Nov. 2

Def Dirichlet kernel

$$D_n(x) = \frac{1}{2} + \sum_{\ell=1}^{n} \cos(\ell x)$$

$$= \frac{\sin((n+\frac{1}{2})x)}{2\sin(\frac{x}{2})}$$

$$\sin\left(\left(n+\frac{1}{2}\right)x\right) =$$

$$= \sin\left(\frac{x}{2}\right) + \sum_{\ell=1}^{n}\left(\sin\left(\left(\ell+\frac{1}{2}\right)x\right) - \sin\left(\left(\ell-\frac{1}{2}\right)x\right)\right)$$

$$= \sin\left(\frac{x}{2}\right) + \sum_{\ell=1}^{n}\left(2\sin\left(\frac{x}{2}\right)\right)\cos(\ell x)$$

$$= \left(2\sin\frac{x}{2}\right) D_n(x)$$

Lemma

$$S_n f(x) = \frac{a_0}{2} + \sum_{1 \le j \le n} \left(a_j \cos(jx) + b_j \sin(jx) \right)$$

$$S_n f(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, D_n(x-t)\, dt$$

Pf

$$S_n f(x) = \frac{1}{2} \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, dt$$

$$+ \sum_{j=1}^{n} \left(\cos(jx) \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(jt) + \sin(jx) \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(jt) \right)$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left(\frac{1}{2} + \sum_{j=1}^{n} \cos(jx) \cos(jt) + \sum_{j=1}^{n} \sin(jx) \sin(jt) \right.$$

$\cos(j(x-t))$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \underbrace{\left(\frac{1}{2} + \sum_{j=1}^{n} \cos(j(x-t)) \right)}_{D_n(x-t)} dt$$

$$C^0(\mathbb{T}) \qquad\qquad G_\delta$$

<u>Theom</u> $\forall \; x \in \mathbb{T} \quad \exists \; f \in C^0(\mathbb{T})$

s.t. $\lim_{n \to +\infty} S_n f(x)$ is not finite

<u>Pf</u> $x = 0$

$$D_n(x) = \frac{\sin((n+\frac{1}{2})x)}{2 \sin(x/2)}$$

$$\|D_n\|_{L^1(\mathbb{T})} \xrightarrow{n \to +\infty} +\infty$$

$$\|D_n\|_{L^1} = 2 \int_0^{\pi} |\sin((n+\tfrac{1}{2})x| \frac{dx}{|2\sin(\frac{x}{2})|}$$

$$\left|\sin\left(\frac{x}{2}\right)\right| \leq \left|\frac{x}{2}\right| \quad \forall \; x$$

$$\geq 2 \int_0^{\pi} \left| \sin(\overbrace{(n+\tfrac{1}{2})x}^{y}) \right| \frac{dx}{x}$$

$$= 2 \int_0^{(n+\frac{1}{2})\pi} |\sin y| \frac{dy}{y}$$

$$> 2 \int_0^{n\pi} |\sin y| \frac{dy}{y}$$

$$= 2 \sum_{j=1}^{n} \int_{(j-1)\pi}^{j\pi} |\sin x| \frac{dx}{x}$$

$$> \frac{2}{\pi} \sum_{j=1}^{n} \frac{1}{j} \underbrace{\int_0^{\pi} |\sin x| \, dx}_{2}$$

$$\|D_n\|_{L^1} \geq \frac{4}{\pi} \sum_{j=1}^{n} \frac{1}{j} \xrightarrow[n \to +\infty]{} +\infty$$

$$g_n(t) = \operatorname{sign}\left(D_n(t)\right)$$

$$S_n g_n(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g_n(t)\, D_n(t)\, dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_n(t)|\, dt$$

$$= \frac{1}{2\pi} |D_n|_{L^1(\mathbb{T})}$$

By Lusin theorem $\forall$ $j \in \mathbb{N}$ $\exists$ $f_j \in C^0(\mathbb{T})$

$$\|f_j\|_{L^\infty(\mathbb{T})} \leq \|g_n\|_{L^\infty(\mathbb{T})} = 1$$

s.t.

$$|\{x : f_j(x) \neq g_n(x)\}| < \frac{1}{j}$$

$$\Rightarrow \quad f_j \xrightarrow{j \to +\infty} g_n \quad \text{in } L^1(\mathbb{T})$$

$$h \to S_n h(0) = ev_0\, S_n h$$

$$L^1(\mathbb{T}) \longrightarrow \mathbb{C}$$

$$ev_0 S_n f_j \xrightarrow{j \to +\infty} S_n g_n(0) = \frac{1}{2\pi} \|D_n\|_{L^1}$$

If $\lim\limits_{n \to +\infty} S_n f(0)$ existed finite

$\forall f \in C^0(\mathbb{T})$ then

$$\sup_n |S_n f(0)| < +\infty$$

$\forall f \in C^0(\mathbb{T})$

$$\Longrightarrow \exists\ C > 0$$

st $\|ev_0 S_n\|_{(C^0(\mathbb{T}))'} \leq C$

$\forall n.$

$$|f_j|_{L^\infty} \leq 1$$

$$C \geq \|ev_0 S_n\|_{(C^0(\mathbb{T}))'} \geq |S_n f_j(0)|$$

$$\xrightarrow{J\to+\infty} \frac{\|D_n\|_{L^1}}{2\pi}$$

$$2\pi C \geq \|D_n\|_{L^1} \xrightarrow{n\to+\infty} +\infty$$

We get a contradiction.

