Prendiamo $m=0$

$$T^{\mu\nu\lambda}(k_1,k_2,q) \equiv i\int d^4x\, d^4y\, d^4z\, e^{ik_1x+ik_2x-iqz} \langle j^\mu(x) j^\nu(y) j_5^\lambda(z)\rangle$$

from $(2\pi)^4\delta(q-k_1-k_2)$

$$\partial_\lambda^z \langle j^\mu(x)\, j^\nu(y)\, j_5^\lambda(z)\rangle = -i\int \frac{dq}{(2\pi)^4}\cancel{\int\frac{dk_1}{(2\pi)^4}}\int\frac{dk_2}{(2\pi)^4}\, e^{-ik_1x-ik_2y+iqz}\, iq_\lambda T^{\mu\nu\lambda}(q,k_1,k_2)$$

$$= \int\frac{dq}{(2\pi)^4}\int\frac{dk_2}{(2\pi)^4}\, e^{-ik_2(y-x)+iq(z-x)}\left(-\frac{1}{2\pi^2}\epsilon^{\mu\nu\alpha\beta}\, iq_\alpha(-i)k_{2\beta}\right)$$

$$= -\frac{1}{2\pi^2}\epsilon^{\mu\nu\alpha\beta}\,\partial_\alpha^z\delta(z-x)\,\partial_\beta^y\delta(y-x)$$

Let's couple the vector current $j^\mu(x)$ to a gauge boson $A_\mu(x)$.

$$\partial_\lambda^z\langle j_A^\lambda(z)\rangle_A = \partial_\lambda^z\langle j_A^\lambda(z)\rangle_{free} + \dots +$$

the only non-trivial

$$-\frac{1}{2}\int d^2x_1\, d^2x_2\, A_{\rho_1}(x_1)\, A_{\rho_2}(x_2)\,\partial_\lambda\langle j^{\rho_1}(x_1)\, j^{\rho_2}(x_2)\, j_A^\lambda(z)\rangle + \dots$$

$$= -\int d^2x_1\, d^2x_2\, A_{\rho_1}(x_1)\, A_{\rho_2}(x_2)\left(-\frac{1}{4\pi^2}\right)\epsilon^{\rho_1\rho_2\alpha\beta}\,\partial_\alpha^z\delta(z-x_1)\,\partial_\beta^{x_2}\delta(x_2-z)$$

$$= +\frac{1}{16\pi^2}\epsilon^{\rho_1\rho_2\alpha\beta}F_{\alpha\rho_1}F_{\beta\rho_2} =$$

$$= -\frac{1}{16\pi^2}\epsilon^{\alpha\rho_1\beta\rho_2}F_{\alpha\rho_1}F_{\beta\rho_2}$$

$$= -\frac{1}{16\pi^2}\epsilon^{\mu_1\nu_1\mu_2\nu_2}F_{\mu_1\nu_1}F_{\mu_2\nu_2}$$

Non-Abelian fields

We now consider fields that sit in R of some group G

$\Rightarrow j^\mu, j_5^\nu$ are in the ADJOINT REP:

$$j^{\mu a} = \bar\psi \gamma^\mu t_R^a \psi$$

$$j_A^{\mu a} = \bar\psi \gamma^\mu \gamma_5 t_R^a \psi$$

In this situation several correlators get anomalous WI.

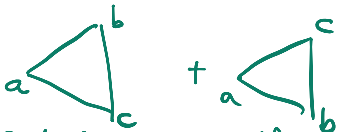
Let's remain on the triangle

$$T^{abc\,\mu\nu\lambda}_{\mu\nu}(k_1,k_2) = i\int d^4x\, d^4y\; e^{ik_1x+ik_2y} \langle j^{a\mu}(x)\, j^{b\nu}(y)\, j_\lambda^{5c}(0)\rangle_{P^c(0)}$$

