31 october

**Def** $X$ t.v.s $M \subseteq X$

$$M^\perp = \{ f \in X' : \langle f, x \rangle_{X' \times X} = 0 \ \ \forall x \in M \}$$

$M^\perp$ is closed in $X'$

$N \subseteq X'$

$$N^\perp = \{ x \in X : \langle x, f \rangle_{X \times X'} = 0 \ \ \forall f \in N \}$$

$N^\perp$ is closed in $X$

**Lemma** $X$ normed space $M \subseteq X$ linear subspace. Then

$$(M^\perp)^\perp = \overline{M}$$

And if $N \subseteq X'$ is vector space

$$(N^\perp)^\perp \supseteq \overline{N}$$

Dim $(M^\perp)^\perp \supseteq \bar{M}$

$$M^\perp = \{ f \in X' : \langle f, x \rangle_{X' \times X} = 0 \ \forall x \in M \}$$

$$f \in M^\perp \Rightarrow \langle f, x \rangle_{X' \times X} = 0 \quad \forall x \in M$$

$$\Rightarrow \left( x \in M \Rightarrow x \in (M^\perp)^\perp \right)$$

$$\Rightarrow \quad (M^\perp)^\perp \supseteq M \Rightarrow (M^\perp)^\perp \supseteq \bar{M}$$

Similarly $(N^\perp)^\perp \supseteq \bar{N}$.

Now we want to show, starting from $M \subseteq X$,

$$(M^\perp)^\perp = \bar{M}$$

By contradiction let $(M^\perp)^\perp \underset{\neq}{\supset} \bar{M}$

$$x_0 \in (M^\perp)^\perp \setminus \bar{M}$$

$\{x_0\}$ $\bar{M}$. By the last geometric version of Hahn Banach $\exists f \in X'$ and $\alpha \in \mathbb{R}$ with

$$f(x_0) < \alpha < f(x) \quad \forall x \in \bar{M}$$

$$\Longrightarrow \quad \alpha < 0 \quad \text{and} \quad f(x) = 0 \ \forall x \in \bar{M}$$

$$f \in M^{\perp}$$

$$\neq 0$$

$$\langle f, x_0 \rangle_{X' \times X} < \alpha \leq 0$$

$$\Longrightarrow \quad x_0 \notin (M^{\perp})^{\perp}$$

$$\Longrightarrow \quad \text{this contradicts } x_0 \in (M^{\perp})^{\perp} \setminus \bar{M}.$$

$$\Longrightarrow \quad (M^{\perp})^{\perp} = \bar{M}$$

$$(N^{\perp})^{\perp} \supseteq \bar{N}$$

Example $\quad (\ell^1(\mathbb{N}))' = \ell^{\infty}(\mathbb{N})$

$$\boxed{c_0(\mathbb{N}) \subsetneq \ell^{\infty}(\mathbb{N})}$$

$$c_0(\mathbb{N}) = \{ f : \mathbb{N} \to \mathbb{R} \text{ s.t. } \lim_{n \to +\infty} f(n) = 0 \}$$

$$(c_0(\mathbb{N}))' = \ell^1(\mathbb{N})$$

$$(c_0(\mathbb{N}))^{\perp} = 0$$

$$\left( (c_0(\mathbb{N}))^{\perp} \right)^{\perp} = 0^{\perp} = \ell^{\infty}(\mathbb{N})$$

$$\underset{\neq}{\supset} \quad \overline{c_0(\mathbb{N})} = c_0(\mathbb{N})$$

<u>Lemma</u> $T : X \to Y$ , $T^* : Y' \to X'$

$R(T) = TX$ $\qquad$ $R(T^*) = T^* Y'$

(1) $\ker T = R(T^*)^{\perp}$

(2) $\ker T^* = R(T)^{\perp}$

(3) $(\ker T)^{\perp} \supseteq \overline{R(T^*)}$

(4) $(\ker T^*)^{\perp} = \overline{R(T)}$

<u>Pf</u> $(R(T))^{\perp} = \ker T^*$

$$\left( (R(T))^{\perp} \right)^{\perp} = (\ker T^*)^{\perp} = \overline{R(T)}$$

(1) $\ker T \overset{\subseteq}{\underset{\supseteq}{=}} R(T^*)^{\perp}$

$$\langle Tx, y' \rangle_{Y \times Y'} = \langle x, T^* y' \rangle_{X \times X'}$$