Therefore there must exist

$f \in C^0(\mathbb{T})$

s.t. $\left| S_n f(0) \right|$ is unbounded

Recall that if

$\{x_n\}$ is a sequence in $\mathbb{R}$

with $\lim\limits_{n\to+\infty} x_n = A$

$$\Downarrow \qquad \cancel{\Uparrow}$$

then $\lim\limits_{n\to+\infty} \dfrac{x_1+\dots+x_n}{n} = A$

$$\sigma_n f(x) = \frac{\sum_{j=0}^{n} S_j f(x)}{n+1}$$

Th. $\forall\ f \in C^0(\mathbb{T})$

$$\sigma_n f \to f \text{ in } C^0(\mathbb{T})$$

(Notice that if $f \in L^\infty(\mathbb{T}) \setminus C^0(\mathbb{T})$

$$\sigma_n f \not\to f$$)

Fejer Kernel

$$K_N(t) = \frac{1}{N+1} \sum_{n=0}^{N} D_n(t)$$

$$= \frac{1}{N+1} \sum_{n=0}^{N} \frac{\sin\left((n+\frac{1}{2})t\right)}{2\sin(\frac{t}{2})}$$

$$= \frac{2}{N+1} \left(\frac{\sin\left(\frac{N+1}{2} t\right)}{4 \sin(\frac{t}{2})} \right)^2$$

$$= \frac{1}{N+1} \sum_{n=0}^{N} \frac{\sin\left(\left(n+\frac{1}{2}\right)t\right) \sin\left(\frac{t}{2}\right)}{2 \sin^2\left(\frac{t}{2}\right)}$$

$$= \frac{1}{N+1} \sum_{n=0}^{N} \frac{\cos(nt) - \cos((n+1)t)}{4 \sin^2\left(\frac{t}{2}\right)}$$

$$\cos\left(\left(n-\frac{1}{2}\right)t + \frac{t}{2}\right) - \cos\left(\left(n+\frac{1}{2}\right)t + \frac{t}{2}\right)$$

$$\frac{1}{(N+1)} \frac{1 - \cos((N+1)t)}{4 \sin^2\left(\frac{t}{2}\right)} =$$

$$\sin^2\left(\frac{N+1}{2} t\right) = \frac{1 - \cos((N+1)t)}{2}$$

$$= \frac{\sin^2\left(\frac{N+1}{2} t\right)}{(N+1)\, 2^3 \sin^2\left(\frac{t}{2}\right)} =$$

$$K_N(t) = \frac{1}{N+1}\, 2 \left(\frac{\sin\left(\frac{N+1}{2} t\right)}{2^2 \sin\left(\frac{t}{2}\right)} \right)^2$$

Lemma 1) $K_N(t) \geq 0$ ✓

2) $\frac{1}{\pi}\int_{-\pi}^{\pi} K_N \, dt = 1$

3) $\mu_N(\delta) = \max\{K_N(t) : \delta \leq t \leq \pi\}$ $\delta > 0$

then $\mu_N(\delta) \xrightarrow[N \to +\infty]{} 0$

$$\int_{-\pi}^{\pi} K_N(t) = \frac{1}{N+1}\sum_{n=0}^{N}\int_{-\pi}^{\pi} D_n(t)$$

$$= \frac{1}{N+1}\sum_{n=0}^{N}\int_{-\pi}^{\pi}\left(\frac{1}{2} + \cancel{\sum_{\ell=1}^{n}\cos(\ell t)}\right) dt$$

$$= \frac{1}{N+1}\sum_{n=0}^{N}\pi = \pi$$

$\delta \leq t \leq \pi$

$|t| \geq \delta > 0$

$$|K_N(t)| \leq \frac{2}{N+1}\left(\frac{\sin\left(\frac{N+1}{2}t\right)}{\sin\left(\frac{t}{2}\right)}\right)^2$$

$$\leq \frac{2}{N+1} \frac{1}{\left(\sin \frac{\delta}{2}\right)^2} \xrightarrow{N \to +\infty} 0$$

$$\sigma_N f(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) K_N(t)\, dt$$

$$f \in C^0(\mathbb{T}) \longrightarrow \sigma_N f \xrightarrow{N \to +\infty} f$$

in $C^0(\mathbb{T})$

$$\left| \sigma_n f(x) - f(x) \right| =$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \left| f(x+t) - f(x) \right| K_n(t)\, dt$$

