ANOMALIES

- An anomaly is " a sym in classical theory is not a sym anymore in the quantum theory "

  Example: YM is scale invariant classically; but at the quantum level a scale is generated (Λ_{QCD})

- A symmetry in classical action $\longrightarrow$ WARD IDENTITY, i.e. relations between CORRELATORS.

  An ANOMALY is a violation in the WARD ID.

CLASSICAL CONSERVATION LAWS AND ANOMALIES

Consider Lagrangian for one Dirac fermion

$$\mathcal{L}_{QED} = \bar{\psi}(i\not{D} - m)\psi$$

let's define the following quantities

VECTOR CURRENT $\quad j^\mu = \bar{\psi}\gamma^\mu\psi$

AXIAL CURRENT $\quad j^\mu_A = \bar{\psi}\gamma^\mu\gamma_5\psi$

PSEUDO SCALAR $\quad P = \bar{\psi}\gamma_5\psi$

E.o.m are $\quad (i\not{\partial} - m + e\not{A})\psi = 0 \qquad \bar{\psi}(i\overleftarrow{\not{\partial}} + m - e\not{A}) = 0$

$\Rightarrow$ CONSERVATION LAWS

$$\partial_\mu j^\mu = \bar{\psi}\overleftarrow{\not{\partial}}\psi + \bar{\psi}\not{\partial}\psi =$$

$$= i\bar{\psi}(m - e\not{A})\psi + i\bar{\psi}(-m + e\not{A}) = 0$$

$$\partial_\mu j^\mu_A = i\bar{\psi}(m - e\not{A})\gamma_5\psi - i\bar{\psi}\gamma_5(-m + e\not{A})\psi$$

$$= 2imP \qquad (=0 \text{ for } m=0)$$

SYMMETRIES

$\psi \mapsto e^{i\alpha}\psi \qquad U(1)_V$

$\psi \mapsto e^{i\beta\gamma_5}\psi \qquad U(1)_A$

$\hookrightarrow$ this is a sym for $m=0$.

One can decompose Dirac spinor in Weyl L or R spinors

using projector $\quad P_\pm = \frac{1}{2}(1 \pm \gamma_5) \quad : \qquad \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$

$$\psi_L = \frac{1}{2}(1+\gamma_5)\psi \qquad \psi_R = \frac{1}{2}(1-\gamma_5)\psi$$

If $m=0$, two conserved currents and symmetries

$j^\mu_L = \frac{1}{2}(j^\mu + j^\mu_A) \qquad \psi_L \mapsto e^{i\Lambda_L}\psi_L \qquad \psi_R \mapsto \psi_R$

$j^\mu_R = \frac{1}{2}(j^\mu - j^\mu_A) \qquad \psi_L \mapsto \psi_L \qquad \psi_R \mapsto e^{i\Lambda_R}\psi_R$

$\rightarrow$ boils down to V-sym when $\Lambda_L = \Lambda_R$

" " " A-sym when $\Lambda_L = -\Lambda_R$

If we have N Dirac fermions, Lagrangian is inv. under $SU(N)_V \times SU(N)_A$, where formally all equal to above but now
$$\left.\begin{aligned}\alpha &= \alpha^a t^a_N\\ \beta &= \beta^a t^a_N\end{aligned}\right\}$$ with t^a gen. of $SU(N)$.

and
$$j^{\mu a} = \bar\psi \gamma^\mu t^a_N \psi$$
$$j_A^{\mu a} = \bar\psi \gamma^\mu \gamma_5 t^a_N \psi$$

Now let's suppose to couple ψ to (external, i.e. non propagating) gauge fields, that make local the sym. above
→ we need V_μ for G_V and A_μ for G_A.

The e.o.m. are now
$$(i\partial\!\!\!/ - m - V\!\!\!/ - A\!\!\!/\gamma_5)\psi = 0$$
$$\bar\psi(i\overleftarrow{\partial\!\!\!/} + m + V\!\!\!/ + A\!\!\!/\gamma_5) = 0$$

$$\partial_\mu j^{\mu a} = \bar\psi \overleftarrow{\partial\!\!\!/} t^a \psi + \bar\psi t^a \partial\!\!\!/\psi =$$
$$= -i\bar\psi\gamma^\mu \underbrace{[t^b,t^a]}_{-if^{abc}t^c}\psi V_\mu - i\bar\psi\gamma^\mu\gamma_5 \underbrace{[t^b,t^a]}_{-if^{abc}t^c}\psi A_\mu$$
$$= -f^{abc}V^b_\mu j^{c\mu} - f^{abc}A^b_\mu j_A^{c\mu}$$

$$\rightarrow \quad D_\mu j^\mu + [A_\mu, j^\mu_5] = 0 \qquad \text{where } D_\mu = \partial_\mu + iV_\mu$$

