Advanced Geometry 3 - Lecture Notes

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Contents

Chapter 1

Regular Functions and Regular Maps on Affine Varieties

1.1 Regular Functions

Definition 1.1. *Let $X \subseteq \mathbb{A}^n$ be an affine variety. A function $f\colon X \to \mathbb{K}$ is* **regular** *it is a polynomial function, that is there exists a polynomial $F \in \mathbb{K}[x_1, \dots, x_n]$ such that*

$$f(a_1, \dots, a_n) = F(a_1, \dots, a_n)$$

for any $(a_1, \dots, a_n) \in X$.

Remark 1.2. *We already noticed that that the set of polynomial functions on an affine variety form a $\mathbb{K}$-algebra, which is isomorphic to the* **coordinate ring**

$$A(X) := \frac{\mathbb{K}[x_1, \dots, x_n]}{I(X)}.$$

Moreover, since X is irreducible, we have that $I(X)$ is prime, so that $A(X)$ is an integral domain. Furthermore, since $\mathbb{K}[x_1, \dots, x_n]$ is noetherian, so is $A(X)$.

Examples 1.3. *Assume $\mathbb{K}$ to be an infinite filed.*

1. *$A(\mathbb{A}^n) = \mathbb{K}[x_1, \dots, x_n]$; indeed, we have $I(\mathbb{A}^n) = (0)$.*
2. *Let $X = V(y - x^2) \subseteq \mathbb{A}^2_{\mathbb{C}}$ be the parabola. We have $I(V(y - x^2)) = \sqrt{\langle y - x^2 \rangle}$ by the NSS, and since $y - x^2$ is an irreducible polynomial, the ideal $\langle y - x^2 \rangle$ is prime, hence radical and $\sqrt{\langle y - x^2 \rangle} = \langle y - x^2 \rangle$. It can be easily checked that*

   $$\begin{aligned} A(X) = \frac{\mathbb{K}[x, y]}{\langle y - x^2 \rangle} &\xrightarrow{\cong} \mathbb{K}[x] \\ \overline{F}(x, y) &\longmapsto F(x, x^2). \end{aligned}$$

   Finally, we observe that X admits the parametrization

   $$V(y - x^2) = \{(t, t^2) : \ t \in \mathbb{A}^1\},$$

   so that is the image of the map $\varphi : \mathbb{A}^1 \to \mathbb{A}^2, \ \varphi(t) = (t, t^2)$.
3. *Let $X = V(x^2 - y^3) \subseteq \mathbb{A}^2_{\mathbb{C}}$ be the cuspidal cubic. Such a curve admits the parametrization*

   $$V(x^2 - y^3) = \{(t^3, t^2) : \ t \in \mathbb{A}^1\}.$$

We observe that in this case

$$A(X) = \frac{\mathbb{K}[x,y]}{\langle x^2 - y^3 \rangle} \ncong \mathbb{K}[t].$$

Indeed, we have that $\mathbb{K}[t]$ is an UFD, while $A(X)$ is not; for instance

$$\overline{x}^2 = \overline{xx} = \overline{yyy}$$

with $\overline{x}, \overline{y}$ irreducible elements. Hence $\overline{x}^2$ admits two different factorizations in irreducible factors.

Definition 1.4. *The element $\overline{x}_i \in A(X)$, class of x_i modulo $I(X)$, will be called the* i**-th coordinate function**. *Under the identification with a polynomial function it corresponds to*

$$\overline{x}_i \colon X \longrightarrow \mathbb{K};\ P = (p_1, \ldots, p_n) \longmapsto p_i.$$

Remark 1.5. *The $\mathbb{K}$-algebra $A(X)$ is finitely generated by $\overline{1}, \overline{x_1}, \ldots, \overline{x_n}$.*

Conversely, we can see that any finitely generated $\mathbb{K}$-algebra A, which is an integral domain, is isomorphic to the coordinate ring of an affine variety. Indeed, we have an exact sequence

$$0 \longrightarrow I \longrightarrow \mathbb{K}[x_1, \ldots, x_n] \xrightarrow{\varphi} A \longrightarrow 0$$

where $I := ker\varphi$, and $A \cong \frac{\mathbb{K}[x_1,\ldots,x_n]}{I}$. Since A is an integral domain, the ideal I is prime. By defining $X := V(I) \subseteq \mathbb{A}^n$, we have $A \cong A(X)$.

1.2 Regular Maps

Definition 1.6. *Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be affine varieties. A map $f \colon X \to Y$ is a* **regular map** *or* **morphism** *if there exist $f_1, \ldots, f_m \in A(X)$ such that*

$$f = (f_1, \ldots, f_m),$$

so that $f(P) = (f_1(P), \ldots, f_m(P)) \in Y$, for any $P \in X$.

