October 28

Geom. version of Hahn Banach

Def X top vector space on $\mathbb{R}$

$A, B \subset X \qquad A \cap B = \phi$

Let $f : X \longrightarrow \mathbb{R}$ linear

$H = f^{-1}(\alpha) \qquad \alpha \in \mathbb{R}$ a hyperplane.

1) H separates A and B if $f \leq \alpha$ in A and $f \geq \alpha$ in B (or viceversa)

2) H separates strictly if $\exists\, \varepsilon > 0$ s.t. $f \leq \alpha - \varepsilon$ in A and

$f \geq \alpha + \varepsilon$ in B

Lemma X top vec sp (t.v.s.) on $\mathbb{R}$,

C open convex and $x_0 \notin C$

$\exists \ f \in X'$ st $f(x) < f(x_0)$

$\forall \ x \in C$.

<u>Pf</u> It is not restrictive to assume $0 \in C$. Let p the associated seminorm

$$p(x) = \inf \{ a > 0 : \frac{x}{a} \in C \}$$

$$C = \{ x \in X : p(x) < 1 \}$$

$$x_0 \notin C \Rightarrow p(x_0) \geq 1.$$

$$Y = sp\{x_0\} = \mathbb{R} x_0$$

$$g : Y \to \mathbb{R}$$

$$g(t x_0) = t \qquad g(x_0) = 1$$

$$g(x_0) = 1 \leq p(x_0)$$

$$\Rightarrow g(t x_0) \leq p(t x_0) \qquad \forall \ t \in \mathbb{R}.$$

$$\exists \ f \in X' \qquad f|_Y = g$$

$$f(x) \leq p(x) \qquad \forall \ x \in X.$$

$x \in C$

$$f(x) \leq p(x) < 1 = g(x_0) = f(x_0)$$

Conclusion. $f^{-1}(1)$ separates

C , $\{x_0\}$.

<u>Thm</u> X t.v.s , A, B ~~open~~ convex sets

$A \cap B = \phi$, A open. $\exists$ a closed

hyperplane H separating them.

<u>Pf</u> $C = A - B = \{a - b : a \in A, b \in B\}$

C is convex

$$C = \bigcup_{b \in B} (A - b)$$

$C \not\ni 0$ because $A \cap B = \phi$.

$\exists \quad f \in X'$ s.t.

$$f(c) < f(0) = 0 \qquad \forall c \in C$$

$$\Longrightarrow \quad f(a) < f(b) \qquad \forall \begin{array}{l} a \in A \\ b \in B \end{array}$$

$$\exists \quad f(a) \leq \alpha \leq f(b)$$

$$H = f^{-1}(\alpha)$$

<u>Remark</u> Exercise 1.9 Brezis shows that given any A B convex and disjoint in a finite dimensional space are separated by a closed hyperplane.

In infinite dimension for $f : X \to \mathbb{R}$ linear but not continuous

$$A = f^{-1}(\mathbb{R}_-) \qquad B = f^{-1}(\mathbb{R}_+)$$

A & B are convex, $A \cap B = \phi$

If there was $g \in \underset{\text{nonzero}}{X'}$ and α st

$$g \leq \alpha \text{ in } A$$

$$g \geq \alpha \text{ in } B$$

then the open set $g < \alpha$
would be a nonempty open set
disjoint from B.
But B is dense
because all the hyperplanes
$f^{-1}(\alpha)$ are dense in X

<u>Teo</u> X locally convex t v s.
A, B convex, disjoint, A closed
B compact.
$\exists$ a closed hyperplane H
separating them strictly

<u>Pf</u> We claim that $\exists$
(1) U open convex balanced neigh of 0 in X
s t $(A+U) \cap (B+U) = \phi$

Then $A+U$, $B+U$ are convex, open

and disjoint. $\exists\ f \in X'$ and $\alpha \in \mathbb{R}$
$f \not\equiv 0$
s.t.