Analogously:
$$\rightarrow \quad D_\mu j^\mu_5 + [A_\mu, j^\mu] = 2imP$$

If $\mathcal{L}$ contains only vector gauge fields:
$$D_\mu j^\mu = 0$$
$$D_\mu j^\mu_5 = 2imP \qquad (=0 \text{ for } m=0)$$

WARD IDENTITIES & ANOMALY

The validity of the classical conservation laws induces RELATIONS among various GREEN'S FNCTS (CORRELATORS) → W.I.

Consider classical sym with $\partial_\mu j^\mu = 0$. Then one expects

$$\partial_x^\mu \langle 0 | T j^\mu(x)\, O^1(y_1) \cdots O^m(y_m) | 0 \rangle = 0 + \text{contact terms}$$

WI important for several things, including renormalizability of the theory.

Here we are interested in Correlators of this form in 4d:

$$\langle j^\mu(x)\, j^\nu(y)\, j_5^\lambda(z) \rangle \qquad \langle j^\mu(x)\, j^\nu(y)\, P(z) \rangle$$

In momentum space

$$T^{\mu\nu\lambda}(k_1, k_2, q) \equiv i \int d^4x\, d^4y\, d^4z\, e^{ik_1x + ik_2x - iqz} \langle j^\mu(x) j^\nu(y) j_5^\lambda(z) \rangle$$

$$T^{\mu\nu}(k_1, k_2, q) \equiv i \int d^4x\, d^4y\, d^4z\, e^{ik_1x + ik_2x - iqz} \langle j^\mu(x) j^\nu(y) P(z) \rangle$$

(Actually these depend on one variable less due to momentum conserv.)

Differentiating the Correlators in momentum space we obtain

$$q_\lambda T^{\mu\nu\lambda} = \int d^4x\, d^4y\, d^4z\, e^{ik_1x + ik_2y - iqz}\, \partial_\lambda^z \langle j^\mu(x) j^\nu(y) j_5^\lambda(z) \rangle$$

$$\underset{\uparrow}{=} 2mi \int d^4x\, d^4y\, d^4z\, e^{ik_1x + ik_2y - iqz} \langle j^\mu(x) j^\nu(y) P(z) \rangle = 2m\, T^{\mu\nu}$$

if WI is satisfied

(contact terms are zero in this case)

↳ AXIAL WARD ID. (AWI) : $q_\lambda T^{\mu\nu\lambda} = 2m\, T^{\mu\nu}$

Analogously differentiating by x and y we get in momentum space

$\hookrightarrow$ VECTOR WARD ID. (VWI) : $k_{1\mu} T^{\mu\nu\lambda} = k_{2\nu} T^{\mu\nu\lambda} = 0$

This is what is expected on the basis of symmetries preserved in the quantum theory.

One can compute correlators $T^{\mu\nu\lambda}$ and $T^{\mu\nu}$ and check whether WI are satisfied. In fact, as we will see, they are NOT $\longrightarrow$ ANOMALY

We state here the results (that we will prove in $d=2$ and $d=4$)

$\rightarrow$ If the VWI is fulfilled, the AWI is ANOMALOUS or viceversa!

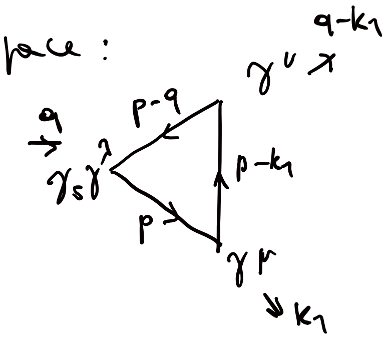
Anomaly in $d=4$

We now want to compute correlators $\langle \underbrace{j^\mu j^\nu j^\lambda}_{T^{\mu\nu\lambda}} \rangle$ and $\langle \underbrace{P j^\mu j^\nu}_{T^{\mu\nu}} \rangle$

The expected WI is $q_\lambda T^{\mu\nu\lambda} = 2m T^{\mu\nu}$

We use Feynman rules to compute the correlators in momentum space:

