{x^2+1} = \left(x - \sqrt{x^2+1}\right) \frac{x+\sqrt{x^2+1}}{x+\sqrt{x^2+1}} =$$

$$= \frac{x^2 - \left(\sqrt{x^2+1}\right)^2}{x+\sqrt{x^2+1}} = \frac{\cancel{x^2} - (\cancel{x^2}+1)}{x+\sqrt{x^2+1}}$$

$$= \frac{-1}{x+\sqrt{x^2+1}} \xrightarrow{x \to +\infty} \frac{-1}{+\infty} = 0$$

$$\lim_{x\to+\infty}\left(-2x^3+x^2-x+1\right)=\lim_{x\to+\infty}\left(-2x^3\right)=-\infty$$

$$-2x^3+x^2-x+1=-2x^3\left(1+\frac{x^2}{-2x^3}-\frac{x}{-2x^3}+\frac{1}{-2x^3}\right)$$

$$=\underbrace{-2x^3}_{\xrightarrow{x\to+\infty}\ -\infty}\left(1\underbrace{-\frac{1}{2x}}_{\to 0}+\underbrace{\frac{1}{2x^2}}_{\to 0}\underbrace{-\frac{1}{2x^3}}_{\to 0}\right)\xrightarrow{x\to+\infty}-\infty$$

$$\lim_{x\to+\infty}\frac{x^6+x+1}{-x^6+x^5-x^3-1}=\lim_{x\to+\infty}\frac{x^6}{-x^6}=\lim_{x\to+\infty}(-1)=-1$$

$$\lim_{x\to+\infty}\frac{x^7+x^4+1}{-x^5-x^3+1}=\lim_{x\to+\infty}\frac{x^7}{-x^5}=\lim_{x\to+\infty}\left(-x^2\right)=-\infty$$

$$\frac{x^6+x+1}{-x^6+x^5-x^3-1}=\frac{x^6\left(1+\frac{1}{x^5}+\frac{1}{x^6}\right)}{-x6\left(1-\frac{1}{x}+\frac{1}{x3}+\frac{1}{x6}\right)}$$

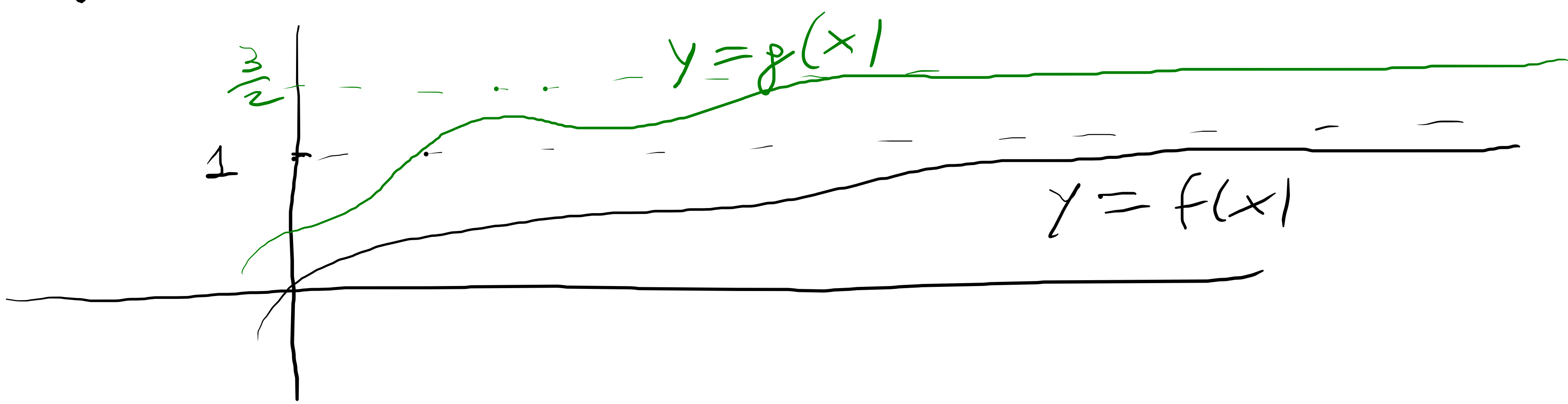
$$=\frac{x^6}{-x6}\;\frac{1+\frac{1}{x5}+\frac{1}{x6}}{1-\frac{1}{x}+\frac{1}{x^3}+\frac{1}{x6}}$$

Teor (Confronto) Siano $f, g: X \xrightarrow{} \overline{\mathbb{R}}$, $X \subseteq \mathbb{R}$, $\sup X = +\infty$

siano $\lim_{x \to +\infty} f(x) = a \in \overline{\mathbb{R}}$, $\lim_{x \to +\infty} g(x) = b \in \overline{\mathbb{R}}$

Supponiamo che $f(x) \leq g(x)$ $\forall x \in X$.