Moreover, we say that f is an **isomorphism** *if it is bijective regular map and its inverse is a regular map.*

Proposition 1.7. *Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be affine varieties and let $f : X \to Y$ be regular. Then f is continuous in the Zariski topology.*

Proof. Let $X = V(g_1, \ldots, g_s),\ f_i \in \mathbb{K}[x_1, \ldots, x_n]$ and $Y = V(h_1, \ldots, h_r), G_i \in \mathbb{K}[y_1, \ldots, y_m]$. A closed subset of Y is of the form $Z = Y \cap V(m_1, \ldots, m_k)$ with $m_i \in \mathbb{K}[y_1, \ldots, y_m]$. Then we have:

$$\begin{aligned} P \in f^{-1}(Z) &\Leftrightarrow f(P) \in Z \Leftrightarrow m_1(f(P)) = \cdots = m_k(f(P)) = 0 \\ &\Leftrightarrow m_1(f_1(P), \ldots, f_m(P)) = \cdots = m_k(f_1(P), \ldots, f_m(P)) = 0 \\ &\Leftrightarrow m_1(f_1, \ldots, f_m)(P) = \cdots = m_k(f_1, \ldots, f_m)(P) = 0. \end{aligned}$$

Since this holds for any $P \in f^{-1}(Z)$, it follows that

$$f^{-1}(Z) = V(m_1(f_1, \ldots, f_m), \ldots, m_k(f_1, \ldots, f_m)) \cap X,$$

which is closed in X. □

Remark 1.8. *We observe that the composition of a regular map with a regular function is again a regular function. Indeed, if $f = (f_1, \ldots, f_m) : X \to Y$ and $h : Y \to \mathbb{K}$, we have that*

$$h \circ f = h(f_1, \ldots, f_m)$$

is a polynomial function.

This allows to define the **pull-back** *$f^\star$ of a regular map $f : X \to Y$:*

$$f^\star : A(Y) \to A(X), \quad f^\star(h) := h \circ f.$$

Note that $f^\star$ is a $\mathbb{K}$-algebra homomorphism.

The pull-back of a map determines a contravariant functor, with the following property.

Proposition 1.9. *The functor associating the coordinate ring $A(X)$ to any affine variety X, and the $\mathbb{K}$-algebra homomorphism $f^\star : A(Y) \to A(X)$ to any regular map $f : X \to Y$, determines a category equivalence between the categories $\mathcal{A} = \{X \text{ affine variety}, f \text{ regular map}\}$ and $\mathcal{B} = \{A(X), u\mathbb{K}\text{-algebra homomorphism}\}$.*

Proof. We need to prove that we have a bijection between $Hom(X,Y)$ and $Hom(A(Y),A(X))$. To this aim, given $u \in Hom(A(Y),A(X))$, we define

$$\begin{aligned} u^{\#} \colon X &\to Y \\ P &\to (u(\overline{y}_1)(P), \ldots, u(\overline{y}_n)(P)) \end{aligned},$$

where $\overline{y_1}, \ldots, \overline{y_n}$ are the coordinate functions on Y.

We observe that $u^{\#}$ is a regular map, since its components are elements of $A(X)$. We need to check now that $u^{\#}(X) \subseteq Y$. Let $g \in I(Y)$ and let us evaluate it in a generic point of $u^{\#}(X)$:

$$g(u^{\#}(P)) = g(u(\overline{y_1})(P), \ldots, u(\overline{y_n})(P)) = g(u(\overline{y_1}), \ldots, u(\overline{y_n}))(P),$$

since the evaluation in a point is a homomorphism. Next we note that $g(u(\overline{y}_1), \ldots, u(\overline{y}_n))$ is a sum of products of images of u, and since u is a $\mathbb{K}$-algebra homomorphism, we have

$$g(u(\overline{y_1}), \ldots, u(\overline{y_n})) = u(g(\overline{y_1}, \ldots, \overline{y_n}).$$

Finally, by the definition of sum and product in a quotient ring, we have

$$u(g(\overline{y_1}, \ldots, \overline{y_n}) = u(\overline{g(y_1, \ldots, y_n)}).$$

By recalling that $g \in I(Y)$ we see that $\overline{g} = 0$ in $A(Y)$, hence

$$g(u^{\#}(P)) = g(u(\overline{y_1})(P), \ldots, u(\overline{y_n})(P)) = u(0)(P) = 0.$$

It follows that $u^{\#}(X) \subseteq V(I(Y)) = Y$.

Finally, it is not difficult to check that $(f^*)^{\#} = f$ e $(u^{\#})^* = u$. The equivalence of the two categories follows. □

Corollary 1.10. *Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be affine varieties. Then X is isomorphic to $Y \iff A(X) \cong A(Y)$.*

Examples 1.11.