$\langle j^\mu j^\nu j_5^\lambda \rangle$



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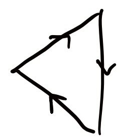
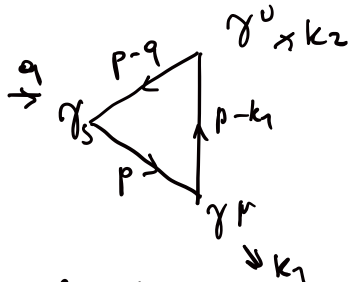


$$\rightarrow T^{\mu\nu\lambda} = i\int \frac{d^4p}{(2\pi)^4}(-)\,\mathrm{tr}\,\frac{i}{\not p - m}\gamma^\lambda\gamma_5\frac{i}{\not p - \not q - m}\gamma^\nu\frac{i}{\not p - \not k_1 - m}\gamma^\mu + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix}$$

$q = k_1 + k_2$

$\langle j^\mu j^\nu P \rangle$



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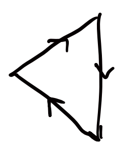


$$\rightarrow T^{\mu\nu} = i\int \frac{d^4p}{(2\pi)^4}(-)\,\mathrm{tr}\,\frac{i}{\not p - m}\gamma_5\frac{i}{\not p - \not q - m}\gamma^\nu\frac{i}{\not p - \not k_1 - m}\gamma^\mu + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix}$$

We want to investigate AWI. We need

$$\not q\gamma_5 = \gamma_5(\not p - \not q - m) + (\not p - m)\gamma_5 + 2m\gamma_5$$

$$\Rightarrow \quad q_\lambda T^{\mu\nu\lambda} = 2m T^{\mu\nu} + R_1^{\mu\nu} + R_2^{\mu\nu}$$

$$R_1^{\mu\nu} = \int \frac{d^4p}{(2\pi)^4} \, \mathrm{tr}\left(\frac{1}{\not p - \not k_2 - m} \gamma_5 \gamma^\nu \frac{1}{\not p - \not q - m} \gamma^\mu - \frac{1}{\not p - m} \gamma_5 \gamma^\nu \frac{1}{\not p - \not k_1 - m} \gamma^\mu \right)$$

$$R_2^{\mu\nu} = \int \frac{d^4p}{(2\pi)^4} \, \mathrm{tr}\left(\frac{1}{\not p - \not k_1 - m} \gamma_5 \gamma^\mu \frac{1}{\not p - \not q - m} \gamma^\nu - \frac{1}{\not p - m} \gamma_5 \gamma^\mu \frac{1}{\not p - \not k_2 - m} \gamma^\nu \right)$$

If R_1 e $R_2 \to 0$ then AWI is fulfilled

Formally, $R_1 \to 0$ if we shift $p \to p + k_2$:

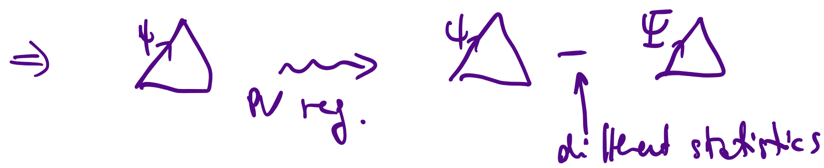
$$R_1^{\mu\nu} = \int \frac{d^4p}{(2\pi)^4} \, \mathrm{tr}\left(\frac{1}{\not p - m} \gamma_5 \gamma^\nu \frac{1}{\not p - \not k_1 - m} \gamma^\mu - \frac{1}{\not p - \not k_2 - m} \gamma_5 \gamma^\nu \frac{1}{\not p - \not q - m} \gamma^\mu \right)$$

$$= -R_1^{\mu\nu} \Rightarrow R_1^{\mu\nu} = 0 \qquad \text{Analogous for } R_2 \quad p \to p + k_1$$

Manipulation inside the integral are fine if integrals are finite. However the integrals are linearly divergent (it appears quadratic, but the term $p^\alpha p^\beta \epsilon_{\alpha\beta\gamma\sigma} = 0$).

$\Rightarrow$ We need to regularize the theory. Here we use a famous regularization :

PAULI – VILLARS REGUL. $\rightarrow$ introduce an auxiliary species of fermions Ψ with opposite statistics and mass M; at the end of computation take $M \to \infty$, that should decouple the new fields.