Allora $a \leq b$.

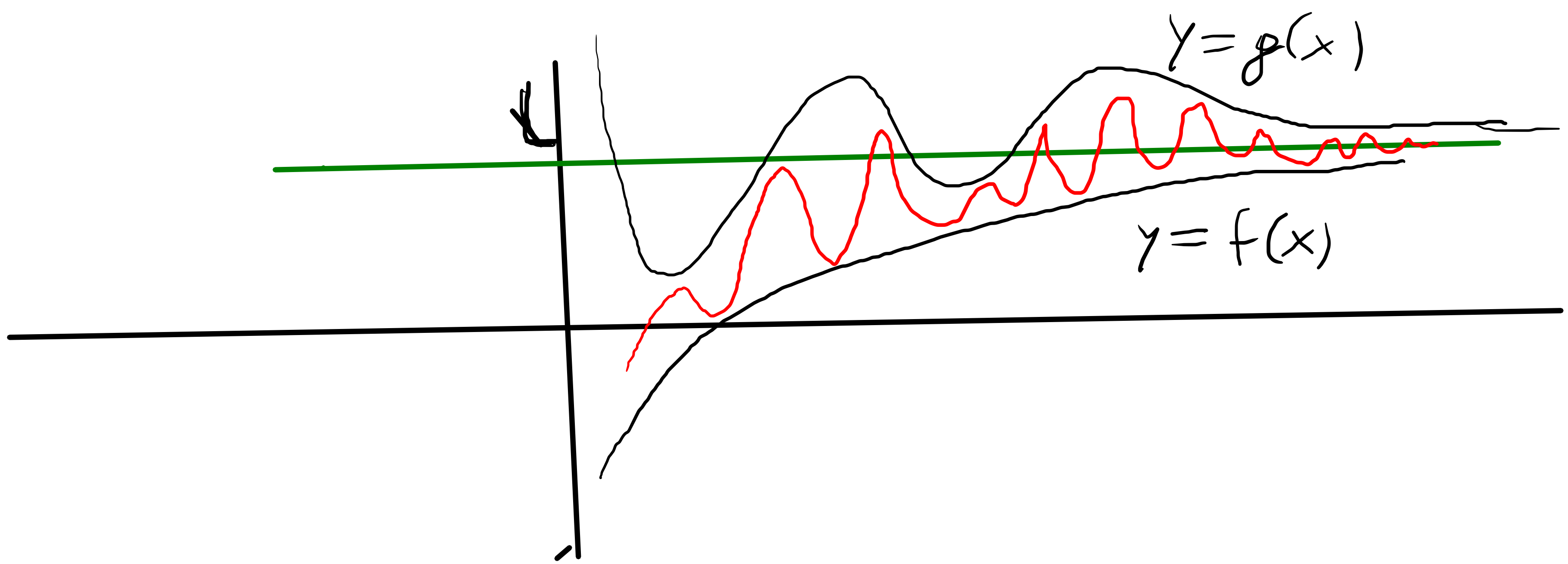


Teor (Carabinieri) Siano $f, g, h : X \xrightarrow{\subseteq \mathbb{R}} \mathbb{R}$ $\sup X = +\infty$

con $\quad f(x) \leq h(x) \leq g(x) \quad \forall x \in X.$

Supponiamo che $\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} g(x) = L \in \overline{\mathbb{R}}$

Allora $\lim\limits_{x \to +\infty} h(x) = L$



Sia $b \in \mathbb{R}_+$ $\lim_{n \to +\infty} b^{\frac{1}{n}} = 1$

Verifichiamo cominciando con $b > 1 \implies b^{\frac{1}{n}} > 1 \quad \forall n \in \mathbb{N}$

$$b^{\frac{1}{n}} = 1 + a_n \quad \text{dove} \quad a_n > 0 \; \forall n \in \mathbb{N}.$$

$$\lim_{n \to +\infty} b^{\frac{1}{n}} = 1 \iff \lim_{n \to +\infty} a_n = 0$$

$$b = \left(b^{\frac{1}{n}}\right)^n = (1 + a_n)^n \geq 1 + n a_n \qquad \text{Bernoulli}$$