Now WI are modified to:

$$q_\lambda T^{abc\,\mu\nu\lambda} = 2m\, T^{abc\,\mu\nu} \underbrace{- \frac{c^{abc}}{2\pi^2}\epsilon^{\mu\nu\alpha\beta} k_{1\alpha} k_{2\beta}}_{\text{ABJ anomaly for non-ab. case}} + \dots$$

where $c^{abc} = \frac{1}{2}\mathrm{tr}_R\{t_R^a, t_R^b\}\, t_R^c$



$\mathrm{tr}_R\, t^a t^b t^c \qquad \mathrm{tr}_R\, t^a t^c t^b$

If we keep the AXIAL current ABELIAN j_A^λ (colour singlet) but the rest in a simple group, we obtain like above, but with t_R^c replaced by $\mathbb{1}$ $\Rightarrow$ $c^{abc} \to c(R)\delta^{ab}$

$$\mathrm{tr}_R(t_R^a t_R^b) = c(R)\delta^{ab}$$

SINGLET ANOMALY

$$q_\lambda T^{ab\,\mu\nu\lambda} = 2m\, T^{ab\,\mu\nu} - \frac{c(R)\delta^{ab}}{2\pi^2}\epsilon^{\mu\nu\alpha\beta} k_{1\alpha} k_{2\beta} + \dots$$

What about gauging t_R^a-vector? Again this contribution produces $(\partial_{[\alpha} A^a_{\beta]})^2$ term. How do the next $(\dots)$ contribute?

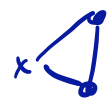
Note that $\partial_\lambda j_5^\lambda$ is a colour SINGLET. Hence the RHS should also be singlet:

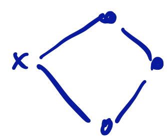
$$\langle \partial_\lambda j_5^\lambda \rangle = -\frac{C(R)\,\delta^{ab}}{16\pi^2}\,\epsilon^{\mu\nu\alpha\beta} F^a_{\mu\nu} F^b_{\alpha\beta} \qquad (m=0)$$

$$= -\frac{1}{8\pi^2}\,\epsilon^{\mu\nu\alpha\beta}\,\mathrm{tr}\left(F_{\mu\nu}F_{\alpha\beta}\right)$$

From this relation all anomalous WI's follow. Since now we have self-interaction terms, we have additional diagrams contributing to anomaly.

$\epsilon^{\mu\nu\alpha\beta}\delta^{ab}F^a_{\mu\nu}F^b_{\alpha\beta} \to$

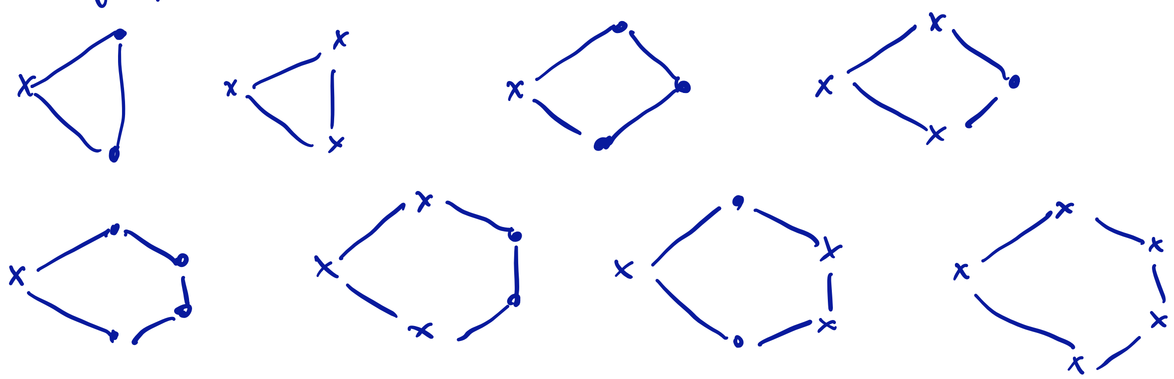
$\epsilon^{\mu\nu\alpha\beta}\partial_\mu A^a_\nu \partial_\alpha A^a_\beta \to$  triangle