Hence $T^{\mu\nu\lambda}_{reg} = T^{\mu\nu\lambda}(m) - T^{\mu\nu\lambda}(M)$ $\qquad T^{\mu\nu\lambda}_{phys} = \lim_{M\to\infty} T^{\mu\nu\lambda}_{reg}$

[If we apply this to $T^{\mu\nu}$:

$$T^{\mu\nu}_{phys} = \lim_{M\to\infty} T^{\mu\nu}_{reg} = \lim_{M\to\infty} \left(T^{\mu\nu}(m) - T^{\mu\nu}(M)\right) = T^{\mu\nu}(m)$$

because $T^{\mu\nu}(M) \sim \frac{1}{M}$. $T^{\mu\nu}$ is convergent and does need regul.]

This REGULARIZATION makes a specific choice on what WI to preserve.

$\rightarrow \quad k_{1\mu} T^{\mu\nu\lambda}_{phys} = k_{2\nu} T^{\mu\nu\lambda}_{phys} = 0 \quad \Rightarrow \quad$ VWI is preserved

Euristically: new $\mathcal{L}_\Psi$ does not break $U(1)_V$.

What about AWI?

$T^{\mu\nu\lambda}_{reg}$ is finite; in particular $R^{\mu\nu}_{1\,reg} = 0 = R^{\mu\nu}_{2\,reg}$:
now R's are finite and then manipulations inside the integral are valid.

$$\Rightarrow \quad q_\lambda T^{\mu\nu\lambda}_{reg} = 2m T^{\mu\nu}(m) - 2M\, T^{\mu\nu}(M)$$

$$\Rightarrow \quad q_\lambda T^{\mu\nu\lambda}_{phys} = 2m\, T^{\mu\nu}(m) - \underbrace{\lim_{M\to\infty} 2M\, T^{\mu\nu}(M)}$$

We now compute this
(that is the possible violation of AWI)

$$T^{\mu\nu}(M) = i\int \frac{d^4p}{(2\pi)^4}(-)\,\mathrm{tr}\,\frac{i}{\not p - M}\,\gamma_5\,\frac{i}{\not p-\not q - M}\,\gamma^\nu\,\frac{i}{\not p - \not k_1 - M}\,\gamma^\mu$$
$$+ \begin{pmatrix}k_1\leftrightarrow k_2\\ \mu\leftrightarrow\nu\end{pmatrix}$$

Use $$\frac{1}{a_1a_2a_3} = 2\int_0^1 dx_1\int_0^{1-x_1}dx_2\,\frac{1}{\left(a_1x_2 + a_2(1-x_1-x_2) + a_3x_1\right)^3}$$

$$= -\int\frac{d^4p}{(2\pi)^4}\,2\int_0^1 dx_1\int_0^{1-x_1}dx_2\,\frac{\mathrm{tr}\left[(\not p+M)\gamma_5(\not p-\not q+M)\gamma^\nu(\not p-\not k_1+M)\gamma^\mu\right]}{\left[(p^2-M^2)x_2 + ((p-q)^2-M^2)(1-x_1-x_2) + ((p-k_1)^2-M^2)x_3\right]^3} + \begin{pmatrix}k_1\leftrightarrow k_2\\ \mu\leftrightarrow\nu\end{pmatrix}$$

$\to \mathrm{Tr}\,\gamma_5\gamma^{\alpha_1}\cdots\gamma^{\alpha_n} = 0$ for $n\neq 4$ and $= 4i\epsilon^{\alpha_1\cdots\alpha_4}$ for $n=4$

$$\mathrm{tr}\left(\not p\gamma_5(\cancel{\not p}-\not q)\gamma^\nu\gamma^\mu\right) + \mathrm{tr}\left(\not p\gamma_5\gamma^\nu(\cancel{\not p}-\not k_1)\gamma^\mu\right) + \mathrm{tr}\left(\gamma_5(\not p-\not q)\gamma^\nu(\not p-\not k_1)\gamma^\mu\right)$$
$$= \mathrm{tr}\left(\gamma_5\cancel{\not p\not q}\gamma^\nu\gamma^\mu\right) + \mathrm{tr}\left(\gamma_5\cancel{\not p\gamma^\nu}\not k_1\gamma^\mu\right) - \mathrm{tr}\left(\cancel{\gamma_5\not p\gamma^\nu\not k_1\gamma^\mu}\right) - \mathrm{tr}\left(\cancel{\gamma_5\not q\gamma^\nu\not p}\gamma^\mu\right)$$
$$+ \mathrm{tr}\left(\gamma_5\underset{\cancel{k_1}+k_2}{\not q}\gamma^\nu\not k_1\gamma^\mu\right) = 4i\epsilon^{\beta\nu\alpha\mu}k_{1\alpha}k_{2\beta}\quad\left(= 4i\epsilon^{\beta\mu\alpha\nu}k_{2\alpha}k_{1\beta}\right)$$