$$x \in \ker T \Rightarrow \langle x, T^* y' \rangle_{X \times X'} = 0 \quad \forall y' \in Y'$$

$$\Rightarrow \quad x \in R(T^*)^\perp$$

$$x \in R(T^*)^\perp$$

$$0 = \langle x, T^* y' \rangle_{X \times X'} = \langle Tx, y' \rangle_{Y \times Y'}$$

$$\forall y' \in Y' \Rightarrow \quad Tx = 0$$

$$\Rightarrow \quad x \in \ker T$$

$X$ Banach

$X'$

$(X')' = X''$ is the bidual of $X$

<u>Lemma</u> Consider $J : X \to X''$, $\forall x \in X$

$$\langle Jx, x' \rangle_{X'' \times X'} = \langle x, x' \rangle_{X \times X'}$$

This map is an isometry

$$\|Jx\|_{X''} = \|x\|_X$$

Pf $\|x'\|_{X'} = 1$

$$|\langle Jx, x' \rangle_{X'' \times X'}| = |\langle x, x' \rangle_{X \times X'}|$$

$$\leq \|x\|_X$$

$$\Rightarrow \quad \|Jx\|_{X''} \overset{\geq}{\leq} \|x\|_X$$

$$\exists \; x' \in X' \qquad \|x'\|_{X'} = 1$$

$$\|x\|_X = |\langle x, x' \rangle_{X \times X'}|$$

$$= |\langle Jx, x' \rangle_{X'' \times X'}|$$

$$\leq \|Jx\|_{X''}$$

If $J : X \longrightarrow X''$ is an isomorphism

We say that $X$ is reflexive.

If $X$ is reflexive then

$$(\ker T)^{\perp} = \overline{R(T^*)}$$

Banach - Steinhaus

Def A top space $X$ is Baire if

1) $\forall$ sequence of open dense subspaces $\{A_n\}$, $\bigcap_{n=1}^{\infty} A_n$ is dense in $X$

2) for any $\{C_n\}$ of closed sets with empty interior, then $\bigcup_{n=1}^{\infty} C_n$ has empty interior

A subspace of a Baire space $X$ which contains an intersection like in 1) will be called a $G_\delta$ subspace of $X$.

Theor Every locally compact and Hausdorff space is Baire

Theor Every complete metric space is Baire.

Def $X, Y$ t.v.s
$\{\Lambda_j\}_{j \in J}$ a family in $\mathcal{L}(X, Y)$.
We say that $\{\Lambda_j\}_{j \in J}$ is equicontinuous if $\forall$ neigh. $V$ of $0$ in $Y$ $\exists$ a neigh. $U$ of $0$ in $X$ s.t.

$$\Lambda_j U \subseteq V \qquad \forall j \in J.$$

Exer If $X$ and $Y$ are normed then $\{\Lambda_j\}$ is equicontinuous iff $\exists M \in \mathbb{R}_+$ s.t.

$$\|\Lambda_j\| \leq M \qquad \forall j \in J.$$

<u>Lemma</u> Let $\{\Lambda_j\}$ be equicontinuous in $\mathcal{L}(X, Y)$

Then for any $E$ bounded in $X$

$\exists$ an $F$ bounded in $Y$ s.t.

$$\bigcup_{j \in J} \Lambda_j E \subseteq F.$$

Pf $F = \bigcup_{j \in J} \Lambda_j E$ and let $V$ be a neigh of 0 in $Y$. By equicont

$\exists$ $U$ neigh. of 0 in $X$ s.t.

$$\Lambda_j U \subseteq V \quad \forall j \in J.$$

Since $E$ is bounded in $X$ $\exists$ $t \in \mathbb{R}_+$ st

$$E \subseteq tU$$

$$\Lambda_j E \subseteq t \Lambda_j U \subseteq tV$$

$$\bigcup_j \Lambda_j E \subseteq tV$$

Thm Consider $\{\Lambda_j\}_{j \in J}$ in $\mathcal{L}(X, Y)$

Set $\overset{\text{orbit of } x}{\Gamma(x)} = \{\Lambda_j x : j \in J\}$ $\forall x \in X$.

Set $B = \{x \in X : \Gamma(x) \text{ is bounded in } Y\}$

Suppose that $CB$ is not a $G_\delta$ set

Then $\{\Lambda_j\}$ is equicontinuous.