$\epsilon^{\mu\nu\alpha\beta}\partial_\mu A^a_\nu A^c_\alpha A^d_\beta f^{acd} \to$  quadrangle

$$\underbrace{\epsilon^{\mu\nu\alpha\beta} A^c_\mu A^d_\nu A^e_\alpha A^f_\beta}_{\text{totally antisym in } cdef}\ \underbrace{f^{acd} f^{aef}}_{\text{it can be replaced by } *} = 0 \quad \text{Jacobi identity}$$

$$* \begin{cases} f^{acd}f^{aef} + \overbrace{f^{aed}f^{afc} + f^{afd}f^{ace}}^{+\text{ cyclic perm in } cef} = 0 \text{ for Jacobi} \\ -f^{acd}f^{afe} - f^{afd}f^{aec} - f^{aed}f^{acf} \end{cases}$$

When we also take j_A to be in adj of G, then several graphs contributes (i.e. several anomalous correlator w/):



If we gauge vector sym, the anomaly becomes

$$\langle (D_\mu j_5^\mu)^a \rangle = -\frac{1}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} \, tr \, T^a F_{\mu\nu} F_{\rho\sigma} \qquad (*)$$

This is called the COVARIANT ANOMALY

Chiral fields

One can couple R-handed and L-handed fermion currents to different gauge fields A^L_μ and A^R_μ.

There exists a type of anomaly, the COVARIANT ANOMALY for L, R currents (with a given REGULARIZATION)

$$\langle (D_\mu j^{H\mu})^a \rangle = \eta_H \frac{1}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} \, \mathrm{tr}\, T^a F^H_{\mu\nu} F^H_{\rho\sigma} \qquad H = L, R$$

Notice that $\quad A^{L,R}_\mu = A^V_\mu \pm A^A_\mu \qquad j^{L,R\,\mu} = \frac{1}{2}(j^{V\mu} \pm j^{A\mu})$

if $A^A_\mu \equiv 0$ (like case above) $\Rightarrow A^{L,R}_\mu = A_\mu$

and $\langle (D_\mu j^{5\mu})^a \rangle = \langle (D_\mu j^{L\mu})^a - (D_\mu j^{R\mu})^a \rangle =$

$$= 2 \cdot \left(-\frac{1}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} \, \mathrm{tr}\, T^a F_{\mu\nu} F_{\rho\sigma} \right)$$

that is (★)

Anomalies are bad for GAUGE theories: if anomaly is in the gauge sym, it spoils the consistency of the theory (remember gauge sym. is redundancy) see Tong

ANOMALY CANCELLATION CONDITIONS

1) VECTOR-LIKE MODEL. A_μ couple to j^μ_V $\Rightarrow$ anomaly can be put only into AXIAL sym that is not gauged $\longrightarrow$ fine!

2) "SAFE GROUPS". Take models with several fermions, L and R, coupled differently to gauge fields. Then we have necessary anomalies on currents coupled to gauge fields.
However anomaly itself may vanish. Remember that anomaly is proportional to

$$C^{abc} = \frac{1}{2}\,\mathrm{tr}\,\{T^a, T^b\}\,T^c$$

If this vanish, both triangle and higher pt funct. vanishes after contracted with q_λ.

The simple safe groups in 4d are:

$$SU(2)\,,\quad SO(2N+1)\,,\quad SO(4N)\quad N \geq 2\,,\ E_6\,,\ E_8$$

(In particular $SU(N)$ with $N \geq 3$ are not safe).

3) STANDARD MODEL : $SU(2)\times U(1)_Y \times SU(3)_c$

This gauge field couple differently to L and R. It's better to take all fields L and complete L ones.