$$f(a+z_1) \leq \alpha \leq f(b+z_2)$$

$$\forall\ a \in A,\ b \in B,\ z_1, z_2 \in U.$$

$\exists\ \varepsilon > 0$ st $f(U) \supseteq [-\varepsilon, \varepsilon]$

If $x_0 \in X$ st $f(x_0) = 1 \Rightarrow \varepsilon > 0$ s.t.

$$\varepsilon x_0 \in U \Rightarrow \lambda \varepsilon x_0 \in U \quad \forall\ |\lambda| \leq 1$$

Conclusion $\lambda x_0 \in U \quad \forall\ |\lambda| \leq \varepsilon.$

$$f(U) = f(\{\lambda x_0 : |\lambda| \leq \varepsilon\}) = [-\varepsilon, \varepsilon]$$

$$f(a+z_1) \leq \alpha \leq f(b+z_2)$$

$$\forall\ a \in A,\ b \in B,\ z_1, z_2 \in U.$$

$$f(a) \leq \alpha - f(z_1) \qquad \forall\ z_1 \in U$$
$$\forall\ a \in A$$

$$\Rightarrow\ f(a) \leq \alpha - \varepsilon \qquad \forall\ a \in A$$

$$\Rightarrow\ f(b) \geq \alpha + \varepsilon \qquad \forall\ b \in B$$

We go back to claim (1)

X normed.

(1') $\exists$ $\varepsilon > 0$ s.t.

$$\left(A + D_X(0,\varepsilon)\right) \cap \left(B + D_X(0,\varepsilon)\right) = \phi$$

If false ~~$\exists$~~ if I take $\varepsilon_n \to 0^+$ $\exists$

z_n in ~~the ob~~

$$z_n \in \left(A + D_X(0,\varepsilon_n)\right) \cap \left(B + D_X(0,\varepsilon_n)\right)$$

$$z_n = \underset{\in A}{a_n} + \underset{\in D_X(0,\varepsilon_n)}{u_n} \qquad \|z_n - a_n\| < \varepsilon_n$$

$$z_n = b_n + v_n \qquad \|z_n - b_n\| < \varepsilon_n$$

Since B is compact it is not restrictive to observe $b_n \longrightarrow b \in B$

$$z_n \to b \qquad a_n \to b$$

A is closed $\Longrightarrow$ $b \in A \Rightarrow b \in A \cap B$

~~but~~ contradicting $A \cap B = \phi$

so (1') is true

Rem A closed B ~~compact~~ closed

Exercise 1.14

there is a pair A, B convex closed not

$$\ell^1(\mathbb{N}) = \{(x_n) : \sum_{n=1}^{\infty} |x_n| < +\infty\}$$

$$X = \{ (x_n) : x_{2n} = 0 \quad \forall n\}$$

$$Y = \{ (y_n) : y_{2n} = \frac{1}{2^n} y_{2n-1} \quad \forall n\}$$

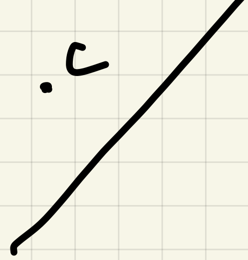
Closed vector spaces.

The key is

$$\overline{X+Y} = \ell^1(\mathbb{N})$$

$$X + Y \subsetneq \ell^1(\mathbb{N})$$

$\exists\, c \notin X+Y$. There is no closed hyperplane separating $\{c\}$ from $X+Y$



$X + Y$

Y

$Z := X - c$

If we had separation, $f \in X'$
$\alpha \in \mathbb{R}$ with

$f \leq \alpha$ in Y

$f \geq \alpha$ $\quad X - c$

$f(y) \leq \alpha \quad \forall y \in Y \Rightarrow f \equiv 0$ in Y

$f(x) - f(c) \geq \alpha \geq 0 \quad \forall x \in X$

$\Rightarrow f|_X = 0$

Then $\quad -f(c) \geq \alpha \geq 0 \qquad f(c) \leq -\alpha \leq 0$

$f^{-1}(-\alpha)$ separates c from $X+Y$

because $f \equiv 0$ in $X+Y$

~~For~~

<u>Corollary</u> X loc convex

$Y \subset X$ $\qquad \overline{Y} \subsetneq X$.