$$= -\int\frac{d^4p}{(2\pi)^4}\,2\int_0^1 dx_1\int_0^{1-x_1}dx_2\,\frac{M\cdot\left(-4i\epsilon^{\mu\nu\alpha\beta}k_{1\alpha}k_{2\beta}\right)}{\left[\cancel{x_2p^2} - \cancel{x_2M^2} + (1-x_1-x_2)(\cancel{p^2} - 2pq + q^2 - \cancel{M^2}) + x_1(\cancel{p^2}-2pk_1+k_1^2-\cancel{M^2})\right]^3} + \begin{pmatrix}\mu\leftrightarrow\nu\\ k_1\leftrightarrow k_2\end{pmatrix}$$
$$+ p^2 - M^2 - 2p\left(q(1-x_1-x_2) + k_1x_1\right) + q^2(1-x_1-x_2) + k_1^2x_1$$
$$\equiv p^2 - 2pK - \bar M^2 \qquad \text{with } K = q(1-x_1-x_2) + k_1x_1,\quad \bar M^2 = M^2 - q^2(1-x_1-x_2) - k_1^2x_1$$

$$= 8Mi\epsilon^{\mu\nu\alpha\beta}k_{1\alpha}k_{2\beta}\int_0^1 dx_1\int_0^{1-x_1}dx_2\,\underbrace{\int\frac{d^4p}{(2\pi)^4}\frac{1}{(p^2-2pK-\bar M^2)^3}}\qquad p^0 = il^0\qquad + \begin{pmatrix}k_1\leftrightarrow k_2\\ \mu\leftrightarrow\nu\end{pmatrix}$$

$$= -i\int\frac{d^4l}{(2\pi)^4}\frac{1}{(l^2+2pK+\bar M^2)^3} = -i\,\frac{\Gamma(3-2)}{(\bar M^2-K^2)^{3-2}(4\pi)^2\,\Gamma(3)}$$

$$\int\frac{d^dk}{(2\pi)^d}\frac{1}{\left[k^2+2kp+b^2\right]^A} = \frac{\Gamma(A-d/2)}{(b^2-p^2)^{A-d/2}(4\pi)^{d/2}\,\Gamma(A)}$$

$$= \frac{-i}{(4\pi)^2}\,\frac{1}{2(\bar M^2-K^2)}$$

$$= \frac{1}{4\pi^2} \epsilon^{\mu\nu\alpha\beta} k_{1\alpha} k_{2\beta} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \, \frac{M}{M^2 + \underset{\substack{\uparrow \\ \text{it does not depend} \\ \text{on } M}}{f(x_1,x_2)}} + \underset{\substack{\uparrow \\ \text{Same expression} \\ \text{but with} \\ f \text{ substituted by} \\ \text{a different } f(x_1,x_2)}}{\begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix}}$$

$$\lim_{M\to\infty} 2M\, T^{\mu\nu}(M) = \frac{1}{4\pi^2} \epsilon^{\mu\nu\alpha\beta} k_{1\alpha} k_{2\beta} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \, \underbrace{\lim_{M\to\infty} \left(\frac{2M^2}{M^2 + f(x_1,x_2)} + \frac{2M^2}{M^2 + g(x_1,x_2)} \right)}_{=4}$$

$$= \frac{1}{\pi^2} \epsilon^{\mu\nu\alpha\beta} k_{1\alpha} k_{2\beta} \underbrace{\int_0^1 dx_1 \int_0^{1-x_1} dx_2}_{= \int_0^1 dx_1 (1-x_1) = 1 - \frac{1}{2} = \frac{1}{2}}$$

$$= \frac{1}{2\pi^2} \epsilon^{\mu\nu\alpha\beta} k_{1\alpha} k_{2\beta}$$

$$\Rightarrow \quad q_\lambda T^{\mu\nu\lambda}_{phys} = 2m\, T^{\mu\nu}(m) \underbrace{- \frac{1}{2\pi^2} \epsilon^{\mu\nu\alpha\beta} k_{1\alpha} k_{2\beta}}_{\text{Anomaly}}$$