R sotto $SU(2)\times U(1)_Y$

$$\begin{array}{ll} l_L & 2_{-3} \\ e_R & 1_{-6} \to 1_{+6} \\ q_L^i & 2_1 \\ u_R^i & 1_4 \to 1_{-4} \\ d_R^i & 1_{-2} \to 1_{+2} \end{array}$$

$i = 1,2,3$

$C^{abc}=0$ when all a,b,c are in $SU(2)$

$SU(2)\times SU(3)\times U(1)_Y$

$\longrightarrow \psi$ in $R = (2,1)_{-3} \oplus (1,1)_{+6} \oplus (2,3)_1 \oplus (1,\bar{3})_{-4} \oplus (1,\bar{3})_{+2}$

Per $SU(2)$ $\quad C^{abc}=0 \Rightarrow tr(\underbrace{T^aT^bT^c}_{\text{gen. di } SU(2)})=0$

Per SU(3) $C^{abc}\neq 0$ ma spettro vector-like

$\to$ consideriamo $tr(T^aT^bT_Y)$ e $tr(T_Y^3)$

$T_R^a=0$ quando $R=1$

$$tr(T^aT^bT_Y) = tr\begin{pmatrix} t^a_{R_1}t^b_{R_1}q_Y^{R_1}\mathbb{1}_{\dim R_1} & & \\ & t^a_{R_2}t^b_{R_2}q_Y^{R_2}\mathbb{1}_{\dim R_2} & \\ & & \ddots \end{pmatrix} =$$

per rap non triviali

$$=\sum_i q_Y^{R_i}\,\delta^{ab}\,c(R_i) = q_Y^{l_L}\,\delta^{ab}c(R_{l_L}) + 3\times q_Y^{q_L}\,\delta^{ab}c(R_{q_L}) =$$

$$= -3\cdot\delta^{ab}\cdot\frac{1}{2} + 3\cdot 1\cdot\delta^{ab}\cdot\frac{1}{2} = 0$$

Per SU(3)

$$tr(T^aT^bT_Y) = 2\,q_Y^{q_L}\,\delta^{ab}c(R_{q_L}) + q_Y^{u_R^c}\,\delta^{ab}c(R_{u_R}) + q_Y^{d_R^{c}}\,\delta^{ab}c(R_{d_R}) =$$

$$= 2\cdot 1\cdot\delta^{ab}\cdot\frac{1}{2} + (-4)\cdot\delta^{ab}\cdot\frac{1}{2} + 2\cdot\delta^{ab}\cdot\frac{1}{2} =$$

$$= \frac{\delta^{ab}}{2}(2-4+2) = 0$$

Ora solo $U(1)_Y$

$$Tr(T_Y^3) = -27\cdot 2 + 216 + 3\cdot(1\cdot 2 - 64 + 8)$$

$$= 162 + 3\cdot(-54) = 0$$

↳ è essenziale per la cancellazione delle ANOMALIE

\# di COLORI

qta cancellazione diede evidenza all'esistenza del quark TOP prima che venisse rilevato

PATH INTEGRAL and ANOMALY

Take $\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi$ $\qquad \not{D} = \gamma^\mu(\partial_\mu + iA_\mu)$

→ Dirac fermions whose vector current coupled to a gauge field, abelian or non-ab. ($A_\mu = A_\mu^a t_R^a$)

We now want to consider the generating functional

$$Z[A] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi}\, e^{i\int dx\, \bar{\psi}(i\not{D} - m)\psi}$$

$\mathcal{L}$ is INVARIANT under axial transform. if $m=0$

CHIRAL ANOMALY

To work out WI, one make a local axial transf.