1. *The map $\varphi : \mathbb{A}^1 \to V(y - x^2)$ is an isomorphism between the affine line and the parabola, since it induces an isomorphism of coordinate rings.*

2. *The map $f \colon \mathbb{A}^1 \to V(y^2 - x^3);\ t \mapsto (t^2, t^3)$ is a regular bijective map, but it is not an isomorphism, since the coordinate rings are not isomorphic.*

3. *Let $\Delta = \{(P,P);\ P \in \mathbb{A}^n\} \subseteq \mathbb{A}^n \times \mathbb{A}^n$ be the diagonal. Then the* **diagonal map**

$$\begin{aligned} \delta \colon \mathbb{A}^n &\longrightarrow \Delta \\ P &\longmapsto (P,P) \\ (p_1, \ldots, p_n) &\longmapsto (p_1, \ldots, p_n, p_1, \ldots, p_n). \end{aligned}$$

is a regular bijective map, since its components are coordinate functions. Moreover, the inverse map $\delta^{-1} : \Delta \to \mathbb{A}^n$ is the restriction of the first projection, hence it has regular components too, so δ is an isomorphism.

We shall make use of the diagonal isomorphism in the following framework: let $X, Y \subseteq \mathbb{A}^n$ be affine varieties; then

$$\delta_{|X \cap Y} \colon X \cap Y \to (X \times Y) \cap \Delta$$

is an isomorphism too.

1.3 Relative NSS

Remark 1.12. *We observe that given an affine variety $X = V(g_1, \dots, g_r) \subseteq \mathbb{A}^n$, the induced Zariski topology is determined by ideals in $A(X)$, and precisely: if $\alpha \subseteq A(X)$ is an ideal, by setting*

$$V(\alpha) := \{P \in X;\ h(P) = 0,\ \forall h \in \alpha\},$$

we obtain all the Zariski closed subsets of X.

Indeed, if $\alpha = \langle h_1, \dots, h_m \rangle$, there exist $H_i \in \mathbb{K}[x_1, \dots, x_n]$ such that $h_i = \overline{H}_i$ and we have

$$V(\alpha) = V(h_1, \dots, h_m) = V(H_1, \dots, H_m) \cap X.$$

Similarly, we can define the ideal of a subset of X: if $S \subseteq X$, we set $I(S) := \{g \in A(X);\ g_{|S} \equiv 0\}$.

Proposition 1.13 (Relative NSS)**.** *If $\mathbb{K} = \overline{\mathbb{K}}$, then in $A(X)$ it holds*

$$I(V(\alpha)) = \sqrt{\alpha} \subseteq A(X).$$

In particluar, there is relative version of the Weak NSS:

$$V(\alpha) = \emptyset \Leftrightarrow \alpha = A(X).$$

Proof. The inclusion $I(V(\alpha)) \supseteq \sqrt{\alpha}$ is easy. For the converse, let $X = V(G_1, \dots, G_r)$ and let $\alpha = \langle h_1, \dots, h_m \rangle$ and let $h \in I(V(\alpha))$, with $h = \overline{H}$ for some $H \in \mathbb{K}[x_1, \dots, x_n]$. By Remark **??** we have $V(\alpha) = V(H_1, \dots, H_m) \cap X = V(H_1, \dots, H_m, G_1, \dots, G_r)$. By the NSS we have

$$H \in I(V(H_1, \dots, H_m, G_1, \dots, G_r)) = \sqrt{(H_1, \dots, H_m, G_1, \dots, G_r)},$$

so there exists $k \in \mathbb{N}$ such that $H^k \in (H_1, \dots, H_m, G_1, \dots, G_r)$. This implies that $\overline{H}^k = h^k \in \alpha \subseteq A(X)$, hence $H \in \sqrt{\alpha}$ and the claim follows. □

1.4 Rational Functions on Affine Varieties

Definition 1.14. *Let $X \subseteq \mathbb{A}^n$ be an affine variety. The* **field of rational functions** *is the quotient field $Q(A(X))$ of $A(X)$, and we will denote it by $\mathbb{K}(X)$. So we have*

$$\mathbb{K}(X) = \left\{ \frac{f}{g}; f, g \in A(X),\ g \neq 0 \right\},$$

where the condition $g \neq 0$ in $A(X)$ means that g is not the zero constant function, and we have

$$\frac{f}{g} = \frac{f'}{g'} \iff f \cdot g' = f' \cdot g \in A(X).$$

In contrast with the regular functions, the rational functions do not define functions on the whole of X.

Definition 1.15. *Let X be an affine variety and let $h \in \mathbb{K}(X)$. We say that h is* **regular** *in $P \in X$ if h admits an expression*

$$h = \frac{f}{g},\ f, g \in A(X),\ g(P) \neq 0.$$

In thsi case we can define the value of h in $P \in X$ as $h(P) := f(P)/g(P) \in \mathbb{K}$.