Pf Suppose $CB \; (= X \setminus B)$ is not a $G_\delta$ set

$\Rightarrow$ $B$ is not contained in the union of a sequence of closed sets with empty interior.

Let $W$ be a balanced neigh of $0$ in $Y$.

And let $V$ be another balanced neigh of $0$ in $Y$ s.t. $\bar{V} + \bar{V} \subseteq W$

$$\left( \bar{V} \subseteq V + V \Rightarrow \left( V + V + V + V \subseteq W \Rightarrow \bar{V} + \bar{V} \subseteq W \right) \right)$$

$$E \doteq \bigcap_{j\in J} \Lambda_j^{-1}\bar{V} \implies \Lambda_j E \subseteq \bar{V} \quad \forall_j$$

We claim $B \subseteq \bigcup_{n\in\mathbb{N}} nE$ closed cover of $B$

For any $x \in B$, $\Gamma(x)$ is bounded in $V \implies \Gamma(x) \subseteq n\bar{V}$

$$\Lambda_j x \in n\bar{V} \qquad \forall_j$$

$$x \in \Lambda_j^{-1}(n\bar{V}) = n\Lambda_j^{-1}\bar{V} \quad \forall_j$$

$$x \in n\bigcap_{j\in J} \Lambda_j^{-1}\bar{V} = nE$$

$E$ has not empty interior

$x \in \mathring{E}$ $\qquad$ $U$ neigh of 0 in $X$

$$x + U \subseteq \mathring{E}$$

$$\Lambda_j(x+U) \subseteq \bar{V} \qquad \forall j \in J$$

$$\Lambda_j x \in \bar{V}$$

$$\Lambda_j U \subseteq \bar{V} - \Lambda_j x \subseteq \bar{V} - \bar{V} = \bar{V} + \bar{V}$$

$\subseteq W$

So we proved that for any neigh $W$ of $0$ in $Y$

$\exists$ a neigh. $U$ of $0$ in $X$

st $\Lambda_j U \subseteq W \quad \forall j$.

$\Rightarrow \{\Lambda_j\}$ is equicont.

<u>Corollary</u> Let $X$ and $Y$ be normed spaces and $T_n \in \mathcal{L}(X, Y)$ a sequence

Then if $\sup_n \|T_n x\|_Y < +\infty \quad \forall x \in X$

$\Rightarrow \exists M \geq 0$ s.t. $\|T_n\| \leq M$

If $\sup_n \|T_n\| = +\infty$ then

$\sup_n \|T_n x\|_Y = +\infty$ for all

the $x$ of a $G_\delta$ subspace of $X$.

$P(\cos x, \sin x)$ when $P(z_1, z_2)$ is a polynomial.

$$f(x) = \frac{a_0}{2} + \sum_{\ell=1}^{n} \left( a_\ell \cos(\ell x) + b_\ell \sin(\ell x) \right)$$

$$= \sum_{\ell=-n}^{n} \hat{f}(\ell)\, e^{i\ell x}$$

$$a_\ell = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(\ell x)\, dx$$

$$b_\ell = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(\ell x)\, dx$$

$$\hat{f}(\ell) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-i\ell x}\, dx$$

<u>Def</u> $\forall\ f \in L^1(-\pi, \pi)$ $\qquad (-\pi, \pi) = \mathbb{T}$

$$\frac{a_0}{2} + \sum_{\ell=1}^{\infty} \left( a_\ell \cos(\ell x) + b_\ell \sin(\ell x) \right)$$

$$\sum_{\ell=-\infty}^{+\infty} \hat{f}(\ell)\, e^{i\ell x} \qquad \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$$

$$\mathbb{T}^d \doteq \frac{\mathbb{R}^d}{2\pi\mathbb{Z}^d}$$

$$f \in L^p(\mathbb{T}^d, \mathbb{C}) \Longleftrightarrow f \in L^p((-\pi,\pi)^d, \mathbb{C})$$

$$\sum_{n \in \mathbb{Z}^d} \hat{f}(n) \, e^{i n \cdot x}$$

$$\hat{f}(n) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} e^{-i n \cdot x} f(x) \, dx$$

$$\widehat{\partial_x^\alpha f} = i^{|\alpha|} n_1^{\alpha_1} \dots n_d^{\alpha_d} \hat{f}(n)$$

$$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_d$$

$$\widehat{\Delta f}(n) = -|n|^2 \hat{f}(n)$$

$$\varphi(n) \hat{f}(n)$$