$$\leq \frac{1}{\pi} \int_{|t| \leq \delta} \left| f(x+t) - f(x) \right| K_n(t)\, dt$$

$$\frac{1}{\pi} \int_{|t| \geq \delta} \left| f(x+t) - f(x) \right| K_n(t)\, dt$$

$$= \mathrm{I} + \mathrm{II}$$

$$\mathrm{II} \leq \frac{2}{\pi} |f|_{L^\infty} \quad \mu_n(\delta) \quad 2\pi$$

$$= 4 \, |f|_{L^\infty} \, \mu_n(\delta) \xrightarrow{n \to +\infty} 0$$

f continuous in $\mathbb{T} \Rightarrow f$ is uniformly continuous. So $\forall \varepsilon > 0 \quad \exists \, \delta > 0$ s.t. for any arc C of length $< \delta \Rightarrow \underset{C}{\operatorname{osc}} |f| < \varepsilon$

$$\mathrm{I} \leq \frac{1}{\pi} \int_{|t| \leq \delta} \underbrace{|f(x+t) - f(x)|}_{< \varepsilon} K_n(t) \, dt$$

$$\leq \varepsilon \underbrace{\frac{1}{\pi} \int_{-\pi}^{\pi} K_n(t) \, dt}_{1}$$

Conclusion $\forall \varepsilon > 0 \quad \exists \, n_0$

(Chosen so that $4 \, |f|_{L^\infty} \mu_{n_0}(\delta) < \varepsilon$)

st $\quad |S_n f(x) - f(x)| < 2\varepsilon$

$\forall x$

if $\quad n > n_0$

$f(x)$

$$\sigma_N f(x) = \frac{2}{\pi(N+1)} \int_{-\pi}^{\pi} dt\, f(x+t) \left(\frac{\sin\left(\frac{N+1}{2}t\right)}{4 \sin\frac{t}{2}} \right)^2$$

$\downarrow$ uniformly

f

Theorem $\quad E, F$ Banach

if $T \in \mathcal{L}(E,F)$ is onto

$R(T) = F$, $\quad$ then T is an open

map

Corollary $\quad E, F$ Banach

and $T \in \mathcal{L}(E, F)$ onto. Then if T is 1-1 then T is an isomorphism and $T^{-1} \in \mathcal{L}(F, E)$

Pf There exists $T^{-1} : F \to E$ we have to prove it is bounded.

Since T is open we know that $\exists\ c > 0$ s.t.

$$T D_E(0,1) \supseteq D_F(0,c)$$

$$D_E(0,1) \supseteq T^{-1} D_F(0,c)$$

$$\Rightarrow \quad \forall\ y \in D_F(0,c)$$

$$\|T^{-1} y\|_E < 1 \quad \Rightarrow$$

sup $\quad \Rightarrow \quad \|T^{-1} y\|_E < \frac{1}{c} \quad \forall y \in D_F(0,1)$

$$\Rightarrow \quad \|T^{-1}\|_{\mathcal{L}(F,E)} \leq \frac{1}{c}$$

Theorem (*) $L^1(\mathbb{T}) \ni f \to \{\hat{f}(n)\}_{n\in\mathbb{Z}} \in c_0(\mathbb{Z})$

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

$$c_0(\mathbb{Z}) = \{ f : \mathbb{Z} \to \mathbb{C} \text{ s.t. } \lim_{n\to\infty} f(n) = 0 \}$$

$$\subseteq \ell^\infty(\mathbb{Z})$$

(*) is not a onto map.

Pf It is easy to see that $f \longrightarrow \hat{f}$ 1-1.

If it is also onto we get an isomorphism

$$L^1(\mathbb{T}) \ni f \longrightarrow \hat{f} \in c_0(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z})$$

$\Rightarrow \quad \hat{f} \longrightarrow f$ is also an isomorphism

$\exists \; C > 0$ s.t.

$$\|f\|_{L^1(\mathbb{T})} \leq C \, \|\hat{f}\|_{\ell^\infty(\mathbb{Z})}$$

But this is false

$$f = D_n(t)$$

$$+\infty \xleftarrow{n \to +\infty} \|D_n\|_{L^1(\mathbb{T})} \leq C \, \|\hat{D}_n\|_{\ell^\infty(\mathbb{Z})} \leq C$$

so this gives a contradiction

<u>Remark</u> $\displaystyle\sum_{n=2}^{\infty} \frac{\sin(nx)}{\log n}$ is not a Fourier series of an $L^1(\mathbb{T})$ function

$\displaystyle\sum_{n=2}^{\infty} \frac{\cos(nx)}{\log n}$ is a Fourier series of an $L^1(\mathbb{T})$ function

$a_n \quad \ell^p(\mathbb{Z})$

$1 \leq p \leq 2 \implies \exists f \in L^1(\mathbb{T})$

$\hat{f}(n) = a_n$

$2 < p < \infty \overset{?}{\implies} \exists f \in L^1(\mathbb{T})$

$\hat{f}(n) = a_n$