$$b \geq 1 + n a_n$$

$$0 < a_n \leq \frac{b-1}{n} \qquad \forall n \in \mathbb{N}$$

$$\downarrow \qquad\qquad \downarrow$$

$$0 \qquad\qquad 0$$

$$\implies \lim_{n \to +\infty} a_n = 0 \implies \lim_{n \to +\infty} b^{\frac{1}{n}} = 1$$

se $b > 1$

Se $0 < b < 1$ $\lim_{n \to +\infty} b^{\frac{1}{n}} = 1$

$$b^{\frac{1}{n}} = \left(\frac{1}{\left(\frac{1}{b}\right)}\right)^{\frac{1}{n}} \qquad b = \frac{1}{\frac{1}{b}} \qquad \frac{1}{b} > 1$$

$$\Downarrow$$

$$= \frac{1}{\left(\frac{1}{b}\right)^{\frac{1}{n}}} \xrightarrow{n \to +\infty} \frac{1}{1} \qquad \lim_{n \to +\infty} \left(\frac{1}{b}\right)^{\frac{1}{n}} = 1$$

Dim teor dei Carab. Solo nel caso $L \in \mathbb{R}$.

$$f(x) \le h(x) \le g(x)$$

$\lim_{x \to +\infty} f(x) = L$ significa

$\forall \varepsilon > 0 \ \exists N_{1\varepsilon}$ t.c. $x > N_{1\varepsilon}$ e $x \in X \Rightarrow |f(x) - L| < \varepsilon$

$\Rightarrow -\varepsilon < f(x) - L < \varepsilon$

$\lim_{x \to +\infty} g(x) = L$ significa

$\forall \varepsilon > 0 \ \exists N_{2\varepsilon}$ t.c. $x > N_{2\varepsilon}$ e $x \in X \Rightarrow -\varepsilon < g(x) - L < \varepsilon$

Posto $N_\varepsilon = \max(N_{1\varepsilon}, N_{2\varepsilon})$, se $x > N_\varepsilon$ e $x \in X$

risulta

$$L - \varepsilon < f(x) \le h(x) \le g(x) < L + \varepsilon$$

Pertanto $x > N_\varepsilon$ e $x \in X$ $\Rightarrow$ $L - \varepsilon \le h(x) < L + \varepsilon$

$$\Updownarrow$$

$$-\varepsilon < h(x) - L < \varepsilon$$

$$\Updownarrow$$

$$|h(x) - L| < \varepsilon$$

Conclusione. Abbiamo dimostrato che

$\forall \varepsilon > 0 \ \exists N_\varepsilon \ (= \max(N_{1\varepsilon}, N_{2\varepsilon}))$

t.c. $x > N_\varepsilon$ e $x \in X \Rightarrow |h(x) - L| < \varepsilon$

$$\Longrightarrow \lim_{x \to +\infty} h(x) = L.$$

Sia $b > 1$. Allora $\lim\limits_{n\to+\infty} \frac{b^n}{n} = +\infty$

Sappiamo $b>1 \Longrightarrow \lim\limits_{n\to+\infty} b^n = +\infty$

E' ... ovvio $\lim\limits_{n\to+\infty} n = +\infty$

$b > 1 \Longrightarrow \sqrt{b} > 1 \Longrightarrow \sqrt{b} = 1 + a$ con $a > 0$

$$+\infty > \frac{b^n}{n} = \frac{\left(\left(\sqrt{b}\right)^2\right)^n}{n} = \frac{\left((1+a)^2\right)^n}{n} = \frac{\left((1+a)^n\right)^2}{n}$$

$$\geq \frac{(1+na)^2}{n} = \qquad \left((1+a)^n\right)^2 \geq (1+na)^2$$

$$= \frac{1+2na+n^2a^2}{n} \xrightarrow{n\to+\infty} +\infty$$

$$\lim_{n\to+\infty} \frac{b^n}{n} = +\infty \qquad +\infty > \frac{b^n}{n} \geq \frac{1+2na+n^2a^2}{n} \to +\infty$$

$$\lim_{n\to+\infty} \frac{b^n}{n^2} = +\infty$$

$$\lim_{n\to+\infty} \frac{b^n}{n^{10^{10}}} = +\infty$$

$$\lim_{n\to+\infty} \frac{b^n}{n^N} = +\infty \qquad \forall\, N \in \mathbb{N}$$