Observe that such a value is well-defined: if $h = \frac{f}{g} = \frac{f'}{g'}$, with $g(p) \neq 0, g'(p) \neq 0$, then $fg' - f'g = 0$ in $A(X)$, that is $FG' - F'G \in I(X)$, where $\bar{F} = f, \bar{G} = g, barF' = f', \bar{G}' = g'$. Hence for any $P \in X$, we have $F(P)G'(P) - F'(P)G(P) = 0$. It follows that

$$h(P) = \frac{f(P)}{g(P)} = \frac{f'(P)}{g'(P)}.$$

Remark 1.16. *Observe that $h \in \mathbb{K}(X)$ is regular on an open subset of X. Indeed, we fix an expression $h = f/g$ with $f, g \in A(X)$; by setting $U := X \setminus V(g)$, we get that h is regular in any $P \in U$, so it defines a function $h\colon U \subseteq X \to \mathbb{K}$. By choosing spme ather representation of h, the domain of regularity can change, but it is still open. The union of all the regularity domains of all representations of h is colled the* **domain of** *h.*

Example 1.17 (Stereographic projection)**.** *Let $X = V(x_1^2 + x_2^2 - 1) \subseteq \mathbb{A}^2$, the unit circle; it is an affine variety. Let*

$$h := \frac{\overline{x}_1}{1 - \overline{x}_2} \in \mathbb{K}(X).$$

In this case $g = 1 - \overline{x}_2$, and

$$V(g) = \{(0, 1)\} \subseteq X.$$

Then the domain of h contains the open subset $X \setminus (0, 1)$. Now we try with another representation of h:

$$h = \frac{(1 + \overline{x}_2)}{\overline{x}_1}.$$

Indeed, we have

$$\overline{x}_1\overline{x}_1 - (1 - \overline{x}_2)(1 + \overline{x}_2) = \overline{x}_1^2 + \overline{x}_2^2 - 1 = 0 \in A(X).$$

In this case

$$V(\overline{x}_1) = \{(0, 1), (0, -1)\} \subset X,$$

so with this representation of h we get a smaller domain. We shall see in the sequel that h can not be extended to the whole of X, and that indeed $X \setminus (0, 1)$ is its domain.

Next we see that the set of rational functions, which are regular in a certain point $P \in X$, and those which are regular on a certain open subset, are given some natural algebraic structures.

Definition 1.18. *Let X be an affine variety and $P \in X$. The* **local ring of rational functions, regular in** *P is the ring*

$$\mathcal{O}_{X,P} := \left\{ h \in \mathbb{K}(X); \exists f, g \in A(X),\ h = \frac{f}{g},\ g(P) \neq 0 \right\}.$$

Moreover, given an open subset $U \subseteq X$, the **ring of rational functions, regular on** *U, is the ring*

$$\mathcal{O}_X(U) := \bigcap_{P \in U} \mathcal{O}_{X,P}.$$

Remark 1.19. *The ring $\mathcal{O}_{X,P}$ is indeed a local ring, as it is the localization $\mathcal{O}_{X,P} = S^{-1}A(X)$ where $S = A(X) \setminus I(P)$. The unique maximal ideal is $\mathcal{M}_P = \{h \in \mathcal{O}_{X,P} :\ h(P) = 0 = f(P)\}$.*

Theorem 1.20. *Let $\mathbb{K} = \overline{\mathbb{K}}$ and let $X \subseteq \mathbb{A}^n$ be an affine variety. Then*

$$\mathcal{O}_X(X) \cong A(X).$$

Proof. We observe that there is an ijection $A(X) \hookrightarrow \mathcal{O}_X(X)$, given by $f \mapsto \frac{f}{1}$. Therefor we can consider $A(X) \subseteq \mathcal{O}_X(X)$ as a subring.

Conversely, let $h \in \mathcal{O}_X(X)$; then for any $P \in X$, there exist f_P, g_P such that $h = \frac{f_P}{g_P}$ with $g_P(P) \neq 0$. Hence for any P we can write $hg_P = f_P$. Consider now the ideal

$$J := \langle \{g_P\}_{P \in X} \rangle \subseteq A(X)$$

generated by the denominators of h. We have taht $V(J) = \emptyset$, since in any point $Q \in X$ there is a nonvanishing denominator $g_Q(Q) \neq 0$.

Since $\mathbb{K} = \overline{\mathbb{K}}$, by the relative Weak NSS $J = A(X)$. So $\overline{1} \in J$, hence we can write

$$\overline{1} = \sum_{finite} h_P\, g_P$$

for some $h_P \in A(X)$. Finally, by multiplying the above equality by h, we obtain

$$h = \sum h_P \, hg_P = \sum h_P \, f_P \in A(X),$$

as $h_P, f_P \in A(X)$. □