It is not the sym that is gauged by A_μ (that is Vector transform.)

$$\psi \mapsto \psi' = e^{i\beta(x)\gamma_5}\psi \qquad \bar{\psi} \mapsto \bar{\psi}' = \bar{\psi}\, e^{i\beta(x)\gamma_5}$$

For infinitesimal β

$$\bar{\psi}(i\not{D} - m)\psi \mapsto \bar{\psi}'(i\not{D} - m)\psi' = \bar{\psi}(i\not{D} - m)\psi' - (\partial_\mu\beta)\bar{\psi}\gamma^\mu\gamma^5\psi - 2im\beta\,\bar{\psi}\gamma^5\psi$$

$$S' = S - \int dx\, \beta(x)\left[\partial_\mu j_A^\mu(x) - 2im\, P(x)\right]$$

Let's apply such transf. inside the P.I., by using it to change integration variables. Let's also introduce sources for $\psi, \bar\psi$

$$Z[\eta,\bar\eta,A] = \int \mathcal{D}\psi\mathcal{D}\bar\psi\, e^{iS[\psi,\bar\psi,A] + i\int dx(\bar\eta\psi + \bar\psi\eta)}$$

$$= \int \mathcal{D}\psi'\mathcal{D}\bar\psi'\, e^{iS[\psi',\bar\psi',A] + i\int(\bar\eta\psi' + \bar\psi'\eta)}$$

change name of integ. var.

$\delta\psi = i\beta\gamma_5\psi \qquad \delta\bar\psi = i\bar\psi\gamma_5\beta$

$$= \int \mathcal{D}\psi\mathcal{D}\bar\psi\, J[\beta,A]\, e^{+iS + i\int(\bar\eta\psi + \bar\psi\eta)} e^{i\int dx\, \beta(x)(\partial_\mu j_5^\mu - 2imP + i\bar\eta\gamma_5\psi + i\bar\psi\gamma_5\eta)}$$

change variables from ψ' to ψ

typically this Jacobian is assumed to be 1 when one derives the WI. However, in this case we will see that $J \neq 1$, this will lead to VIOLATION of expected WI, i.e. to an ANOMALY.

One can see that the Jacobian takes the following form

$$J[\beta,A] = e^{-\int dx\, \beta(x)\, \mathcal{A}[A](x)} \qquad \text{(see QFT 3)}$$

Let's derive the anomalous WI. Take into account $\beta \ll 1$

$$Z[\eta,\bar\eta,A] = \int \mathcal{D}\psi\mathcal{D}\bar\psi\, e^{+iS + i\int(\bar\eta\psi + \bar\psi\eta)}\Big(1 + i\int dx\, \beta(x)\big[\partial_\mu j_5^\mu - 2imP - \mathcal{A}[A] + i\bar\eta\gamma_5\psi + i\bar\psi\gamma_5\eta\big]\Big)$$

$$= Z[\eta,\bar\eta,A] + i\int dx\, \beta(x) \langle \partial_\mu j_5^\mu - 2imP - \mathcal{A}[A] + i\bar\eta\gamma_5\psi + i\bar\psi\gamma_5\eta \rangle_{\eta,\bar\eta}$$

$$\Rightarrow \quad \langle \partial_\mu j_5^\mu - 2imP - \mathcal{A}[A] + i\bar\eta\gamma_5\psi + i\bar\psi\gamma_5\eta \rangle_{\eta,\bar\eta} = 0 \qquad (\mathcal{A})$$

$(*) \Rightarrow \quad \partial_\mu \langle j_5^\mu \rangle = 2im\langle P \rangle + \mathcal{A}[A] \qquad$ (setting $\eta = \bar{\eta} = 0$)

$\mathcal{A}[A]$ depends by the spacetime dimension.

In 4d, $\mathcal{A}$ is quadratic in A, hence it contributes to 3pt funct.

In 2d, $\mathcal{A}$ " linear " ", " " " 2pt ".

By performing the computation in 4d one obtains:

$$\mathcal{A}[A] = -\frac{1}{16\pi^2} \epsilon^{\mu\nu\alpha\beta} \, \mathrm{tr} \, F_{\mu\nu} F_{\alpha\beta}$$ (for the singlet anomaly)