$\exists\ f \in X'$ $\qquad f \not\equiv 0$ in X

s.t. $f\big|_Y = 0$

Pf Let $x_0 \notin \overline{Y}$. Then apply the theorem to $\{x_0\}$ and $\overline{Y}$

$\Rightarrow\ f \in X'$ $\qquad \alpha \in \mathbb{R}$ st

$$f(y) < \alpha < f(x_0) \quad \forall\ y \in \overline{Y}$$

$f(y) = 0\ \forall\ y \in \overline{Y}$

This is our $f \in X'$!

<u>Remark</u> $L^p(0,1)$ $0<p<1$

$\left(L^p(0,1)\right)' = 0$

$\cancel{\forall}$ $f \in L^p(0,1)$ $f \neq 0$

$Y = \mathrm{sp}\{f\}$

<u>Ex</u> Muntz - Szasz Theorem

Theor $I = [0,1]$ and let

$0 < \lambda_1 < \lambda_2 < \dots$ $\lambda_n \xrightarrow{n \to +\infty} +\infty$

In $C^0(I)$

$Y = \overline{\mathrm{sp}\{1, t^{\lambda_1}, t^{\lambda_2}, \dots\}}$

1) If $\sum \frac{1}{\lambda_n} = +\infty \Rightarrow Y = C^0(I)$

2) If $\sum \lambda_n < +\infty$

and if $\lambda \neq 0$ $\lambda \notin \{\lambda_n\}$

then $t^\lambda \notin Y$

Notice that $\lambda_n = n$ gives us Weierstrass approx th.

Notice that in case (1) if we eliminate N elements

$$\lambda_{n_1} < \dots < \lambda_{n_N}$$ from the

sequence (1) continues to be true.

Y $\qquad (C^0(I))'$ is the space of Borel measures on I.

For any μ

$$\int_I t^{\lambda_n} d\mu = \int_I d\mu = 0 \quad \forall n$$

$$\Rightarrow \int_I t^n \, d\mu = 0 \quad \forall n$$

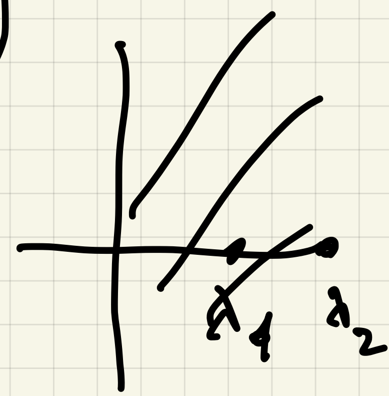
$$\int d\mu \Big|_Y \equiv 0 \rightarrow \cancel{\int} \mu = 0$$

$$f(z) \doteq \int_I t^z \, d\mu(t)$$

$$\mathrm{Re}\, z > 0$$

$$f(\lambda_n) = 0 \qquad \forall n$$

$$f \in H(\{z : \mathrm{Re}\, z > 0\})$$



$$g(z) \doteq f\left(\frac{1+z}{1-z}\right)$$

$$z \longrightarrow \frac{1+z}{1-z}$$

$$U \longrightarrow \{z : \mathrm{Re}\, z > 0\}$$

$$g \in H^\infty(U)$$

$$g(\alpha_n) = 0 \qquad \alpha_n \doteq \frac{\lambda_n - 1}{1 + \lambda_n}$$

$$\alpha_n \nearrow 1$$

$$\left(\frac{t-1}{t+1}\right)' = \frac{2}{(t+1)^2} \qquad \frac{1}{1+\alpha_1} \geq \frac{1}{1+\alpha_n}$$

$$+\infty = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \sum_{n=1}^{\infty} \frac{1-\alpha_n}{1+\alpha_n} = \sum_{n=1}^{\infty} \frac{1-|\alpha_n|}{1+\alpha_n}$$

$$\leq \frac{1}{1+\alpha_1} \sum_{n=1}^{\infty} \left(1-|\alpha_n|\right) = +\infty$$

$$\Longrightarrow \quad g \equiv 0$$

4C

4D